

Research Article

Existence and Uniqueness of Solutions for Higher-Order Three-Point Boundary Value Problems

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We are concerned with the higher-order nonlinear three-point boundary value problems: $x^{(n)} = f(t, x, x', \dots, x^{(n-1)})$, $n \geq 3$, with the three point boundary conditions $g(x(a), x'(a), \dots, x^{(n-1)}(a)) = 0$; $x^{(i)}(b) = \mu_i$, $i = 0, 1, \dots, n-3$; $h(x(c), x'(c), \dots, x^{(n-1)}(c)) = 0$, where $a < b < c$, $f : [a, c] \times \mathbb{R}^n \rightarrow \mathbb{R} = (-\infty, +\infty)$ is continuous, $g, h : \mathbb{R}^n \rightarrow \mathbb{R}$ are continuous, and $\mu_i \in \mathbb{R}$, $i = 0, 1, \dots, n-3$ are arbitrary given constants. The existence and uniqueness results are obtained by using the method of upper and lower solutions together with Leray-Schauder degree theory. We give two examples to demonstrate our result.

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1. Introduction

Higher-order boundary value problems were discussed in many papers in recent years; for instance, see [1–22] and references therein. However, most of all the boundary conditions in the above-mentioned references are for two-point boundary conditions [2–11, 14, 17–22], and three-point boundary conditions are rarely seen [1, 12, 13, 16, 18]. Furthermore works for nonlinear three point boundary conditions are quite rare in literatures.

The purpose of this article is to study the existence and uniqueness of solutions for higher order nonlinear three point boundary value problem

$$x^{(n)} = f\left(t, x, x', \dots, x^{(n-1)}\right), \quad n \geq 3, \quad (1.1)$$

with nonlinear three point boundary conditions

$$\begin{aligned} g(x(a), x'(a), \dots, x^{(n-1)}(a)) &= 0, \\ x^{(i)}(b) &= \mu_i, \quad i = 0, 1, \dots, n-3, \\ h(x(c), x'(c), \dots, x^{(n-1)}(c)) &= 0, \end{aligned} \tag{1.2}$$

where $a < b < c$, $f : [a, c] \times \mathbb{R}^n \rightarrow \mathbb{R} = (-\infty, +\infty)$ is a continuous function, $g, h : \mathbb{R}^n \rightarrow \mathbb{R}$ are continuous functions, and $\mu_i \in \mathbb{R}$, $i = 0, 1, \dots, n-3$ are arbitrary given constants. The tools we mainly used are the method of upper and lower solutions and Leray-Schauder degree theory.

Note that for the cases of $a = b$ or $b = c$ in the boundary conditions (1.2), our theorems hold also true. However, for brevity we exclude such cases in this paper.

2. Preliminary

In this section, we present some definitions and lemmas that are needed to our main results.

Definition 2.1. $\alpha(t), \beta(t) \in C^n[a, c]$ are called lower and upper solutions of BVP (1.1), (1.2), respectively, if

$$\begin{aligned} \alpha^{(n)}(t) &\geq f(t, \alpha(t), \alpha'(t), \dots, \alpha^{(n-1)}(t)), \quad t \in [a, c], \\ g(\alpha(a), \alpha'(a), \dots, \alpha^{(n-1)}(a)) &\leq 0, \\ \alpha^{(i)}(b) &\leq \mu_i, \quad i = 0, 1, \dots, n-3, \\ h(\alpha(c), \alpha'(c), \dots, \alpha^{(n-1)}(c)) &\leq 0, \\ \beta^{(n)}(t) &\leq f(t, \beta(t), \beta'(t), \dots, \beta^{(n-1)}(t)), \quad t \in [a, c], \\ g(\beta(a), \beta'(a), \dots, \beta^{(n-1)}(a)) &\geq 0, \\ \beta^{(i)}(b) &\geq \mu_i, \quad i = 0, 1, \dots, n-3, \\ h(\beta(c), \beta'(c), \dots, \beta^{(n-1)}(c)) &\geq 0. \end{aligned} \tag{2.1}$$

Definition 2.2. Let E be a subset of $[a, c] \times \mathbb{R}^n$. We say that $f(t, x_0, x_1, \dots, x_{n-1})$ satisfies the Nagumo condition on E if there exists a continuous function $\phi : [0, +\infty) \rightarrow (0, +\infty)$ such that

$$\begin{aligned} |f(t, x_0, x_1, \dots, x_{n-1})| &\leq \phi(|x_{n-1}|), \quad (t, x_0, x_1, \dots, x_{n-1}) \in E, \\ \int_0^{+\infty} \frac{s ds}{\phi(s)} &= +\infty. \end{aligned} \tag{2.2}$$

Lemma 2.3 (see [10]). *Let $f : [a, c] \times \mathbb{R}^n \rightarrow \mathbb{R}$ be a continuous function satisfying the Nagumo condition on*

$$E = \{(t, x_0, x_1, \dots, x_{n-1}) \in [a, c] \times \mathbb{R}^n : \gamma_i(t) \leq x_i \leq \Gamma_i(t), i = 0, 1, \dots, n-2\}, \tag{2.3}$$

where $\gamma_i(t), \Gamma_i(t) : [a, c] \rightarrow \mathbb{R}$ are continuous functions such that

$$\gamma_i(t) \leq \Gamma_i(t), \quad i = 0, 1, \dots, n-2, \quad t \in [a, c]. \quad (2.4)$$

Then there exists a constant $r > 0$ (depending only on $\gamma_{n-2}(t), \Gamma_{n-2}(t)$ and $\phi(t)$) such that every solution $x(t)$ of (1.1) with

$$\gamma_i(t) \leq x^{(i)}(t) \leq \Gamma_i(t), \quad i = 0, 1, \dots, n-2, \quad t \in [a, c] \quad (2.5)$$

satisfies $\|x^{(n-1)}\|_\infty \leq r$.

