Hindawi Publishing Corporation
Boundary Value Problems
Volume 2009, Article ID 509801, 18 pages
doi:10.1155/2009/509801

Research Article

Existence and Stability of Steady Waves for the Hasegawa-Mima Equation

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Received 28 September 2008; Accepted 18 February 2009

Recommended by Sandro Salsa

By introducing a compactness lemma and considering a constrained variational problem, we obtain a set $G_{\mathbb{R}^2}$ of steady waves for Hasegawa-Mima equation, which describes the motion of drift waves in plasma. Moreover, we prove that $G_{\mathbb{R}^2}$ is a stable set for the initial value problem of the equation, in the sense that a solution $\psi(t)$ which starts near $G_{\mathbb{R}^2}$ will remain near it for all time.

1. Introduction

It is commonly believed that drift waves and drift-wave turbulence play a major role in the understanding of anomalous transport at the plasma edge of a tokamak fusion reactor. One-field equation describing the electrostatic potential fluctuations in this regime is Hasegawa-Mima equation:

$$\frac{\partial(\Delta\psi-\psi)}{\partial t}+\frac{\partial\psi}{\partial x}\frac{\partial(\Delta\psi-\psi)}{\partial y}-\frac{\partial\psi}{\partial y}\frac{\partial(\Delta\psi-\psi)}{\partial x}=0, \tag{1.1}$$

where $(x,y)\in\mathbb{R}^2$, ψ describes the electrostatic fluctuation and $\psi(x,y)\to 0$ as $|x|+|y|\to+\infty$. The derivation of (1.1) can be found in [1].

There are many works about analytical mathematical study for (1.1); see, for example, [2, 3] and references therein. In [2], Grauer proved that the energy for a perturbed Hasegawa-Mima equation saturates at a finite level, which was observed by numerical simulations. Guo and Han in [3] studied the global well-posedness of Cauchy problem for (1.1). One of their results is that the solution $\psi(t)$ of (1.1) with $\psi(0)\in W^{2,2}(\mathbb{R}^2)\cap W^{2,\infty}(\mathbb{R}^2)$ exists globally and is unique. However, the global well-posedness of (1.1) with $\psi(0) \in W^{2,q}(\mathbb{R}^2)$ is still not attacked, where $W^{2,q}(\mathbb{R}^2)$ is the usual Sobolev space with norm $\|\cdot\|_{2,q}$ and $1 < q < 2$. A natural problem is whether the solution of (1.1) with the initial data $\psi(0)$ is close to a steady wave ψ_0 for all time or not, if $\psi(0)$ is sufficiently close to ψ_0 in $W^{2,q}(\mathbb{R}^2)$. The problem is concerned with the existence and stability of steady waves of (1.1).

Here, we are interested in studying the above problem. ψ_0 is a steady wave for (1.1) if and only if there exists some function f such that $\psi_0 = f(\Delta\psi_0 - \psi_0)$. In order to prove the existence and nonlinear stability of steady waves for (1.1), we consider the existence and property of critical points of the so-called energy-Casimir functional:

$$I_{\mathbb{R}^2}(\psi) = \Phi_{\mathbb{R}^2}(\psi) + \Psi_{\mathbb{R}^2}(\psi), \tag{1.2}$$

where

$$\begin{aligned} \Phi_{\mathbb{R}^2}(\psi) &= \frac{1}{2}\int_{\mathbb{R}^2} \left(|\nabla\psi|^2 + |\psi|^2\right), \\ \Psi_{\mathbb{R}^2}(\psi) &= \int_{\mathbb{R}^2} F(\Delta\psi - \psi) \end{aligned} \tag{1.3}$$

are two conserved quantities of (1.1), called the total energy and the generalized enstrophy, respectively. Here $\int_{\mathbb{R}^2} \cdot\, dx\, dy$ is denoted by $\int_{\mathbb{R}^2} \cdot$, and F is an arbitrary C^1 function. The critical points ψ_0 of $I_{\mathbb{R}^2}$ are steady waves of (1.1), given by $\psi_0 = f(\Delta\psi_0 - \psi_0)$, where $f = F'$.

A usual approach to prove the existence of stable critical points of $I_{\mathbb{R}^2}$ is to find extremum points of it, which is the well-known Liapunov method. If ψ_0 is a global or local extremum point of $I_{\mathbb{R}^2}$ in an appropriate defined function space X, then it follows that ψ_0 is a steady nonlinearly stable solutions of (1.1); see, for example, [4]. There are two examples for F such that $I_{\mathbb{R}^2}$ have a global extremum. One is F satisfying that $F''(x) \geq 0$ for any $x \in \mathbb{R}$ and there exists ψ_0 such that $\psi_0 = F'(\Delta\psi_0 - \psi_0)$. In this case,

$$I_{\mathbb{R}^2}(\psi) - I_{\mathbb{R}^2}(\psi_0) \geq \frac{1}{2}\int_{\mathbb{R}^2} \left(\left|\nabla(\psi - \psi_0)\right|^2 + \left|\psi - \psi_0\right|^2\right), \tag{1.4}$$

which implies that ψ_0 is a global minimizer of $I_{\mathbb{R}^2}$. Therefore, the steady wave ψ_0 is nonlinearly stable in the following sense: for any $\varepsilon > 0$, there exists $\delta > 0$ such that if $\|\psi(0) - \psi_0\|_X < \delta$ and $\psi(t) \in C([0,T), X)$ is a solution of (1.1) with initial data $\psi(0)$, then for any $t \in [0,T)$, $\|\psi(t) - \psi_0\|_1 < \varepsilon$, where $\|\cdot\|_1$ is the norm of $H^1(\mathbb{R}^2)$ and $T > 0$. The other example is F, which has the properties: $F''(x) \leq -c$ for large enough $c > 0$ and there exists ψ_0 such that $F'(\Delta\psi_0 - \psi_0) = \psi_0$. In this example,

$$\begin{aligned} I_{\mathbb{R}^2}(\psi) - I_{\mathbb{R}^2}(\psi_0) &\leq \frac{1}{2}\int_{\mathbb{R}^2} \left(\left|\nabla(\psi - \psi_0)\right|^2 + \left|\psi - \psi_0\right|^2\right) - c\int_{\mathbb{R}^2} \left|\Delta(\psi - \psi_0) - (\psi - \psi_0)\right|^2 \\ &\leq -c'\int_{\mathbb{R}^2} \left|\Delta(\psi - \psi_0) - (\psi - \psi_0)\right|^2, \end{aligned} \tag{1.5}$$

where $c' > 0$. So ψ_0 is a global maximizer of the functional $I_{\mathbb{R}^2}$ and nonlinearly stable in the above sense.

However, for some F, all critical points of $I_{\mathbb{R}^2}$ are neither global nor local extremum points of $I_{\mathbb{R}^2}$. Among them, some critical points are saddle points regarded as an unstable equilibria or transient excited state of (1.1). In the present paper, we consider the existence and stability defined later of saddle points of $I_{\mathbb{R}^2}$ for $F(x) = -(1/q)|x|^q$, where $q = p/(p-1)$, $2 < p < \infty$, and $x \in \mathbb{R}$. Since $F''(x) < 0$ and there does not exist a positive constant $c > 0$ such that $F''(x) < -c$ for any $x \neq 0$, F is not within the range of the above two examples. As shown in Proposition A.1 (Proposition A.1, Definition A.2, Proposition A.3, and Remark A.4 are given in the appendix), the functional (1.2) is neither bounded from above nor from below in $W^{2,q}(\mathbb{R}^2)$, that is, it is impossible to prove the existence of critical points by finding global extremum points of $I_{\mathbb{R}^2}$. However, through studying the constrained variational problem

$$M_{\mathbb{R}^2} = \inf_{\tilde{\varphi} \in W^{2,q}(\mathbb{R}^2), \|\tilde{\varphi}\|_1 = 1} \int_{\mathbb{R}^2} |\Delta \tilde{\varphi} - \tilde{\varphi}|^q, \tag{$M_{\mathbb{R}^2}$}$$

we obtain the existence of critical points of (1.2). In fact, if $\tilde{\varphi}_0$ is a minimizer for ($M_{\mathbb{R}^2}$), then, according to Lagrange Multiplier Principle, with the transform $\varphi_0 = M_{\mathbb{R}^2}^{1/(2-q)} \tilde{\varphi}_0$, φ_0 is a steady wave of (1.1) in $\mathbb{R}^2$. With Definition A.2 and Proposition A.3, φ_0 is a ground state and saddle point of $I_{\mathbb{R}^2}$. Let $Z_{\mathbb{R}^2}$ be the set of all minimizers for ($M_{\mathbb{R}^2}$), that is,

$$Z_{\mathbb{R}^2} = \left\{ \tilde{\varphi}_0;\ \int_{\mathbb{R}^2} \left(\left|\nabla \tilde{\varphi}_0\right|^2 + \left|\tilde{\varphi}_0\right|^2 \right) = 1,\ \int_{\mathbb{R}^2} \left(\left|\Delta \tilde{\varphi}_0 - \tilde{\varphi}_0\right|^q \right) = M_{\mathbb{R}^2} \right\}, \tag{1.6}$$

and let $G_{\mathbb{R}^2}$ be the set of steady waves of (1.1) corresponding to minimizers of ($M_{\mathbb{R}^2}$), that is,

$$G_{\mathbb{R}^2} = \left\{ \varphi_0;\ \varphi_0 = M_{\mathbb{R}^2}^{1/(2-q)} \tilde{\varphi}_0,\ \tilde{\varphi}_0 \in Z_{\mathbb{R}^2} \right\}. \tag{1.7}$$

As is presented in Remark A.4, $G_{\mathbb{R}^2}$ is the set of all ground states of the functional $I_{\mathbb{R}^2}$. Although the elements of $G_{\mathbb{R}^2}$ are saddle points of $I_{\mathbb{R}^2}$ regarded as an unstable state of (1.1), we prove that $G_{\mathbb{R}^2}$ is a stable set in the sense of Definition 1.1, that is, a solution $\psi(t)$ of (1.1) which starts near $G_{\mathbb{R}^2}$ will remain near it for all time.