Lemma 2.4. Let $\phi : [0, +\infty) \rightarrow (0, +\infty)$ be a continuous function. Then boundary value problem

$$x^{(n)} = x^{(n-2)}\phi(|x^{(n-1)}|), \quad t \in [a, c], \quad (2.6)$$

$$x^{(n-2)}(a) = x^{(i)}(b) = x^{(n-2)}(c) = 0, \quad i = 0, 1, \dots, n-3 \quad (2.7)$$

has only the trivial solution.

Proof. Suppose that $x_0(t)$ is a nontrivial solution of BVP (2.6), (2.7). Then there exists $t_0 \in (a, c)$ such that $x_0^{(n-2)}(t_0) > 0$ or $x_0^{(n-2)}(t_0) < 0$. We may assume $x_0^{(n-2)}(t_0) > 0$. There exists $t_1 \in (a, c)$ such that

$$\max_{t \in [a, c]} x_0^{(n-2)}(t) := x_0^{(n-2)}(t_1) > 0. \quad (2.8)$$

Then $x_0^{(n-1)}(t_1) = 0$, $x_0^{(n)}(t_1) \leq 0$. From (2.6) we have

$$0 \geq x_0^{(n)}(t_1) = x_0^{(n-2)}(t_1)\phi(|x_0^{(n-1)}(t_1)|) > 0, \quad (2.9)$$

which is a contradiction. Hence BVP (2.6), (2.7) has only the trivial solution. \square

3. Main Results

We may now formulate and prove our main results on the existence and uniqueness of solutions for n th-order three point boundary value problem (1.1), (1.2).

Theorem 3.1. Assume that

(i) there exist lower and upper solutions $\alpha(t), \beta(t)$ of BVP (1.1), (1.2), respectively, such that

$$\begin{aligned} (-1)^{n-i}\alpha^{(i)}(t) &\leq (-1)^{n-i}\beta^{(i)}(t), \quad t \in [a, b], \quad i = 0, 1, \dots, n-2, \\ \alpha^{(i)}(t) &\leq \beta^{(i)}(t), \quad t \in [b, c], \quad i = 0, 1, \dots, n-2; \end{aligned} \quad (3.1)$$

- (ii) $f(t, x_0, \dots, x_{n-1})$ is continuous on $[a, c] \times \mathbb{R}^n$, $(-1)^{n-i} f(t, x_0, \dots, x_{n-1})$ is nonincreasing in x_i ($i = 0, 1, \dots, n-3$) on D_a^b , and $f(t, x_0, \dots, x_{n-1})$ is nonincreasing in x_i ($i = 0, 1, \dots, n-3$) on D_b^c and satisfies the Nagumo condition on D_a^c , where

$$\begin{aligned} \varphi_i(t) &= \min\{\alpha^{(i)}(t), \beta^{(i)}(t)\}, & \psi_i(t) &= \max\{\alpha^{(i)}(t), \beta^{(i)}(t)\}, & i &= 0, \dots, n-2, \\ D_a^b &= \{(t, x_0, \dots, x_{n-1}) \in [a, b] \times \mathbb{R}^n : \varphi_i(t) \leq x_i \leq \psi_i(t), i = 0, \dots, n-2\}, \\ D_b^c &= \{(t, x_0, \dots, x_{n-1}) \in [b, c] \times \mathbb{R}^n : \varphi_i(t) \leq x_i \leq \psi_i(t), i = 0, \dots, n-2\}, \\ D_a^c &= \{(t, x_0, \dots, x_{n-1}) \in [a, c] \times \mathbb{R}^n : \varphi_i(t) \leq x_i \leq \psi_i(t), i = 0, \dots, n-2\}; \end{aligned} \quad (3.2)$$

- (iii) $g(x_0, x_1, \dots, x_{n-1})$ is continuous on \mathbb{R}^n , and $(-1)^{n-i} g(x_0, x_1, \dots, x_{n-1})$ is nonincreasing in x_i ($i = 0, 1, \dots, n-3$) and nondecreasing in x_{n-1} on $\prod_{i=0}^{n-2} [\varphi_i(a), \psi_i(a)] \times \mathbb{R}$;
 (iv) $h(x_0, x_1, \dots, x_{n-1})$ is continuous on \mathbb{R}^n , and nonincreasing in x_i ($i = 0, 1, \dots, n-3$) and nondecreasing in x_{n-1} on $\prod_{i=0}^{n-2} [\varphi_i(c), \psi_i(c)] \times \mathbb{R}$.

Then BVP (1.1), (1.2) has at least one solution $x(t) \in C^n[a, c]$ such that for each $i = 0, 1, \dots, n-2$,

$$\begin{aligned} (-1)^{n-i} \alpha^{(i)}(t) &\leq (-1)^{n-i} x^{(i)}(t) \leq (-1)^{n-i} \beta^{(i)}(t), & t &\in [a, b], \\ \alpha^{(i)}(t) &\leq x^{(i)}(t) \leq \beta^{(i)}(t), & t &\in [b, c]. \end{aligned} \quad (3.3)$$

Proof. For each $i = 0, 1, \dots, n-2$ define

$$w_i(t, x) = \begin{cases} \psi_i(t), & x > \psi_i(t), \\ x, & \varphi_i(t) \leq x \leq \psi_i(t), \\ \varphi_i(t), & x < \varphi_i(t), \end{cases} \quad (3.4)$$

where $\varphi_i(t) = \min\{\alpha^{(i)}(t), \beta^{(i)}(t)\}$, $\psi_i(t) = \max\{\alpha^{(i)}(t), \beta^{(i)}(t)\}$.

For $\lambda \in [0, 1]$, we consider the auxiliary equation

$$\begin{aligned} x^{(n)}(t) &= \lambda f\left(t, w_0(t, x(t)), \dots, w_{n-2}\left(t, x^{(n-2)}(t)\right), x^{(n-1)}(t)\right) \\ &\quad + \left[x^{(n-2)}(t) - \lambda w_{n-2}\left(t, x^{(n-2)}(t)\right)\right] \phi\left(\left|x^{(n-1)}(t)\right|\right), \end{aligned} \quad (3.5)$$

where ϕ is given by the Nagumo condition, with the boundary conditions

$$\begin{aligned} x^{(n-2)}(a) &= \lambda \left[w_{n-2}\left(a, x^{(n-2)}(a)\right) - g\left(w_0(a, x(a)), \dots, w_{n-2}\left(a, x^{(n-2)}(a)\right), x^{(n-1)}(a)\right) \right], \\ x^{(i)}(b) &= \lambda \mu_i, & i &= 0, 1, \dots, n-3, \\ x^{(n-2)}(c) &= \lambda \left[w_{n-2}\left(c, x^{(n-2)}(c)\right) - h\left(w_0(c, x(c)), \dots, w_{n-2}\left(c, x^{(n-2)}(c)\right), x^{(n-1)}(c)\right) \right]. \end{aligned} \quad (3.6)$$

Then we can choose a constant $M_{n-2} > 0$ such that

$$-M_{n-2} < \alpha^{(n-2)}(t) \leq \beta^{(n-2)}(t) < M_{n-2}, \quad t \in [a, c], \quad (3.7)$$

$$f(t, \alpha(t), \dots, \alpha^{(n-2)}(t), 0) - [M_{n-2} + \alpha^{(n-2)}(t)]\phi(0) < 0, \quad t \in [a, c], \quad (3.8)$$

$$f(t, \beta(t), \dots, \beta^{(n-2)}(t), 0) + [M_{n-2} - \beta^{(n-2)}(t)]\phi(0) > 0, \quad t \in [a, c],$$

$$\left| \alpha^{(n-2)}(a) - g(\alpha(a), \dots, \alpha^{(n-2)}(a), 0) \right| < M_{n-2}, \quad (3.9)$$

$$\left| \beta^{(n-2)}(a) - g(\beta(a), \dots, \beta^{(n-2)}(a), 0) \right| < M_{n-2},$$

$$\left| \alpha^{(n-2)}(c) - h(\alpha(c), \dots, \alpha^{(n-2)}(c), 0) \right| < M_{n-2}, \quad (3.10)$$

$$\left| \beta^{(n-2)}(c) - h(\beta(c), \dots, \beta^{(n-2)}(c), 0) \right| < M_{n-2}.$$

In the following, we will complete the proof in four steps.

Step 1. Show that every solution $x(t)$ of BVP (3.5), (3.6) satisfies

$$\left| x^{(n-2)}(t) \right| < M_{n-2}, \quad t \in [a, c], \quad (3.11)$$

independently of $\lambda \in [0, 1]$.

Suppose that the estimate $|x^{(n-2)}(t)| < M_{n-2}$ is not true. Then there exists $t_0 \in [a, c]$ such that $x^{(n-2)}(t_0) \geq M_{n-2}$ or $x^{(n-2)}(t_0) \leq -M_{n-2}$. We may assume $x^{(n-2)}(t_0) \geq M_{n-2}$. There exists $t_1 \in [a, c]$ such that

$$\max_{t \in [a, c]} x^{(n-2)}(t) := x^{(n-2)}(t_1) (\geq M_{n-2} > 0). \quad (3.12)$$

There are three cases to consider.

Case 1 ($t_1 \in (a, c)$). In this case, $x^{(n-1)}(t_1) = 0$ and $x^{(n)}(t_1) \leq 0$. For $\lambda \in (0, 1]$, by (3.8), we get the following contradiction:

$$\begin{aligned} 0 &\geq x^{(n)}(t_1) \\ &= \lambda f(t_1, w_0(t_1, x(t_1)), \dots, w_{n-2}(t_1, x^{(n-2)}(t_1)), x^{(n-1)}(t_1)) \\ &\quad + [x^{(n-2)}(t_1) - \lambda w_{n-2}(t_1, x^{(n-2)}(t_1))] \phi(|x^{(n-1)}(t_1)|) \\ &= \lambda f(t_1, w_0(t_1, x(t_1)), \dots, w_{n-3}(t_1, x^{(n-3)}(t_1)), \beta^{(n-2)}(t_1), 0) \\ &\quad + [x^{(n-2)}(t_1) - \lambda \beta^{(n-2)}(t_1)] \phi(0) \\ &\geq \lambda \left\{ f(t_1, \beta(t_1), \dots, \beta^{(n-2)}(t_1), 0) + [M_{n-2} - \beta^{(n-2)}(t_1)] \phi(0) \right\} > 0, \end{aligned} \quad (3.13)$$

and for $\lambda = 0$, we have the following contradiction:

$$0 \geq x^{(n)}(t_1) = x^{(n-2)}(t_1)\phi(0) \geq M_{n-2}\phi(0) > 0. \quad (3.14)$$