Definition 1.1. Let E be a function space with norm $\|\cdot\|_E$, and $T \in (0, \infty]$. A set $G \subset E$ is called E-stable with respect to (1.1) in $\mathbb{R}^2$, if for any $\varepsilon > 0$, there exists a $\delta > 0$ such that if $\psi \in C([0,T), E)$ is a solution to (1.1) with initial data $\psi(0)$ satisfying $\inf_{\varphi_0 \in G} \|\psi(0) - \varphi_0\|_E \leq \delta$, then for any $t \in [0,T)$, $\inf_{\varphi_0 \in G} \|\psi(t,\cdot) - \varphi_0\|_E \leq \varepsilon$.

One gives some explanations for the above definition as follows. If G has only one element φ_0, then the steady wave φ_0 is nonlinearly stable in the usual sense. But, in general, the elements of G might not be unique. For example, as shown in Theorem 1.3, $G_{\mathbb{R}^2} = \{M_{\mathbb{R}^2}^{1/(2-q)} \tilde{\varphi}_0(\cdot + y), -M_{\mathbb{R}^2}^{1/(2-q)} \tilde{\varphi}_0(\cdot + y);\ y \in \mathbb{R}^2\}$. In this case, if $\psi(0)$ is sufficiently close to $G_{\mathbb{R}^2}$, then, for any $t > 0$, the form of $\psi(t)$ is almost similar to that of $M_{\mathbb{R}^2}^{1/(2-q)} \tilde{\varphi}_0$ or $-M_{\mathbb{R}^2}^{1/(2-q)} \tilde{\varphi}_0$.

Now one turns to describe his two main results.

Theorem 1.2. *If $\{\tilde{\varphi}_m\}$ is a minimizing sequence for ($M_{\mathbb{R}^2}$), then there exist $\{y_{m_k}\} \subset \mathbb{R}^2$ and a subsequence $\{\tilde{\varphi}_{m_k}\}$ such that $\{\tilde{\varphi}_{m_k}(\cdot + y_{m_k})\}$ is a convergent sequence in $W^{2,q}(\mathbb{R}^2)$. In particular, the minimization problem ($M_{\mathbb{R}^2}$) has a minimizer $\tilde{\varphi}_0$.*

Theorem 1.3. *According to Definition 1.1, the ground state set $G_{\mathbb{R}^2}$ is $W^{2,q}(\mathbb{R}^2)$-stable with respect to* (1.1) *in $\mathbb{R}^2$. Moreover, there is a unique, up to translation, positive radially symmetric C^2 minimizer $\tilde{\varphi}_0$ of $(M_{\mathbb{R}^2})$, and*

$$G_{\mathbb{R}^2} = \left\{ M_{\mathbb{R}^2}^{1/(2-q)} \tilde{\varphi}_0(\cdot + y), -M_{\mathbb{R}^2}^{1/(2-q)} \tilde{\varphi}_0(\cdot + y);\ y \in \mathbb{R}^2 \right\}. \tag{1.8}$$

The important step to obtain Theorems 1.2 and 1.3 is to prove that the infimum is achieved. If $\{\tilde{\varphi}_m\}$ is a minimizing sequence of $(M_{\mathbb{R}^2})$, then

$$\|\tilde{\varphi}_m\|_1 = 1, \quad \int_{\mathbb{R}^2} |\Delta \tilde{\varphi}_m - \tilde{\varphi}_m|^q \longrightarrow M_{\mathbb{R}^2}, \quad m \longrightarrow +\infty. \tag{1.9}$$

Going if necessary to a subsequence, we may assume $\tilde{\varphi}_m \rightarrow \tilde{\varphi}_0$ weakly in $W^{2,q}(\mathbb{R}^2)$, so that

$$\int_{\mathbb{R}^2} |\Delta \tilde{\varphi}_0 - \tilde{\varphi}_0|^q \le \liminf \int_{\mathbb{R}^2} |\Delta \tilde{\varphi}_m - \tilde{\varphi}_m|^q = M_{\mathbb{R}^2}. \tag{1.10}$$

Thus $\tilde{\varphi}_0$ is a minimizer of $(M_{\mathbb{R}^2})$ provided $\|\tilde{\varphi}_0\|_1 = 1$. Since $W^{2,q}(\mathbb{R}^2) \hookrightarrow H^1(\mathbb{R}^2)$ is not compact, we cannot derive $\|\tilde{\varphi}_0\|_1 = 1$ from $\|\tilde{\varphi}_m\|_1 = 1$ and $\tilde{\varphi}_m \rightarrow \tilde{\varphi}_0$ weakly in $W^{2,q}(\mathbb{R}^2)$. Therefore, we cannot directly derive the existence of minimizer from any minimizing sequence. However, if we obtain the result that for any minimizing sequence $\{\tilde{\varphi}_m\}$ there exist $\{y_{m_k}\} \subset \mathbb{R}^2$ and a subsequence $\{\tilde{\varphi}_{m_k}\}$ such that $\{\tilde{\varphi}_{m_k}(\cdot + y_{m_k})\}$ is a convergent sequence in $W^{2,q}(\mathbb{R}^2)$, which is the first part of Theorem 1.2, then we prove that the infimum is achieved.

In order to prove Theorem 1.2, we construct Lemma 2.1, which is used to study the behavior at infinity of the minimizing sequence $\{\tilde{\varphi}_m\}$ and to overcome the loss of compactness of $(M_{\mathbb{R}^2})$. Theorem 1.2 is proved by two steps. Firstly, using Lemma 2.4, we prove that for any minimizing sequence $\{\tilde{\varphi}_m\}$, there exist a subsequence $\{\tilde{\varphi}_{m_k}\}$ and $\{y_{m_k}\} \subset \mathbb{R}^2$ such that $\tilde{\varphi}_{m_k}(\cdot + y_{m_k}) \rightarrow \tilde{\varphi}_0 \neq 0$ weakly in $W^{2,q}(\mathbb{R}^2)$, which denotes $0 \le \alpha_\infty < 1$. Here α_∞ is a quantity related to $\{\tilde{\varphi}_{m_k}(\cdot + y_{m_k})\}$ defined in Lemma 2.1. Secondly, according to Lemmas 2.1 and 2.3 based on Ekeland Principle, we know that if $\alpha_\infty > 0$, then $\alpha_\infty \ge 1$. Therefore, putting together the results of the above steps, we obtain $\alpha_\infty = 0$, which implies that there exists a sequence $y_{m_k} \subset \mathbb{R}^2$ such that the sequence $\{\tilde{\varphi}_{m_k}(\cdot + y_{m_k})\}$ is convergent in $W^{2,q}(\mathbb{R}^2)$. Applying Theorem 1.2, we prove that $G_{\mathbb{R}^2}$ is a stable set with respect to (1.1), which is the first part of Theorem 1.3. The second part about the structure of $G_{\mathbb{R}^2}$ is obtained by studying the properties of the elliptic equation satisfied by $M_{\mathbb{R}^2}^{1/(2-q)} \tilde{\varphi}_0$. Our method in proving the existence and stability of steady waves for (1.1) is different from that in [5]. In [5], Albert considered constrained variational problems with concentration-compactness Lemma introduced by Lions [6, 7] and proved the existence and stability of solitary waves to Kdv equation and some nonlocal equations.

The paper is organized as follows. In Section 2, we establish three lemmas for proving Theorem 1.2. In Section 3, we give the proofs of Theorems 1.2 and 1.3. In Section 4, we consider the existence and stability of steady waves for Hasegawa-Mima equation in general periodic domains and give the application of Lemmas 2.1 and 2.3 to study the existence and stability of steady waves for two-dimensional incompressible fluid in an infinite strip channel. Two propositions about the properties of the functional $I_{\mathbb{R}^2}$ for $F(x) = -(1/q)|x|^q$ and the definition of the ground state of $I_{\mathbb{R}^2}$ are presented in the appendix.

2. Three Lemmas

At first, we give some notations used later. Let $L^q(\Omega)$ be the usual Lebesgue space with norm $|\cdot|_q$, where Ω is an unbounded domain in $\mathbb{R}^2$ and $q = p/(p-1)$, $2 < p < \infty$. The space $W_0^{2,q}(\Omega)$ is the completion of $D(\Omega)$ with respect to $\|\cdot\|_{2,q}$, where $D(\Omega)$ is the set of all C^∞-functions with compact support in Ω. The space $H_0^1(\Omega)$ is the completion of $D(\Omega)$ with respect to $\|\cdot\|_1$.

Now we give Lemma 2.1, which is used to study the behavior at infinity of minimizing sequence $\{\tilde{\varphi}_m\}$ for $(M_{\mathbb{R}^2})$.

Lemma 2.1 (compactness lemma). *Let $\{\tilde{\varphi}_m\} \subset W_0^{2,q}(\Omega)$ be a sequence such that $\tilde{\varphi}_m \to \tilde{\varphi}$ weakly in $W_0^{2,q}(\Omega)$ and define*

$$\alpha_\infty := \lim_{R\to\infty} \limsup_{m\to\infty} \int_{\Omega\cap\{|x|>R\}} \left(\left|\nabla\tilde{\varphi}_m\right|^2 + \left|\tilde{\varphi}_m\right|^2\right),$$
$$\beta_\infty := \lim_{R\to\infty} \limsup_{m\to\infty} \int_{\Omega\cap\{|x|>R\}} \left|\Delta\tilde{\varphi}_m - \tilde{\varphi}_m\right|^q. \tag{2.1}$$

Then one has

(1) $\limsup_{m\to\infty}\int_\Omega(|\nabla\tilde{\varphi}_m|^2 + |\tilde{\varphi}_m|^2) = \int_\Omega(|\nabla\tilde{\varphi}|^2 + |\tilde{\varphi}|^2) + \alpha_\infty$,

(2) $\limsup_{m\to\infty}\int_\Omega|\Delta\tilde{\varphi}_m - \tilde{\varphi}_m|^q \geq \int_\Omega|\Delta\tilde{\varphi} - \tilde{\varphi}|^q + \beta_\infty$,

(3) $(\alpha_\infty)^{q/2} M_\Omega \leq \beta_\infty$, *where*

$$M_\Omega = \inf_{\tilde{\varphi}\in W_0^{2,q}(\Omega),\|\tilde{\varphi}\|_1=1} \int_\Omega \left|\Delta\tilde{\varphi} - \tilde{\varphi}\right|^q. \tag{M_Ω}$$

Proof. (i) For any $R > 0$, let $B_R = \{x; |x| < R\}$, $B_R^c = \{x; |x| \geq R\}$:

$$\limsup_{m\to\infty} \int_\Omega \left(\left|\nabla\tilde{\varphi}_m\right|^2 + \left|\tilde{\varphi}_m\right|^2\right)$$
$$= \limsup_{m\to\infty} \left[\int_{\Omega\cap B_R} \left(\left|\nabla\tilde{\varphi}_m\right|^2 + \left|\tilde{\varphi}_m\right|^2\right) + \int_{\Omega\cap B_R^c} \left(\left|\nabla\tilde{\varphi}_m\right|^2 + \left|\tilde{\varphi}_m\right|^2\right)\right]. \tag{2.2}$$

Since the imbedding $W^{2,q}(\Omega\cap B_R) \hookrightarrow H^1(\Omega\cap B_R)$ is compact and $\tilde{\varphi}_m \to \tilde{\varphi}$ weakly in $W_0^{2,q}(\Omega)$,

$$\lim_{m\to\infty} \int_{\Omega\cap B_R} \left(\left|\nabla\tilde{\varphi}_m\right|^2 + \left|\tilde{\varphi}_m\right|^2\right) = \int_{\Omega\cap B_R} \left(\left|\nabla\tilde{\varphi}\right|^2 + \left|\tilde{\varphi}\right|^2\right). \tag{2.3}$$

Combining (2.2) with (2.3), we get

$$\limsup_{m\to\infty}\int_{\Omega}\left(\left|\nabla\widetilde{\varphi}_m\right|^2+\left|\widetilde{\varphi}_m\right|^2\right) = \int_{\Omega\cap B_R}\left(\left|\nabla\widetilde{\varphi}\right|^2+\left|\widetilde{\varphi}\right|^2\right)+\limsup_{m\to\infty}\int_{\Omega\cap B_R^c}\left(\left|\nabla\widetilde{\varphi}_m\right|^2+\left|\widetilde{\varphi}_m\right|^2\right). \tag{2.4}$$

Letting $R \to \infty$ in the above formula, we obtain (1).

(ii) Using the weakly lower semicontinuity of a norm, we have

$$\liminf_{m\to\infty}\int_{\Omega\cap B_R}\left|\Delta\widetilde{\varphi}_m-\widetilde{\varphi}_m\right|^q \geq \int_{\Omega\cap B_R}\left|\Delta\widetilde{\varphi}-\widetilde{\varphi}\right|^q, \quad \text{for any } R>0. \tag{2.5}$$

Applying (2.5), we have

$$\begin{aligned}\limsup_{m\to\infty}&\int_{\Omega}\left|\Delta\widetilde{\varphi}_m-\widetilde{\varphi}_m\right|^q\\ &\geq \liminf_{m\to\infty}\int_{\Omega\cap B_R}\left|\Delta\widetilde{\varphi}_m-\widetilde{\varphi}_m\right|^q+\limsup_{m\to\infty}\int_{\Omega\cap B_R^c}\left|\Delta\widetilde{\varphi}_m-\widetilde{\varphi}_m\right|^q\\ &\geq \int_{\Omega\cap B_R}\left|\Delta\widetilde{\varphi}-\widetilde{\varphi}\right|^q+\limsup_{m\to\infty}\int_{\Omega\cap B_R^c}\left|\Delta\widetilde{\varphi}_m-\widetilde{\varphi}_m\right|^q.\end{aligned} \tag{2.6}$$

Letting $R \to \infty$ in (2.6), we deduce (2).

(iii) Applying elementary inequalities, we can prove that the norm $|(\Delta-1)\cdot|_q$ is equivalent to $\|\cdot\|_{2,q}$ in $W_0^{2,q}(\Omega)$. Let $\varphi_R \in C^\infty(\mathbb{R}^2)$ such that

$$\varphi_R(x) = \begin{cases} 0 & |x|<R, \\ 1 & |x|>R+1 \end{cases} \tag{2.7}$$

and $0 \leq \varphi_R(x) \leq 1$ on $\mathbb{R}^2$. It follows from the definition of M_Ω and the convexity of the function $g(x)=|x|^q$ $(x\in\mathbb{R})$ that

$$\begin{aligned} M_\Omega&\left[\int_{\Omega}\left(\left|\nabla\widetilde{\varphi}_m\varphi_R\right|^2+\left|\widetilde{\varphi}_m\varphi_R\right|^2\right)\right]^{q/2}\\ &\leq \int_{\Omega}\left|\Delta\left(\widetilde{\varphi}_m\varphi_R\right)-\widetilde{\varphi}_m\varphi_R\right|^q\\ &= \int_{\Omega}\left|\varphi_R\left(\Delta\widetilde{\varphi}_m-\widetilde{\varphi}_m\right)+\widetilde{\varphi}_m\Delta\varphi_R+2\nabla\widetilde{\varphi}_m\cdot\nabla\varphi_R\right|^q\\ &= \int_{\Omega}\left|(1-\varepsilon)\frac{\varphi_R}{1-\varepsilon}\left(\Delta\widetilde{\varphi}_m-\widetilde{\varphi}_m\right)+\varepsilon\left(\widetilde{\varphi}_m\Delta\frac{\varphi_R}{\varepsilon}+2\nabla\widetilde{\varphi}_m\cdot\nabla\frac{\varphi_R}{\varepsilon}\right)\right|^q\\ &\leq (1-\varepsilon)\int_{\Omega}\left(\frac{\varphi_R}{1-\varepsilon}\right)^q\left|\Delta\widetilde{\varphi}_m-\widetilde{\varphi}_m\right|^q+\varepsilon\int_{\Omega}\left|\widetilde{\varphi}_m\Delta\frac{\varphi_R}{\varepsilon}+2\nabla\widetilde{\varphi}_m\cdot\nabla\frac{\varphi_R}{\varepsilon}\right|^q,\end{aligned} \tag{2.8}$$

where $0 < \varepsilon < 1$. Since $\tilde{\varphi}_m \to \tilde{\varphi}$ weakly in $W_0^{2,q}(\Omega)$ and the embedding $W^{2,q}(\Omega \cap \{x : R < |x| < R+1\}) \hookrightarrow H^1(\Omega \cap \{x : R < |x| < R+1\})$ is compact,

$$\tilde{\varphi}_m \Delta \frac{\varphi_R}{\varepsilon} \longrightarrow \tilde{\varphi} \Delta \frac{\varphi_R}{\varepsilon}, \quad \nabla \tilde{\varphi}_m \nabla \frac{\varphi_R}{\varepsilon} \longrightarrow \nabla \tilde{\varphi} \nabla \frac{\varphi_R}{\varepsilon} \quad \text{in } L^q(\Omega \cap \{x : R < |x| < R+1\}), \tag{2.9}$$

which implies

$$\lim_{R\to\infty} \limsup_{m\to\infty} \int_\Omega \left| \tilde{\varphi}_m \Delta \frac{\varphi_R}{\varepsilon} + 2\nabla \tilde{\varphi}_m \cdot \nabla \frac{\varphi_R}{\varepsilon} \right|^q = 0. \tag{2.10}$$

With the definitions of α_∞, β_∞,

$$\begin{gathered}
\lim_{R\to\infty} \limsup_{m\to\infty} \left[\int_\Omega \left(\left|\nabla \tilde{\varphi}_m \varphi_R\right|^2 + \left|\tilde{\varphi}_m \varphi_R\right|^2 \right) \right]^{q/2} = (\alpha_\infty)^{q/2}, \\
\lim_{R\to\infty} \limsup_{m\to\infty} \int_\Omega \varphi_R^q \left|\Delta \tilde{\varphi}_m - \tilde{\varphi}_m\right|^q = \beta_\infty.
\end{gathered} \tag{2.11}$$

We derive from (2.8)–(2.11)

$$M_\Omega (\alpha_\infty)^{q/2} \le (1-\varepsilon)^{1-q} \beta_\infty. \tag{2.12}$$

Since ε is arbitrary, we have

$$M_\Omega (\alpha_\infty)^{q/2} \le \beta_\infty. \tag{2.13}$$

□

Remark 2.2. In [8], Huang and Li have used a concentration-compactness principle at infinity, similar to Lemma 2.1, to study the existence of positive solutions for some quasilinear equations on unbounded domains in $\mathbb{R}^N$.