Case 2 ($t_1 = a$). In this case,

$$\max_{t \in [a, c]} x^{(n-2)}(t) := x^{(n-2)}(a) (\geq M_{n-2} > 0), \quad (3.15)$$

and $x^{(n-1)}(a) \leq 0$. For $\lambda = 0$, by (3.6) we have the following contradiction:

$$0 < M_{n-2} \leq x^{(n-2)}(a) = 0. \quad (3.16)$$

For $\lambda \in (0, 1]$, by (3.9) and condition (iii) we can get the following contradiction:

$$\begin{aligned} M_{n-2} &\leq x^{(n-2)}(a), \\ &= \lambda \left[w_{n-2}(a, x^{(n-2)}(a)) - g(w_0(a, x(a)), \dots, w_{n-2}(a, x^{(n-2)}(a)), x^{(n-1)}(a)) \right], \\ &\leq \lambda \left[\beta^{(n-2)}(a) - g(\beta(a), \dots, \beta^{(n-2)}(a), 0) \right] < M_{n-2}. \end{aligned} \quad (3.17)$$

Case 3 ($t_1 = c$). In this case,

$$\max_{t \in [a, c]} x^{(n-2)}(t) := x^{(n-2)}(c) (\geq M_{n-2} > 0), \quad (3.18)$$

and $x^{(n-1)}(c) \geq 0$. For $\lambda = 0$, by (3.6) we have the following contradiction:

$$0 < M_{n-2} \leq x^{(n-2)}(c) = 0. \quad (3.19)$$

For $\lambda \in (0, 1]$, by (3.10) and condition (iv) we can get the following contradiction:

$$\begin{aligned} M_{n-2} &\leq x^{(n-2)}(c), \\ &= \lambda \left[w_{n-2}(c, x^{(n-2)}(c)) - h(w_0(c, x(c)), \dots, w_{n-2}(c, x^{(n-2)}(c)), x^{(n-1)}(c)) \right] \\ &\leq \lambda \left[\beta^{(n-2)}(c) - h(\beta(c), \dots, \beta^{(n-2)}(c), 0) \right] < M_{n-2}. \end{aligned} \quad (3.20)$$

By (3.6), the estimates

$$\left| x^{(i)}(t) \right| < M_i := (c - a)M_{i+1} + |\mu_i|, \quad i = 0, 1, \dots, n-3, \quad t \in [a, c] \quad (3.21)$$

are obtained by integration.

Step 2. Show that there exists $M_{n-1} > 0$ such that every solution $x(t)$ of BVP (3.5), (3.6) satisfies

$$|x^{(n-1)}(t)| < M_{n-1}, \quad t \in [a, c], \quad (3.22)$$

independently of $\lambda \in [0, 1]$.

Let

$$E = \{(t, x_0, \dots, x_{n-1}) \in [a, c] \times \mathbb{R}^n : |x_i| \leq M_i, i = 0, 1, \dots, n-2\}, \quad (3.23)$$

and define the function $F_\lambda : [a, c] \times \mathbb{R}^n \rightarrow \mathbb{R}$ as follows:

$$\begin{aligned} F_\lambda(t, x_0, \dots, x_{n-1}) &= \lambda f(t, w_0(t, x_0), \dots, w_{n-2}(t, x_{n-2}), x_{n-1}) \\ &\quad + [x_{n-2} - \lambda w_{n-2}(t, x_{n-2})] \phi(|x_{n-1}|). \end{aligned} \quad (3.24)$$

In the following, we show that $F_\lambda(t, x_0, \dots, x_{n-1})$ satisfies the Nagumo condition on E , independently of $\lambda \in [0, 1]$. In fact, since f satisfies the Nagumo condition on D_a^c , we have

$$\begin{aligned} |F_\lambda(t, x_0, \dots, x_{n-1})| &= |\lambda f(t, w_0(t, x_0), \dots, w_{n-2}(t, x_{n-2}), x_{n-1}) \\ &\quad + [x_{n-2} - \lambda w_{n-2}(t, x_{n-2})] \phi(|x_{n-1}|)| \\ &\leq [1 + 2M_{n-2}] \phi(|x_{n-1}|) := \phi_E(|x_{n-1}|). \end{aligned} \quad (3.25)$$

Furthermore, we obtain

$$\int_0^{+\infty} \frac{s}{\phi_E(s)} ds = \int_0^{+\infty} \frac{s}{(1 + 2M_{n-2})\phi(s)} ds = +\infty. \quad (3.26)$$

Thus, F_λ satisfies the Nagumo condition on E , independently of $\lambda \in [0, 1]$. Let

$$\gamma_i(t) = -M_i, \quad \Gamma_i(t) = M_i, \quad i = 0, 1, \dots, n-2, \quad t \in [a, c]. \quad (3.27)$$

By Step 1 and Lemma 2.3, there exists $M_{n-1} > 0$ such that $|x^{(n-1)}(t)| < M_{n-1}$ for $t \in [a, c]$. Since M_{n-2} and ϕ_E do not depend on λ , the estimate $|x^{(n-1)}(t)| < M_{n-1}$ on $[a, c]$ is also independent of λ .

Step 3. Show that for $\lambda = 1$, BVP (3.5), (3.6) has at least one solution $x_1(t)$.

Define the operators as follows:

$$L : C^n[a, c] \subset C^{n-1}[a, c] \longrightarrow C[a, c] \times \mathbb{R}^n, \quad (3.28)$$

by

$$\begin{aligned} Lx &= (x^{(n)}(t), x^{(n-2)}(a), x(b), \dots, x^{(n-3)}(b), x^{(n-2)}(c)), \\ N_\lambda &: C^{n-1}[a, c] \longrightarrow C[a, c] \times \mathbb{R}^n, \end{aligned} \quad (3.29)$$

by

$$N_\lambda x = \left(F_\lambda(t, x(t), \dots, x^{(n-1)}(t)), A_\lambda, \lambda\mu_0, \dots, \lambda\mu_{n-3}, C_\lambda \right), \quad (3.30)$$

with

$$\begin{aligned} A_\lambda &:= \lambda [w_{n-2}(a, x^{(n-2)}(a)) - g(w_0(a, x(a)), \dots, w_{n-2}(a, x^{(n-2)}(a)), x^{(n-1)}(a))] \\ C_\lambda &:= \lambda [w_{n-2}(c, x^{(n-2)}(c)) - h(w_0(c, x(c)), \dots, w_{n-2}(c, x^{(n-2)}(c)), x^{(n-1)}(c))]. \end{aligned} \quad (3.31)$$

Since L^{-1} is compact, we have the following compact operator:

$$T_\lambda : C^{n-1}[a, c] \longrightarrow C^{n-1}[a, c], \quad (3.32)$$

defined by

$$T_\lambda(x) = L^{-1}N_\lambda(x). \quad (3.33)$$

Consider the set $\Omega = \{x \in C^{n-1}[a, c] : \|x^{(i)}\|_\infty < M_i, i = 0, 1, \dots, n-1\}$.

By Steps 1 and 2, the degree $\deg(I - T_\lambda, \Omega, 0)$ is well defined for every $\lambda \in [0, 1]$, and by homotopy invariance, we get

$$\deg(I - T_0, \Omega, 0) = \deg(I - T_1, \Omega, 0). \quad (3.34)$$

Since the equation $x = T_0(x)$ has only the trivial solution from Lemma 2.4, by the degree theory we have

$$\deg(I - T_1, \Omega, 0) = \deg(I - T_0, \Omega, 0) = \pm 1. \quad (3.35)$$

Hence, the equation $x = T_1(x)$ has at least one solution. That is, the boundary value problem

$$\begin{aligned} x^{(n)}(t) &= f\left(t, w_0(t, x(t)), \dots, w_{n-2}\left(t, x^{(n-2)}(t)\right), x^{(n-1)}(t)\right) \\ &\quad + \left[x^{(n-2)}(t) - w_{n-2}\left(t, x^{(n-2)}(t)\right)\right] \phi\left(|x^{(n-1)}(t)|\right), \end{aligned} \quad (3.36)$$

with the boundary conditions

$$\begin{aligned} x^{(n-2)}(a) &= w_{n-2}(a, x^{(n-2)}(a)) - g(w_0(a, x(a)), \dots, w_{n-2}(a, x^{(n-2)}(a)), x^{(n-1)}(a)), \\ x^{(i)}(b) &= \mu_i, \quad i = 0, 1, \dots, n-3, \\ x^{(n-2)}(c) &= w_{n-2}(c, x^{(n-2)}(c)) - h(w_0(c, x(c)), \dots, w_{n-2}(c, x^{(n-2)}(c)), x^{(n-1)}(c)), \end{aligned} \quad (3.37)$$

has at least one solution $x_1(t)$ in Ω .

Step 4. Show that $x_1(t)$ is a solution of BVP (1.1), (1.2).

In fact, the solution $x_1(t)$ of BVP (3.36), (3.37) will be a solution of BVP (1.1), (1.2), if it satisfies

$$\varphi_i(t) \leq x_1^{(i)}(t) \leq \psi_i(t), \quad i = 0, 1, \dots, n-2, \quad t \in [a, c]. \quad (3.38)$$

By contradiction, suppose that there exists $t_0 \in [a, c]$ such that $x_1^{(n-2)}(t_0) > \psi_{n-2}(t_0)$. There exists $t_1 \in [a, c]$ such that

$$\max_{t \in [a, c]} [x_1^{(n-2)}(t) - \psi_{n-2}(t)] := x_1^{(n-2)}(t_1) - \psi_{n-2}(t_1) > 0. \quad (3.39)$$

Now there are three cases to consider.

Case 1 ($t_1 \in (a, c)$). In this case, since $\psi_{n-2}(t) = \beta^{(n-2)}(t)$ on $[a, c]$, we have $x_1^{(n-1)}(t_1) = \beta^{(n-1)}(t_1)$ and $x_1^{(n)}(t_1) \leq \beta^{(n)}(t_1)$. By conditions (i) and (ii), we get the following contradiction:

$$\begin{aligned} 0 &\geq x_1^{(n)}(t_1) - \beta^{(n)}(t_1) \\ &\geq f(t_1, w_0(t_1, x_1(t_1)), \dots, w_{n-2}(t_1, x_1^{(n-2)}(t_1)), x_1^{(n-1)}(t_1)) \\ &\quad + [x_1^{(n-2)}(t_1) - w_{n-2}(t_1, x_1^{(n-2)}(t_1))] \phi(|x_1^{(n-1)}(t_1)|) - f(t_1, \beta(t_1), \dots, \beta^{(n-1)}(t_1)) \\ &\geq f(t_1, \beta(t_1), \dots, \beta^{(n-1)}(t_1)) + [x_1^{(n-2)}(t_1) - \beta^{(n-2)}(t_1)] \phi(|x_1^{(n-1)}(t_1)|) \\ &\quad - f(t_1, \beta(t_1), \dots, \beta^{(n-1)}(t_1)) \\ &= [x_1^{(n-2)}(t_1) - \beta^{(n-2)}(t_1)] \phi(|x_1^{(n-1)}(t_1)|) > 0. \end{aligned} \quad (3.40)$$