In the following, we give Lemma 2.3, which is used to find a Palais-Smale sequence $\{\tilde{\varphi}_m^1\}$ of $I_{\mathbb{R}^2}$ through the minimizing sequence $\{\tilde{\varphi}_m\}$ for $(M_{\mathbb{R}^2})$. Firstly, we give some notations. Let

$$\begin{gathered}
I_\Omega(\varphi) = \frac{1}{2} \int_\Omega \left(|\nabla \varphi|^2 + |\varphi|^2 \right) - \frac{1}{q} \int_\Omega |\Delta \varphi - \varphi|^q, \\
J_\Omega(\tilde{\varphi}) = \frac{\int_\Omega \left|\Delta \tilde{\varphi} - \tilde{\varphi}\right|^q}{\left[\int_\Omega \left(\left|\nabla \tilde{\varphi}\right|^2 + \left|\tilde{\varphi}\right|^2 \right)\right]^{q/2}}.
\end{gathered} \tag{2.14}$$

By the definition of Fréchet derivative, it is easy to verify that $I_\Omega \in C^1(W_0^{2,q}(\Omega), \mathbb{R})$, $J_\Omega \in C^1(W_0^{2,q}(\Omega) \setminus \{0\}, \mathbb{R})$ and

$$\begin{aligned} \langle I'_\Omega(\psi), h\rangle &= \int_\Omega (\nabla\psi\nabla h + \psi h) - \int_\Omega |\Delta\psi - \psi|^{q-2}(\Delta\psi - \psi)(\Delta h - h), \\ \langle J'_\Omega(\tilde{\varphi}), h\rangle &= \frac{q\|\tilde{\varphi}\|_1^{q-2}\left[\|\tilde{\varphi}\|_1^2 \int_\Omega \left|A\tilde{\varphi}\right|^{q-2} A\tilde{\varphi} Ah - \|\tilde{\varphi}\|_{2,q}^q \int_\Omega (\nabla\tilde{\varphi}\nabla h + \tilde{\varphi}h)\right]}{\left[\int_\Omega \left(\left|\nabla\tilde{\varphi}\right|^2 + \left|\tilde{\varphi}\right|^2\right)\right]^q}, \end{aligned} \tag{2.15}$$

where $\psi, \tilde{\varphi}, h \in W_0^{2,q}(\Omega)$ and $A\tilde{\varphi} = \Delta\tilde{\varphi} - \tilde{\varphi}$.

Lemma 2.3. *If $\{\tilde{\varphi}_m\}$ is a minimizing sequence of (M_Ω), then there is a minimizing sequence $\{\tilde{\varphi}_m^1\}$ such that $\|\tilde{\varphi}_m^1 - \tilde{\varphi}_m\|_{2,q} < 1/m$, $J_\Omega(\tilde{\varphi}_m^1) \to M_\Omega$, $J'_\Omega(\tilde{\varphi}_m^1) \to 0$ in $W^{-2,q'}(\Omega)$ as $m \to \infty$, and*

$$I_\Omega(\psi_m) \longrightarrow \left(\frac{1}{2} - \frac{1}{q}\right) M_\Omega^{2/(2-q)}, \quad I'_\Omega(\psi_m) \longrightarrow 0 \quad \text{in } W^{-2,q'}(\Omega), \text{ as } m \longrightarrow \infty, \tag{2.16}$$

where $\psi_m = M_\Omega^{1/(2-q)} \tilde{\varphi}_m^1$, and $W^{-2,q'}(\Omega)$ is the dual space of $W_0^{2,q}(\Omega)$, $1/q + 1/q' = 1$. Moreover, if α_∞, β_∞ are quantities related to $\tilde{\varphi}_m^1$ in Lemma 2.1, then

$$M_\Omega \cdot \alpha_\infty = \beta_\infty. \tag{2.17}$$

Proof. Using the definitions of M_Ω and J_Ω, we have

$$M_\Omega = \inf_{0 \neq \tilde{\varphi} \in W_0^{2,q}(\Omega)} J_\Omega(\tilde{\varphi}) = \lim_{m \to \infty} J_\Omega(\tilde{\varphi}_m). \tag{2.18}$$

Applying the Ekeland Variational Principle (cf. [9, page 51]) to (2.18), we get a Palais-Smale sequence $\{\tilde{\varphi}_m^1\}$, which satisfies

$$\begin{aligned} &\|\tilde{\varphi}_m^1 - \tilde{\varphi}_m\|_{2,q} < \frac{1}{m}, \qquad J_\Omega(\tilde{\varphi}_m^1) \longrightarrow M_\Omega, \\ &J'_\Omega(\tilde{\varphi}_m^1) \longrightarrow 0 \quad \text{in } W^{-2,q'}(\Omega), \text{ as } m \longrightarrow \infty. \end{aligned} \tag{2.19}$$

Then, according to the definitions of I_Ω and φ_m, for any $h \in W_0^{2,q}(\Omega)$, we get

$$\begin{aligned}
I_\Omega(\varphi_m) &= \frac{1}{2} M_\Omega^{2/(2-q)} \int_\Omega \left(|\nabla \widetilde{\varphi}_m^1|^2 + |\widetilde{\varphi}_m^1|^2 \right) - \frac{1}{q} M_\Omega^{q/(2-q)} \int_\Omega |\Delta \widetilde{\varphi}_m^1 - \widetilde{\varphi}_m^1|^q \\
&\longrightarrow \frac{1}{2} M_\Omega^{2/(2-q)} - \frac{1}{q} M_\Omega^{q/(2-q)+1} \\
&= \left(\frac{1}{2} - \frac{1}{q} \right) M_\Omega^{2/(2-q)}, \\
\langle I'_\Omega(\varphi_m), h \rangle &= M_\Omega^{1/(2-q)} \int_\Omega \left(\nabla \widetilde{\varphi}_m^1 \nabla h + \widetilde{\varphi}_m^1 h \right) - M_\Omega^{(q-1)/(2-q)} \int_\Omega |A\widetilde{\varphi}_m^1|^{q-2} A\widetilde{\varphi}_m^1 Ah \\
&= M_\Omega^{(q-1)/(2-q)} \left[M_\Omega \int_\Omega \left(\nabla \widetilde{\varphi}_m^1 \nabla h + \widetilde{\varphi}_m^1 h \right) - \int_\Omega |A\widetilde{\varphi}_m^1|^{q-2} A\widetilde{\varphi}_m^1 Ah \right] \\
&\longrightarrow 0 \left(\text{because of } \langle J'_\Omega(\widetilde{\varphi}_m^1), h \rangle \longrightarrow 0 \right), \quad \text{as } m \longrightarrow \infty.
\end{aligned} \tag{2.20}$$

Since (2.20) implies

$$I'_\Omega\left(M_\Omega^{1/(2-q)} \widetilde{\varphi}_m^1 \right) \longrightarrow 0 \quad \text{in } W^{-2,q'}(\Omega), \text{ as } m \longrightarrow \infty, \tag{2.21}$$

it follows that

$$\left\langle I'_\Omega\left(M_\Omega^{1/(2-q)} \widetilde{\varphi}_m^1 \right), M_\Omega^{1/(2-q)} \widetilde{\varphi}_m^1 \varphi_R \right\rangle \longrightarrow 0, \quad \text{as } m \longrightarrow \infty \text{ uniformly for } R \geq 1, \tag{2.22}$$

where φ_R is the function defined in the proof of Lemma 2.1. With the definition of I_Ω, we have

$$\begin{aligned}
&\left\langle I'_\Omega\left(M_\Omega^{1/(2-q)} \widetilde{\varphi}_m^1 \right), M_\Omega^{1/(2-q)} \widetilde{\varphi}_m^1 \varphi_R \right\rangle \\
&\quad = M_\Omega^{2/(2-q)} \int_\Omega \left(\nabla \widetilde{\varphi}_m^1 \nabla \left(\widetilde{\varphi}_m^1 \varphi_R \right) + \widetilde{\varphi}_m^1 \widetilde{\varphi}_m^1 \varphi_R \right) \\
&\qquad - M_\Omega^{q/(2-q)} \int_\Omega |\Delta \widetilde{\varphi}_m^1 - \widetilde{\varphi}_m^1|^{q-2} \left(\Delta \widetilde{\varphi}_m^1 - \widetilde{\varphi}_m^1 \right) \left[\Delta \left(\widetilde{\varphi}_m^1 \varphi_R \right) - \widetilde{\varphi}_m^1 \varphi_R \right].
\end{aligned} \tag{2.23}$$

Using (2.22), and letting $R \to \infty$ after $m \to \infty$ in (2.23), we obtain

$$M_\Omega \cdot \alpha_\infty = \beta_\infty. \tag{2.24}$$

□

At last, we give another lemma for proving Theorem 1.2.