Case 2 ($t_1 = a$). In this case, we have

$$\max_{t \in [a, c]} [x_1^{(n-2)}(t) - \psi_{n-2}(t)] := x_1^{(n-2)}(a) - \beta^{(n-2)}(a) > 0, \quad (3.41)$$

and $x_1^{(n-1)}(a) \leq \beta^{(n-1)}(a)$. By (3.37) and conditions (i) and (iii) we can get the following contradiction:

$$\begin{aligned} \beta^{(n-2)}(a) &< x_1^{(n-2)}(a), \\ &= w_{n-2}\left(a, x_1^{(n-2)}(a)\right) - g\left(w_0(a, x_1(a)), \dots, w_{n-2}\left(a, x_1^{(n-2)}(a)\right), x_1^{(n-1)}(a)\right) \\ &\leq \beta^{(n-2)}(a) - g\left(\beta(a), \dots, \beta^{(n-2)}(a), \beta^{(n-1)}(a)\right) \leq \beta^{(n-2)}(a). \end{aligned} \quad (3.42)$$

Case 3 ($t_1 = c$). In this case, we have

$$\max_{t \in [a, c]} \left[x_1^{(n-2)}(t) - \varphi_{n-2}(t) \right] := x_1^{(n-2)}(c) - \beta^{(n-2)}(c) > 0, \quad (3.43)$$

and $x_1^{(n-1)}(c) \geq \beta^{(n-1)}(c)$. By (3.37) and conditions (i) and (iv) we can get the following contradiction:

$$\begin{aligned} \beta^{(n-2)}(c) &< x_1^{(n-2)}(c) \\ &= w_{n-2}\left(c, x_1^{(n-2)}(c)\right) - h\left(w_0(c, x_1(c)), \dots, w_{n-2}\left(c, x_1^{(n-2)}(c)\right), x_1^{(n-1)}(c)\right) \\ &\leq \beta^{(n-2)}(c) - h\left(\beta(c), \dots, \beta^{(n-2)}(c), \beta^{(n-1)}(c)\right) \leq \beta^{(n-2)}(c). \end{aligned} \quad (3.44)$$

Similarly, we can show that $\varphi_{n-2}(t) \leq x_1^{(n-2)}(t)$ on $[a, c]$. Hence

$$\alpha^{(n-2)}(t) = \varphi_{n-2}(t) \leq x_1^{(n-2)}(t) \leq \varphi_{n-2}(t) = \beta^{(n-2)}(t), \quad t \in [a, c]. \quad (3.45)$$

Also, by boundary condition (3.37) and condition (i), we have

$$\begin{aligned} \alpha^{(i)}(b) &= x_1^{(i)}(b) = \beta^{(i)}(b), \quad i = n-1-2j, \quad j = 1, 2, \dots, \left\lfloor \frac{n-1}{2} \right\rfloor, \\ \alpha^{(i)}(b) &\leq x_1^{(i)}(b) \leq \beta^{(i)}(b), \quad i = n-2-2j, \quad j = 1, 2, \dots, \left\lfloor \frac{n-2}{2} \right\rfloor. \end{aligned} \quad (3.46)$$

Therefore by integration we have for each $i = 0, 1, \dots, n-2$,

$$\begin{aligned} (-1)^{n-i} \alpha^{(i)}(t) &\leq (-1)^{n-i} x_1^{(i)}(t) \leq (-1)^{n-i} \beta^{(i)}(t), \quad t \in [a, b], \\ \alpha^{(i)}(t) &\leq x_1^{(i)}(t) \leq \beta^{(i)}(t), \quad t \in [b, c], \end{aligned} \quad (3.47)$$

that is,

$$\varphi_i(t) \leq x_1^{(i)}(t) \leq \psi_i(t), \quad i = 0, 1, \dots, n-2, \quad t \in [a, c]. \quad (3.48)$$

Hence $x_1(t)$ is a solution of BVP (1.1), (1.2) and satisfies (3.3). \square

Now we give a uniqueness theorem by assuming additionally the differentiability for functions f , g and h , and a kind of estimating condition in Theorem 3.1.

Theorem 3.2. *Assume that*

(i) *there exist lower and upper solutions $\alpha(t), \beta(t)$ of BVP (1.1), (1.2), respectively, such that*

$$\begin{aligned} (-1)^{n-i} \alpha^{(i)}(t) &\leq (-1)^{n-i} \beta^{(i)}(t), \quad t \in [a, b], \quad i = 0, 1, \dots, n-2, \\ \alpha^{(i)}(t) &\leq \beta^{(i)}(t), \quad t \in [b, c], \quad i = 0, 1, \dots, n-2; \end{aligned} \quad (3.49)$$

(ii) *$f(t, x_0, \dots, x_{n-1})$ and its first-order partial derivatives in x_i ($i = 0, 1, \dots, n-1$) are continuous on $[a, c] \times \mathbb{R}^n$, $(-1)^{n-i}(\partial f / \partial x_i) \leq 0$ ($i = 0, 1, \dots, n-3$) on D_a^b , $\partial f / \partial x_i \leq 0$ ($i = 0, 1, \dots, n-3$) on D_b^c and satisfy the Nagumo condition on D_a^c ;*

(iii) *$g(x_0, x_1, \dots, x_{n-1})$ is continuous on \mathbb{R}^n and continuously partially differentiable on $\prod_{i=0}^{n-2} [\varphi_i(a), \psi_i(a)] \times \mathbb{R}$, and*

$$(-1)^{n-i} \frac{\partial g}{\partial x_i} \leq 0, \quad i = 0, 1, \dots, n-3, \quad (3.50)$$

$$\frac{\partial g}{\partial x_{n-1}} \leq 0, \quad \text{on } \prod_{i=0}^{n-2} [\varphi_i(a), \psi_i(a)] \times \mathbb{R};$$