Lemma 2.4. *If $\{\widetilde{\varphi}_m\}$ is bounded in $W^{2,q}(\mathbb{R}^2)$, and*

$$\sup_{y \in \mathbb{R}^2} \int_{B(y,R)} |\widetilde{\varphi}_m|^q \longrightarrow 0, \quad \textit{for some } R > 0, \tag{2.25}$$

then $\widetilde{\varphi}_m \to 0$ in $H^1(\mathbb{R}^2)$.

Proof. Applying interpolation inequalities, for $\widetilde{\varphi} \in W^{2,q}(\mathbb{R}^2)$, we have

$$\|\nabla\widetilde{\varphi}\|^2_{L^2(B(y,R))} \leq c\|\widetilde{\varphi}\|^{2(1-\gamma_1)}_{L^q(B(y,R))}\|\widetilde{\varphi}\|^{2\gamma_1}_{W^{2,q}(B(y,R))}, \tag{2.26}$$

$$\|\widetilde{\varphi}\|^2_{L^2(B(y,R))} \leq c\|\widetilde{\varphi}\|^{2(1-\gamma_2)}_{L^q(B(y,R))}\|\widetilde{\varphi}\|^{2\gamma_2}_{W^{2,q}(B(y,R))}, \tag{2.27}$$

where $\gamma_1 = 1/q$, $\gamma_2 = 1/q-1/2$, and c is a positive constant. Letting $B_1 = B(0,R)$, $B_2 = B(y_2,R)$ where $y_2 \in \partial B(0,R)$, $B_3 = B(y_3,R)$, $B_4 = B(y_4,R)$ where $\{y_3,y_4\} = \partial B_1 \cap \partial B_2,\ldots$, we cover $\mathbb{R}^2$ by the above balls of radius R such that each point of $\mathbb{R}^2$ is contained in at most 3 balls. Therefore, combining (2.26) with (2.27), we obtain

$$\begin{aligned}\|\nabla\widetilde{\varphi}_m\|^2_{L^2(\mathbb{R}^2)} &\leq 3c\left(\sup_{y\in\mathbb{R}^2}\int_{B(y,R)}|\widetilde{\varphi}_m|^q\right)^{2(1-\gamma_1)}\|\widetilde{\varphi}_m\|^{2\gamma_1}_{W^{2,q}(\mathbb{R}^2)},\\ \|\widetilde{\varphi}_m\|^2_{L^2(\mathbb{R}^2)} &\leq 3c\left(\sup_{y\in\mathbb{R}^2}\int_{B(y,R)}|\widetilde{\varphi}_m|^q\right)^{2(1-\gamma_2)}\|\widetilde{\varphi}_m\|^{2\gamma_2}_{W^{2,q}(\mathbb{R}^2)}.\end{aligned} \tag{2.28}$$

According to the assumptions of Lemma 2.4 and the above two inequalities, $\widetilde{\varphi}_m \to 0$ in $H^1(\mathbb{R}^2)$. □

3. Proof of Theorems 1.2 and 1.3

Now we turn to prove our main results.

Proof of Theorem 1.2. Using Lemma 2.3 and $I_{\mathbb{R}^2} \in C^1(W^{2,q}(\mathbb{R}^2),\mathbb{R})$, for any minimizing sequence $\{\widetilde{\varphi}_m\}$ of $(M_{\mathbb{R}^2})$, we have

$$\begin{aligned} I_{\mathbb{R}^2}\left(M^{1/(2-q)}_{\mathbb{R}^2}\widetilde{\varphi}_m\right) &\longrightarrow \left(\frac{1}{2}-\frac{1}{q}\right)M^{2/(2-q)}_{\mathbb{R}^2},\\ I'_{\mathbb{R}^2}\left(M^{1/(2-q)}_{\mathbb{R}^2}\widetilde{\varphi}_m\right) &\longrightarrow 0 \quad \text{in } W^{-2,q'}(\mathbb{R}^2) \text{ as } m \longrightarrow \infty.\end{aligned} \tag{3.1}$$

Moreover, $\{\widetilde{\varphi}_m\}$ is bounded in $W^{2,q}(\mathbb{R}^2)$.

Lemma 2.4 implies that there exists $\{y_{m_k}\} \subset \mathbb{R}^2$ such that

$$\widetilde{\varphi}^1_{m_k} := \widetilde{\varphi}_{m_k}(\cdot + y_{m_k}) \longrightarrow \widetilde{\varphi}_0 \neq 0 \quad \text{weakly in } W^{2,q}(\mathbb{R}^2). \tag{3.2}$$

In fact, since $\|\widetilde{\varphi}_m\|_1 = 1$, by Lemma 2.4, there exists $\nu > 0$ such that

$$\liminf_{m\to\infty}\sup_{y\in\mathbb{R}^2}\int_{B(y,R)}|\widetilde{\varphi}_m|^q > \nu. \tag{3.3}$$

Then, the above inequality and the boundedness of $\{\tilde{\varphi}_m\}$ in $W^{2,q}(\mathbb{R}^2)$ imply that there exist a subsequence $\{\tilde{\varphi}_{m_k}\}$ and $y_{m_k} \in \mathbb{R}^2$ such that

$$\int_{B(y_{m_k},R)} |\tilde{\varphi}_{m_k}|^q > \frac{\nu}{2} \tag{3.4}$$

and $\tilde{\varphi}_{m_k}(\cdot + y_{m_k}) \to \tilde{\varphi}_0$ weakly in $W^{2,q}(\mathbb{R}^2)$. Letting $\tilde{\varphi}^1_{m_k} := \tilde{\varphi}_{m_k}(\cdot + y_{m_k})$, we know that $\{\tilde{\varphi}^1_{m_k}\}$ is a minimizing sequence of $(M_{\mathbb{R}^2})$ and $\int_{B(0,R)} |\tilde{\varphi}^1_{m_k}|^q > \nu/2$. Since the embedding $W^{2,q}(B(0,R)) \hookrightarrow L^q(B(0,R))$ is compact, $\tilde{\varphi}^1_{m_k} \to \tilde{\varphi}_0$ in $L^q(B(0,R))$, which implies $\|\tilde{\varphi}_0\|^q_{L^q(B(0,R))} \geq \nu/2$. So $\tilde{\varphi}^1_{m_k} \to \tilde{\varphi}_0 \neq 0$ weakly in $W^{2,q}(\mathbb{R}^2)$.

Let $\Omega = \mathbb{R}^2$ and α_∞, β_∞ be quantities related to $\tilde{\varphi}^1_{m_k}$ in Lemma 2.1. With Lemma 2.1, we have

$$\int_{\mathbb{R}^2} \left(|\nabla \tilde{\varphi}_0|^2 + |\tilde{\varphi}_0|^2\right) + \alpha_\infty = 1. \tag{3.5}$$

In order to prove Theorem 1.2, we have to show $\alpha_\infty = 0$. Since $\tilde{\varphi}_0 \neq 0 \, in \, W^{2,q}(\mathbb{R}^2)$ implies $0 \leq \alpha_\infty < 1$, arguing by contradiction, we assume $0 < \alpha_\infty < 1$. Using Lemmas 2.1 and 2.3, we have

$$M_{\mathbb{R}^2}(\alpha_\infty)^{q/2} \leq \beta_\infty = M_{\mathbb{R}^2}\alpha_\infty. \tag{3.6}$$

Since Sobolev Theorem implies $M_{\mathbb{R}^2} > 0$, we derive from (3.6) $\alpha_\infty \geq 1$, which contradicts the assumption that $0 < \alpha_\infty < 1$. Thus $\tilde{\varphi}_0$ is a minimizer for the minimization problem $(M_{\mathbb{R}^2})$. Since $W^{2,q}(\mathbb{R}^2)$ is a uniformly convex space, $\|\tilde{\varphi}^1_{m_k}\|_{2,q} \to \|\tilde{\varphi}_0\|_{2,q}$ and $\tilde{\varphi}^1_{m_k} \to \tilde{\varphi}_0$ weakly in $W^{2,q}(\mathbb{R}^2)$, we know that $\tilde{\varphi}^1_{m_k} \to \tilde{\varphi}_0$ in $W^{2,q}(\mathbb{R}^2)$. □

Proof of Theorem 1.3. We divide the proof into two steps.
Step 1. We prove that $G_{\mathbb{R}^2}$ *is a stable set*. Assume that the ground state set $G_{\mathbb{R}^2}$ is not $W^{2,q}(\mathbb{R}^2)$-stable. Then there exist $\varepsilon_0 > 0$, $\psi^0_m \subset W^{2,q}(\mathbb{R}^2)$ and $t_m \in [0,T)$ such that

$$\inf_{\psi_0 \in G_{\mathbb{R}^2}} \|\psi^0_m - \psi_0\|_E \leq \frac{1}{m}, \tag{3.7}$$

$$\inf_{\psi_0 \in G_{\mathbb{R}^2}} \|\psi_m(t_m) - \psi_0\|_E \geq \varepsilon_0, \tag{3.8}$$

where $\psi_m \in C([0,T), E)$ is a solution to (1.1) with $\psi_m(0) = \psi^0_m$. Let $\tilde{\varphi}_m = M_{\mathbb{R}^2}^{-1/(2-q)}\psi^0_m$. Equation (3.7) implies that

$$\int_{\mathbb{R}^2} \left(|\nabla \tilde{\varphi}_m|^2 + |\tilde{\varphi}_m|^2\right) \longrightarrow 1, \qquad \int_{\mathbb{R}^2} |\Delta \tilde{\varphi}_m - \tilde{\varphi}_m|^q \longrightarrow M_{\mathbb{R}^2}. \tag{3.9}$$

Then there exists $\{r_m\} \subset \mathbb{R}$, $r_m \to 1$ as $m \to +\infty$ such that

$$\{r_m \tilde{\varphi}_m\} \text{ is a minimizing sequence for } (M_{\mathbb{R}^2}). \tag{3.10}$$

Using (1.3), we have

$$\|\varphi_m^0\|_1 = \|\varphi_m(t_m)\|_1, \qquad \|\varphi_m^0\|_{2,q} = \|\varphi_m(t_m)\|_{2,q}. \tag{3.11}$$

With (3.10) and (3.11), we know that $\{r_m M_{\mathbb{R}^2}^{-1/(2-q)}\varphi_m(t_m)\}$ is a minimizing sequence for $(M_{\mathbb{R}^2})$. By Theorem 1.2, there exist $y_{m_k} \in \mathbb{R}^2$ and $\widetilde{\varphi}_{m_k}^1 \in Z_{\mathbb{R}^2}$ such that

$$\begin{aligned}
&\left\|r_{m_k} M_{\mathbb{R}^2}^{-1/(2-q)}\varphi_{m_k}(t_{m_k})(\,\cdot + y_{m_k}) - \widetilde{\varphi}_{m_k}^1\right\|_{2,q} \\
&\quad = \left\|r_{m_k} M_{\mathbb{R}^2}^{-1/(2-q)}\varphi_{m_k}(t_{m_k}) - \widetilde{\varphi}_{m_k}^1(\,\cdot - y_{m_k})\right\|_{2,q} \\
&\quad \le \frac{\varepsilon_0}{2M_{\mathbb{R}^2}^{1/(2-q)}},
\end{aligned} \tag{3.12}$$

for sufficiently large m_k. Since $r_m \to 1$ and $\|\varphi_m(t_m)\|_{2,q}$ is bounded, we derive from (3.8) and (3.12)

$$\begin{aligned}
\varepsilon_0 &\le \left\|\varphi_{m_k}(t_{m_k}) - M_{\mathbb{R}^2}^{1/(2-q)}\widetilde{\varphi}_{m_k}^1(\,\cdot - y_{m_k})\right\|_{2,q} \\
&\le \left\|\varphi_{m_k}(t_{m_k}) - r_{m_k}\varphi_{m_k}(t_{m_k})\right\|_{2,q} + \left\|r_{m_k}\varphi_{m_k}(t_{m_k}) - M_{\mathbb{R}^2}^{1/(2-q)}\widetilde{\varphi}_{m_k}^1(\,\cdot - y_{m_k})\right\|_{2,q} \\
&\le \left\|\varphi_{m_k}(t_{m_k}) - r_{m_k}\varphi_{m_k}(t_{m_k})\right\|_{2,q} + M_{\mathbb{R}^2}^{1/(2-q)}\left\|r_{m_k} M_{\mathbb{R}^2}^{-1/(2-q)}\varphi_{m_k}(t_{m_k}) - \widetilde{\varphi}_{m_k}^1(\,\cdot - y_{m_k})\right\|_{2,q} \\
&\le \frac{3}{4}\varepsilon_0,
\end{aligned} \tag{3.13}$$

for sufficiently large m_k. Equation (3.13) is a contradiction. Therefore, the ground state set $G_{\mathbb{R}^2}$ is $W^{2,q}(\mathbb{R}^2)$-stable with respect to (1.1).

Step 2. Show the structure of $G_{\mathbb{R}^2}$. If $\widetilde{\varphi}_0$ is a minimizer for $(M_{\mathbb{R}^2})$, then $|\widetilde{\varphi}_0|$ is also a minimizer for $(M_{\mathbb{R}^2})$. We can assume that $\widetilde{\varphi}_0$ is a nonnegative minimizer for $(M_{\mathbb{R}^2})$. Let $\varphi_0 = M_{\mathbb{R}^2}^{1/(2-q)}\widetilde{\varphi}_0$, then $I'_{\mathbb{R}^2}(\varphi_0) = 0$ in $W^{-2,q'}(\mathbb{R}^2)$, that is,

$$-\Delta\varphi_0 + \varphi_0 = \varphi_0^{p-1}, \qquad \varphi_0(x) \longrightarrow 0 \quad \text{as } |x| \longrightarrow +\infty, \tag{3.14}$$

where $p > 2$ is given in Section 2. In fact, using the implicit function theorem, we know that $M = \{\widetilde{\varphi} : \widetilde{\varphi} \in W^{2,q}(\mathbb{R}^2), \int_{\mathbb{R}^2}|\nabla\widetilde{\varphi}|^2 + |\widetilde{\varphi}|^2 = 1\}$ is a C^1-submanifold of $W^{2,q}(\mathbb{R}^2)$. Then, according to the Lagrange Multiplier Principle, there exists $\lambda \in \mathbb{R}$ such that

$$-q\Psi'_{\mathbb{R}^2}(\widetilde{\varphi}_0) = 2\lambda\Phi'_{\mathbb{R}^2}(\widetilde{\varphi}_0) \quad \text{in } W^{-2,q'}(\mathbb{R}^2), \tag{3.15}$$

where $\Psi_{\mathbb{R}^2}(\widetilde{\varphi}_0) = -(1/q)\int_{\mathbb{R}^2}|\Delta\widetilde{\varphi}_0 - \widetilde{\varphi}_0|^q$, $\Phi_{\mathbb{R}^2}(\widetilde{\varphi}_0) = (1/2)\int_{\mathbb{R}^2}(|\nabla\widetilde{\varphi}_0|^2 + |\widetilde{\varphi}_0|^2)$. Since $\widetilde{\varphi}_0$ is a minimizer for $(M_{\mathbb{R}^2})$, with the definition of $M_{\mathbb{R}^2}$, we obtain $\lambda = (q/2)M_{\mathbb{R}^2}$. So, we conclude $I'_{\mathbb{R}^2}(\varphi_0) = 0$.

Applying elliptic regularity theory (cf., e.g., [10], Lemma 1.30), we prove that $\psi_0 \in C^2(\mathbb{R}^2)$. According to the strong maximum principle, ψ_0 is positive. Using the moving plane method (cf. [11]), we show that ψ_0 is radially symmetric. Moreover, by the uniqueness result in [12], ψ_0 is unique up to translations. Therefore, $\widetilde{\varphi}_0$ is a unique, up to translation, positive radially symmetric C^2 minimizer of $(M_{\mathbb{R}^2})$, and

$$G_{\mathbb{R}^2} = \left\{ M_{\mathbb{R}^2}^{1/(2-q)} \widetilde{\varphi}_0(\cdot + y), -M_{\mathbb{R}^2}^{1/(2-q)} \widetilde{\varphi}_0(\cdot + y);\ y \in \mathbb{R}^2 \right\}. \tag{3.16}$$

□

Remark 3.1. The existence of nontrivial critical points of (1.2) for $F(x) = -(1/q)|x|^q$ can also be proved by the following method. Since $W_r^{2,q}(\mathbb{R}^2) \hookrightarrow H_r^1(\mathbb{R}^2)$ is compact, where $W_r^{2,q}(\mathbb{R}^2)$ and $H_r^1(\mathbb{R}^2)$ are the set of the radially symmetric functions of $W^{2,q}(\mathbb{R}^2)$ and $H^1(\mathbb{R}^2)$, respectively, with Principle of Symmetric Criticality proposed by Palais (cf. [13]) and Fountain Theorem (cf. [10], Chapter 3), we can prove that there are infinitely many solutions to (1.2). However, we do not know how to consider the stability of these solutions. So, we cannot study the existence of nontrivial critical points of (1.2) with this method. Through considering the constrained variational problem $(M_{\mathbb{R}^2})$, we obtain the existence of a set $G_{\mathbb{R}^2}$ of steady waves for (1.1) and know that $G_{\mathbb{R}^2}$ is $W^{2,q}(\mathbb{R}^2)$-stable with respect to (1.1).

4. Hasegawa-Mima Equation in Periodic Domains

In this section, we consider the existence and stability of steady waves for Hasegawa-Mima equation in periodic domains:

$$\frac{\partial(\Delta\psi - \psi)}{\partial t} + \frac{\partial\psi}{\partial x}\frac{\partial(\Delta\psi - \psi)}{\partial y} - \frac{\partial\psi}{\partial y}\frac{\partial(\Delta\psi - \psi)}{\partial x} = 0, \tag{4.1}$$

where $(x, y) \in \Omega$, Ω is a periodic domain defined later, $\psi(x, y) \to 0$ as $|x| + |y| \to +\infty$, and $\psi|_{(x,y)\in\partial\Omega} = 0$. Moreover, as a byproduct, we prove the existence and stability of steady two-dimensional incompressible flows in infinite strip channel.