(iv) *$h(x_0, x_1, \dots, x_{n-1})$ is continuous on \mathbb{R}^n and continuously partially differentiable on $\prod_{i=0}^{n-2} [\varphi_i(c), \psi_i(c)] \times \mathbb{R}$, and*

$$\frac{\partial h}{\partial x_i} \leq 0, \quad i = 0, 1, \dots, n-3, \quad (3.51)$$

$$\frac{\partial h}{\partial x_{n-1}} \geq 0, \quad \text{on } \prod_{i=0}^{n-2} [\varphi_i(c), \psi_i(c)] \times \mathbb{R};$$

(v) there exists a function $\gamma(t) \in C^n[a, c]$ such that $\gamma^{(n-2)}(t) > 0$ on $[a, c]$, and

$$\begin{aligned} \gamma^{(n)}(t) &< \sum_{i=0}^{n-1} \frac{\partial f}{\partial x_i} \cdot \gamma^{(i)}(t), \quad \text{on } D_a^c \\ \sum_{i=0}^{n-1} \frac{\partial g}{\partial x_i} \cdot \gamma^{(i)}(a) &> 0, \quad \text{on } \prod_{i=0}^{n-2} [\varphi_i(a), \psi_i(a)] \times \mathbb{R}, \\ \sum_{i=0}^{n-1} \frac{\partial h}{\partial x_i} \cdot \gamma^{(i)}(c) &> 0, \quad \text{on } \prod_{i=0}^{n-2} [\varphi_i(c), \psi_i(c)] \times \mathbb{R}, \\ \gamma^{(i)}(b) &= 0, \quad \text{if } n-i : \text{ odd}, \quad i = 0, 1, \dots, n-3, \\ \gamma^{(i)}(b) &\geq 0, \quad \text{if } n-i : \text{ even}, \quad i = 0, 1, \dots, n-3. \end{aligned} \tag{3.52}$$

Then BVP (1.1), (1.2) has a unique solution $x(t)$ satisfying (3.3).

Proof. The existence of a solution for BVP (1.1), (1.2) satisfying (3.3) follows from Theorem 3.1.

Now, we prove the uniqueness of solution for BVP (1.1), (1.2). To do this, we let $x_1(t)$ and $x_2(t)$ are any two solutions of BVP (1.1), (1.2) satisfying (3.3). Let $z(t) = x_2(t) - x_1(t)$. It is easy to show that $z(t)$ is a solution of the following boundary value problem

$$z^{(n)}(t) = \sum_{i=0}^{n-1} d_i(t) z^{(i)}(t), \tag{3.53}$$

$$\sum_{i=0}^{n-1} a_i z^{(i)}(a) = 0, \quad \sum_{i=0}^{n-1} c_i z^{(i)}(c) = 0, \tag{3.54}$$

$$z^{(i)}(b) = 0, \quad i = 0, 1, \dots, n-3, \tag{3.55}$$

where for each $i = 0, 1, \dots, n-1$,

$$\begin{aligned} d_i(t) &= \int_0^1 \frac{\partial}{\partial x_i} f\left(t, x_1(t) + \theta z(t), x_1'(t) + \theta z'(t), \dots, x_1^{(n-1)}(t) + \theta z^{(n-1)}(t)\right) d\theta, \\ a_i &= \int_0^1 \frac{\partial}{\partial x_i} g\left(x_1(a) + \theta z(a), x_1'(a) + \theta z'(a), \dots, x_1^{(n-1)}(a) + \theta z^{(n-1)}(a)\right) d\theta, \\ c_i &= \int_0^1 \frac{\partial}{\partial x_i} h\left(x_1(c) + \theta z(c), x_1'(c) + \theta z'(c), \dots, x_1^{(n-1)}(c) + \theta z^{(n-1)}(c)\right) d\theta. \end{aligned} \tag{3.56}$$

By conditions (ii), (iii), and (iv), we have that $d_i(t) \in C[a, c]$, $i = 0, 1, \dots, n-3$, and

$$\begin{aligned} (-1)^{n-i}d_i(t) &\leq 0, \quad i = 0, 1, \dots, n-3, \quad t \in [a, b], \\ d_i(t) &\leq 0, \quad i = 0, 1, \dots, n-3, \quad t \in [b, c], \\ (-1)^{n-i}a_i &\leq 0, \quad i = 0, 1, \dots, n-3, \quad a_{n-1} \leq 0, \\ c_i &\leq 0, \quad i = 0, 1, \dots, n-3, \quad c_{n-1} \geq 0. \end{aligned} \tag{3.57}$$

Now suppose that there exists $t_0 \in [a, c]$ such that $z^{(n-2)}(t_0) \neq 0$. Without loss of generality assume $z^{(n-2)}(t_0) > 0$, and let

$$\Omega = \left\{ M : Mz^{(n-2)}(t) < \gamma^{(n-2)}(t), t \in [a, c] \right\}. \tag{3.58}$$

It is easy to see that $0 \in \Omega$ by condition (v), hence $\Omega \neq \emptyset$. Let $M_0 = \sup \Omega$. We have that $0 < M_0 < +\infty$, $M_0z^{(n-2)}(t) \leq \gamma^{(n-2)}(t)$ on $[a, c]$, and there exists a point $t_1 \in [a, c]$ such that $M_0z^{(n-2)}(t_1) = \gamma^{(n-2)}(t_1)$. Furthermore $t_1 \neq a, c$. In fact, if $t_1 = a$, then $M_0z^{(n-1)}(a) \leq \gamma^{(n-1)}(a)$. By condition (v) and (3.55) we can easily show that

$$(-1)^{n-i} \left[M_0z^{(i)}(t) - \gamma^{(i)}(t) \right] \leq 0, \quad i = 0, 1, \dots, n-3, \quad t \in [a, b]. \tag{3.59}$$

In particular

$$(-1)^{n-i} \left[M_0z^{(i)}(a) - \gamma^{(i)}(a) \right] \leq 0, \quad i = 0, 1, \dots, n-3. \tag{3.60}$$

Hence

$$\sum_{i=0}^{n-1} M_0 a_i z^{(i)}(a) \geq \sum_{i=0}^{n-1} a_i \gamma^{(i)}(a) > 0, \tag{3.61}$$

which contradicts to (3.54). Thus $t_1 \neq a$. Similarly we can show that $t_1 \neq c$. Consequently $M_0z^{(n-1)}(t_1) = \gamma^{(n-1)}(t_1)$.