At first, we give the definition and two examples of periodic domain.

Definition 4.1 (periodic domain). If Ω is a domain in $\mathbb{R}^2$, and there are a partition $\{\Omega_n\}$ of Ω and points $\{y_n\}$ in $\mathbb{R}^2$ satisfying the following conditions: (1) $\{y_n\}$ forms a subgroup of $\mathbb{R}^2$, (2) Ω_0 is a bounded domain in $\mathbb{R}^2$, (3) $\Omega_n = y_n + \Omega_0$, then Ω is called a periodic domain.

It is clear that $\mathbb{R}^2$ is a periodic domain. The other example of periodic domain is $Q = \{(x, y) \in \mathbb{R} \times \mathbb{R} : |y| < 1\}$.

Similarly to the method used in studying (1.1), in order to prove the existence and stability of steady waves for (4.1), we consider the following minimization problem:

$$M_\Omega = \inf_{\widetilde{\varphi} \in W_0^{2,q}(\Omega), \|\widetilde{\varphi}\|_1 = 1} \int_\Omega \left|\Delta\widetilde{\varphi} - \widetilde{\varphi}\right|^q. \tag{M_Ω}$$

Let

$$Z_\Omega = \left\{\widetilde{\varphi}_0;\ \int_\Omega \left(\left|\nabla\widetilde{\varphi}_0\right|^2 + \left|\widetilde{\varphi}_0\right|^2\right) = 1,\ \int_\Omega \left(\left|\Delta\widetilde{\varphi}_0 - \widetilde{\varphi}_0\right|^q\right) = M_\Omega\right\},$$
$$G_\Omega = \left\{\varphi_0;\ \varphi_0 = M_\Omega^{1/(2-q)}\widetilde{\varphi}_0,\ \widetilde{\varphi}_0 \in Z_\Omega\right\}. \tag{4.2}$$

If $\widetilde{\varphi}_0 \in Z_\Omega$, then, according to Lagrange Multiplier Principle, $\varphi_0 = M_\Omega^{1/(2-q)}\widetilde{\varphi}_0$ is a steady wave for (4.1). Moreover, as presented in Proposition A.3, φ_0 is a ground state and saddle point of I_Ω.

Through considering the above minimization problem (M_Ω), we obtain the following result similar to Theorems 1.2 and 1.3.

Theorem 4.2. *If $\{\widetilde{\varphi}_m\}$ is a minimizing sequence for (M_Ω), then there exist a subsequence $\{\widetilde{\varphi}_{m_k}\}$ and $\{y_{n_{m_k}}\}$ such that $\widetilde{\varphi}_{m_k}(\cdot + y_{n_{m_k}}) \rightarrow \widetilde{\varphi} \neq 0$ weakly in $W_0^{2,q}(\Omega)$. Moreover, the ground state set G_Ω is $W_0^{2,q}(\Omega)$-stable with respect to* (4.1).

Proof. Without loss of generality, we assume that $\Omega = \bigcup_{n\in\mathbb{Z}}\Omega_n$, where Ω_i and Ω_j are disjoint if $i \neq j$, and there exists $\Omega_0 \subset \mathbb{R}^2$ such that $\Omega_n = y_n + \Omega_0$. Similar to Lemma 2.3, using $I_\Omega \in C^1(W_0^{2,q}(\Omega), \mathbb{R})$, we know that the minimizing sequence $\{\widetilde{\varphi}_m\}$ satisfies

$$I_\Omega\left(M_\Omega^{1/(2-q)}\widetilde{\varphi}_m\right) \longrightarrow \left(\frac{1}{2} - \frac{1}{q}\right)M_\Omega^{2/(2-q)},$$
$$I'_\Omega\left(M_\Omega^{1/(2-q)}\widetilde{\varphi}_m\right) \longrightarrow 0 \quad \text{in } W^{-2,q'}(\Omega),\ \text{as } m \longrightarrow \infty. \tag{4.3}$$

If we let $\varphi_m = M_\Omega^{1/(2-q)}\widetilde{\varphi}_m$, then (4.3) implies

$$\int_\Omega \left(\left|\nabla\varphi_m\right|^2 + \left|\varphi_m\right|^2\right) = \int_\Omega \left|\Delta\varphi_m - \varphi_m\right|^q + o(1). \tag{4.4}$$

Letting $d_m = \max_{n\in\mathbb{Z}}\left[\int_{\Omega_n}(|\nabla\varphi_m|^2 + |\varphi_m|^2)\right]^{1/2}$, applying Sobolev inequality, (4.3) and (4.4), we have

$$\begin{aligned}
o(1) + I_\Omega(\varphi_m) &= \left(\frac{1}{2} - \frac{1}{q}\right)\sum_{n\in\mathbb{Z}}\int_{\Omega_n}\left(\left|\nabla\varphi_m\right|^2 + \left|\varphi_m\right|^2\right) \\
&\geq \left(\frac{1}{2} - \frac{1}{q}\right)\max_{n\in\mathbb{Z}}\left[\int_{\Omega_n}\left(\left|\nabla\varphi_m\right|^2 + \left|\varphi_m\right|^2\right)\right]^{(2-q)/2}\sum_{n\in\mathbb{Z}}\left[\int_{\Omega_n}\left(\left|\nabla\varphi_m\right|^2 + \left|\varphi_m\right|^2\right)\right]^{q/2} \\
&\geq \left(\frac{1}{2} - \frac{1}{q}\right)d_m^{2-q}C\sum_{n\in\mathbb{Z}}\int_{\Omega_n}\left|\Delta\varphi_m - \varphi_m\right|^q \\
&\geq \left(\frac{1}{2} - \frac{1}{q}\right)d_m^{2-q}C\sum_{n\in\mathbb{Z}}\int_{\Omega_n}\left(\left|\nabla\varphi_m\right|^2 + \left|\varphi_m\right|^2\right) + o(1),
\end{aligned} \tag{4.5}$$

for some constant $C > 0$ independent of m. Since $\int_\Omega(|\nabla\widetilde{\varphi}_m|^2+|\widetilde{\varphi}_m|^2) = 1$ for all m, (4.5) implies that there exists $\mu > 0$ such that $d_m \geq \mu$ for $m = 1, 2, \ldots$. Therefore, for each m, there exists Ω_{n_m} such that

$$\left[\iint_{\Omega_{n_m}} \left(|\nabla\varphi_m|^2 + |\varphi_m|^2\right)\right]^{1/2} \geq \frac{\mu}{2}. \tag{4.6}$$

Equation (4.6) means that there exists a point y_{n_m} such that

$$\left[\iint_{\Omega_{n_m}} \left(|\nabla\varphi_m|^2 + |\varphi_m|^2\right)\right]^{1/2} = \left[\iint_{\Omega_0} \left(|\nabla\varphi_m(x+y_{n_m})|^2 + |\varphi_m(x+y_{n_m})|^2\right)\right]^{1/2} \geq \frac{\mu}{2}. \tag{4.7}$$

Since $\{\varphi_m(x+y_{n_m})\}$ is bounded in $W_0^{2,q}(\Omega)$, there exists a subsequence $\{\varphi_{m_k}(x+y_{n_{m_k}})\}$ such that

$$\varphi_{m_k}(x+y_{n_{m_k}}) \longrightarrow \varphi \neq 0 \quad \text{weakly in } W_0^{2,q}(\Omega). \tag{4.8}$$

So $\widetilde{\varphi}_{m_k}(x+y_{n_{m_k}}) = M_\Omega^{-1/(2-q)}\varphi_{m_k}(x+y_{n_{m_k}}) \to M_\Omega^{-1/(2-q)}\varphi = \widetilde{\varphi} \neq 0$ weakly in $W_0^{2,q}(\Omega)$.

Let α_∞, β_∞ be quantities related to $\{\widetilde{\varphi}_{m_k}(x+y_{n_{m_k}})\}$ in Lemma 2.1. Similarly to the proof of the first part of Theorem 1.3, we can prove the second part of Theorem 4.2. □

In the following, we give an application of Lemmas 2.1 and 2.3 to study the existence and stability of steady two-dimensional incompressible waves. The well-known vorticity equation governing the two-dimensional incompressible flow in an infinite strip channel is

$$\frac{\partial\Delta\varphi}{\partial t} + \frac{\partial\varphi}{\partial x}\frac{\partial\Delta\varphi}{\partial y} - \frac{\partial\varphi}{\partial y}\frac{\partial\Delta\varphi}{\partial x} = 0, \quad (x,y) \in Q = \mathbb{R}\times(0,1), \tag{4.9}$$

where φ is a stream function.