Now, there are two cases to consider, that is

$$t_1 \in (a, b] \quad \text{or} \quad t_1 \in [b, c). \tag{3.62}$$

If $t_1 \in (a, b]$, then by (3.59) we have

$$(-1)^{n-i} \left[M_0z^{(i)}(t_1) - \gamma^{(i)}(t_1) \right] \leq 0, \quad i = 0, 1, \dots, n-3. \tag{3.63}$$

Thus, by (3.53) and condition (v) we have

$$M_0 z^{(n)}(t_1) = \sum_{i=0}^{n-1} M_0 d_i(t_1) z^{(i)}(t_1) \geq \sum_{i=0}^{n-1} d_i(t_1) \gamma^{(i)}(t_1) > \gamma^{(n)}(t_1). \quad (3.64)$$

Consequently, by Taylor's theorem there exists $t_2 \in (t_1, c)$ such that

$$M_0 z^{(n-2)}(t) > \gamma^{(n-2)}(t), \quad \forall t \in (t_1, t_2], \quad (3.65)$$

which is a contradiction.

A similar contradiction can be obtained if $t_1 \in [b, c)$. Hence $z^{(n-2)}(t) \equiv 0$ on $[a, c]$. By (3.55), we obtain $z(t) \equiv 0$ on $[a, c]$. This completes the proof of the theorem. \square

Next we give two examples to demonstrate the application of Theorem 3.2.

Example 3.3. Consider the following third-order three point BVP:

$$x''' = -tx + (2t^2 + 1)x' + \frac{1}{3}(x')^3 - t^4 \sin(t + x''), \quad t \in [-1, 1], \quad (3.66)$$

$$1 + 3x'(-1) + (x'(-1))^3 - (x''(-1) + 1)^3 = 0,$$

$$x(0) = 0, \quad (3.67)$$

$$-1 - x(1) + 2x'(1) + (x'(1))^3 + (x''(1) + 1)^3 = 0.$$

Let

$$f(t, x_0, x_1, x_2) = -tx_0 + (2t^2 + 1)x_1 + \frac{1}{3}x_1^3 - t^4 \sin(t + x_2),$$

$$g(x_0, x_1, x_2) = 1 + 3x_1 + x_1^3 - (x_2 + 1)^3, \quad (3.68)$$

$$h(x_0, x_1, x_2) = -1 - x_0 + 2x_1 + x_1^3 + (x_2 + 1)^3.$$

Choose $\alpha(t) = -t$, $\beta(t) = t$ and $\gamma(t) = t$. It is easy to check that $\alpha(t) = -t$, and $\beta(t) = t$ are lower and upper solutions of BVP (3.66), (3.67) respectively, and all the assumptions in Theorem 3.2 are satisfied. Therefore by Theorem 3.2 BVP (3.66), (3.67) has a unique solution $x = x(t)$ satisfying

$$\begin{aligned} t \leq x(t) \leq -t, \quad t \in [-1, 0], \quad -t \leq x(t) \leq t, \quad t \in [0, 1], \\ -1 \leq x'(t) \leq 1, \quad t \in [-1, 1]. \end{aligned} \quad (3.69)$$

Example 3.4. Consider the following fourth-order three point BVP:

$$x^{(4)} = -t^2x + x'' + (x'')^3, \quad t \in [-1, 1], \quad (3.70)$$

$$-x(-1) + (x'(-1))^3 + 13x''(-1) = 0,$$

$$x(0) = 0, \quad x'(0) = 0, \quad (3.71)$$

$$-x(1) - 4x'(1) + (x'(1))^2 + 9x''(1) = 0.$$

Let

$$f(t, x_0, x_1, x_2, x_3) = -t^2x_0 + x_2 + x_2^3,$$

$$g(x_0, x_1, x_2, x_3) = -x_0 + x_1^3 + 13x_2, \quad (3.72)$$

$$h(x_0, x_1, x_2, x_3) = -x_0 - 4x_1 + x_1^2 + 9x_2.$$

Choose $\alpha(t) = -t^2$, $\beta(t) = t^2$ and $\gamma(t) = t^2$. It is easy to check that $\alpha(t) = -t^2$, and $\beta(t) = t^2$ are lower and upper solutions of BVP (3.70), (3.71), respectively, and all the assumptions in Theorem 3.2 are satisfied. Therefore by Theorem 3.2 BVP (3.70), (3.71) has a unique solution $x = x(t)$ satisfying

$$\begin{aligned} -2 &\leq x(t) \leq 2, \quad t \in [-1, 1], \\ 2t \leq x'(t) \leq -2t, \quad t \in [-1, 0], \quad -2t \leq x'(t) \leq 2t, \quad t \in [0, 1], \\ -t^2 &\leq x''(t) \leq t^2, \quad t \in [-1, 1]. \end{aligned} \quad (3.73)$$

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