If the stream function φ satisfies

$$\varphi(x,0) = \varphi(x,1) = 0 \quad \text{for any } x \in \mathbb{R}, \qquad \varphi(x,y) \longrightarrow 0 \text{ as } |x| \longrightarrow +\infty, \tag{4.10}$$

then there are two conserved quantities of (4.9) and (4.10): the total energy $\Phi_Q(\varphi) = (1/2)\int_Q|\nabla\varphi|^2$, the generalized enstrophy $\Psi_Q(\varphi) = \int_Q F(\Delta\varphi)$, where F is an arbitrary C^1 function. Letting $F(x) = -(1/q)|x|^q$ for $x \in \mathbb{R}$, where $q = p/(p-1)$ and $2 < p < +\infty$, we consider the minimization problem:

$$M_Q := \inf\left\{\int_Q\left(|\Delta\widetilde{\varphi}|^q\right);\ \varphi \in W_0^{2,q}(Q),\ \int_Q|\nabla\widetilde{\varphi}|^2 = 1\right\}. \tag{M_Q}$$

Let $Z_Q = \{\widetilde{\varphi};\ \int_Q|\Delta\widetilde{\varphi}|^q = M_Q,\ \textit{and} \int_Q|\nabla\widetilde{\varphi}|^2 = 1\}$, $G_Q = \{\varphi; \varphi = M_Q^{1/(2-q)}\widetilde{\varphi},\ \widetilde{\varphi} \in Z_Q\}$. Then we have the following result, which is a corollary of Theorem 4.2.

Corollary 4.3. *The ground state set G_Q is not empty. Moreover, it is $W_0^{2,q}(Q)$-stable with respect to* (4.9) *and* (4.10).

Appendix

Proposition A.1. *The functional*

$$I_\Omega(\psi) = \frac{1}{2}\int_\Omega \left(|\nabla\psi|^2 + |\psi|^2\right) - \frac{1}{q}\int_\Omega |\Delta\psi - \psi|^q \tag{A.1}$$

is neither bounded from above nor from below in $W_0^{2,q}(\Omega)$.

Proof. For given $\psi \neq 0$ in $W_0^{2,q}(\Omega)$, from $I_\Omega(t\psi) = (t^2/2)\int_\Omega(|\nabla\psi|^2 + |\psi|^2) - (|t|^q/q)\int_\Omega|\Delta\psi - \psi|^q$ and $1 < q < 2$, we have

$$I_\Omega(t\psi) \longrightarrow +\infty, \quad \text{as } |t| \longrightarrow +\infty. \tag{A.2}$$

In the following, we prove that I_Ω is not bounded from below in $W_0^{2,q}(\Omega)$.

Let $E_r = \{\psi;\ \psi \in W_0^{2,q}(\Omega),\ \text{and}\ \psi \text{ is radially symmetric}\}$, $\{e_i\}_{i=1}^{+\infty}$ be a Schauder basis for E_r (cf. [14]), $X_i = \text{span}\{e_i\}$, and $Y_k = \overline{\oplus_{i=k}^{+\infty} X_i}$. Firstly, we claim

$$\alpha_k = \sup_{u\in Y_k, \|\psi\|_{2,q}=1} \|\psi\|_1 \longrightarrow 0, \quad \text{as } k \longrightarrow +\infty. \tag{A.3}$$

It is clear that α_k is a nonnegative decreasing sequence in $\mathbb{R}$. Let $\alpha_k \to \alpha \geq 0$, as $k \to +\infty$. For given k, there exists ψ_k such that $\|\psi_k\|_{2,q} = 1$ and $\|\psi_k\|_1 \geq \alpha_k/2$. The fact that $\|\psi_k\|_{2,q} = 1$ and $\psi_k \in Y_k$ implies that $\psi_k \rightharpoonup \psi = 0$ in E_r as $k \to +\infty$. Since $E_r \hookrightarrow H_{0,r}^1(\Omega)$ is compact, $\psi_k \to 0$ in $H_{0,r}^1(\Omega)$ as $k \to +\infty$, where $H_{0,r}^1(\Omega) = \{\psi : \psi \in H_0^1(\Omega),\ \text{and}\ u \text{ is radially symmetric}\}$. So $\alpha_k \to \alpha = 0$. With the definition of α_k,

$$I_\Omega(\psi) \leq \frac{1}{2}\alpha_k^2\|\psi\|_{2,q}^2 - \frac{1}{q}\|\psi\|_{2,q}^q \quad \text{on the subspace } Y_k. \tag{A.4}$$

Let $\psi \in Y_k$, $\|\psi\|_{2,q} = \alpha_k^{-2/(2-q)}$,

$$I_\Omega(\psi) \leq \frac{1}{2}\alpha_k^2\alpha_k^{-4/(2-q)} - \frac{1}{q}\alpha_k^{-2q/(2-q)} = \left(\frac{1}{2} - \frac{1}{q}\right)\alpha_k^{-2q/(2-q)} \longrightarrow -\infty, \quad \text{as } k \longrightarrow +\infty. \tag{A.5}$$

□

Let $N = \{\psi;\ \psi \in W_0^{2,q}(\Omega), \langle I'_\Omega(\psi), \psi\rangle = 0\}$, which is usually called Nehari manifold, $N_1 = \{\psi;\ \psi \in W_0^{2,q}(\Omega), \langle I'_\Omega(\psi), h\rangle = 0 \text{ for any } h \in W_0^{2,q}(\Omega)\}$, which is the set of all critical points for I_Ω. It is clear that $N \supset N_1$.

Definition A.2. A function ψ_0 is called a ground state for I_Ω, if ψ_0 is a critical point of I_Ω and $I_\Omega(\psi_0) \geq I_\Omega(\psi)$, that is, $\|\psi_0\|_1^2 \leq \|\psi\|_1^2$, for any $\psi \in N_1$.

Proposition A.3. *If $\widetilde{\psi}_0$ is a minimizer for (M_Ω), that is, $\widetilde{\psi} \in Z_\Omega$, then $\psi_0 = M_\Omega^{1/(2-q)}\widetilde{\psi}_0$ is a ground state solution of I_Ω. Moreover, ψ_0 is a "Mountain Pass type" critical point (saddle point) of I_Ω.*

Proof. As is presented in the proof of Theorem 1.3, ψ_0 is a critical point of I_Ω and

$$I_\Omega(\psi_0) = \frac{1}{2}\int_\Omega \left(|\nabla\psi_0|^2 + |\psi_0|^2\right) - \frac{1}{q}\int_\Omega |\Delta\psi_0 - \psi_0|^q = \left(\frac{1}{2} - \frac{1}{q}\right)M_\Omega^{2/(2-q)}. \tag{A.6}$$

The definition of M_Ω implies that $\|h\|_{2,q}^q \geq M_\Omega \|h\|_1^q$ for all $h \in W_0^{2,q}(\Omega)$. If $\psi \in N$, which implies that $\|\psi\|_1^2 = \|\psi\|_{2,q}^q$, then $\|\psi\|_1^2 \geq M_\Omega^{2/(2-q)}$. Therefore, for any $\psi \in N$,

$$I_\Omega(\psi) = \left(\frac{1}{2} - \frac{1}{q}\right)\|\psi\|_1^2 \leq \left(\frac{1}{2} - \frac{1}{q}\right)M_\Omega^{2/(2-q)} = I_\Omega(\psi_0). \tag{A.7}$$

Since $N_1 \subset N$, by Definition A.2, ψ_0 is a ground state solution of I_Ω.

For $0 \neq \psi \in E$ and $t \geq 0$, let

$$g(t) = I_\Omega(t\psi) = \frac{t^2}{2}\int_\Omega \left(|\nabla\psi|^2 + |\psi|^2\right) - \frac{t^q}{q}\int_\Omega |\Delta\psi - \psi|^q. \tag{A.8}$$

Then there exists a unique $t(\psi) > 0$ such that $I_\Omega(t(\psi)\psi) = \inf_{t\geq 0} I_\Omega(t\psi)$ and $(dg(t)/dt)|_{t=t(\psi)} = 0$, that is, $t(\psi)\int_\Omega(|\nabla\psi|^2 + |\psi|^2) - t(\psi)^{q-1}\int_\Omega |\Delta\psi - \psi|^q = 0$, which implies $t(\psi)\psi \in N$. Let

$$c_1 = \sup_{\psi\in N} I_\Omega(\psi), \qquad c_2 = \sup_{0\neq\psi\in E}\ \inf_{t\geq 0} I_\Omega(t\psi). \tag{A.9}$$

Therefore, $I_\Omega(\psi_0) = c_1 = c_2$. It follows from the definition of c_2 that ψ_0 is a "Mountain Pass type" critical point (saddle point) of I_Ω. □

Remark A.4. From the proof of Proposition A.3, we obtain

$$\begin{aligned} G_\Omega &= \left\{\psi_0;\ \psi_0 = M_\Omega^{1/(2-q)}\widetilde{\psi}_0,\ \widetilde{\psi}_0 \in Z_\Omega\right\} \\ &= \left\{\psi_0;\ \psi_0 \in N_1,\ \|\psi_0\|_1^2 \leq \|\psi\|_1^2 \text{ for any } \psi \in N_1\right\}. \end{aligned} \tag{A.10}$$

In fact, if $\psi_0 = M_\Omega^{1/(2-q)}\widetilde{\psi}_0$, where $\widetilde{\psi}_0 \in Z_\Omega$, then $\psi_0 \in N_1$ and $\|\psi_0\|_1^2 \leq \|\psi\|_1^2$ for any $\psi \in N_1$. On the other hand, if $\psi_0 \in N_1$ and $\|\psi_0\|_1^2 \leq \|\psi\|_1^2$ for any $\psi \in N_1$, then $\|\psi_0\|_1^2 = \|\psi_0\|_{2,q}^q = M_\Omega^{2/(2-q)}$. Letting $\widetilde{\psi}_0 = M_\Omega^{-1/(2-q)}\psi_0$, we know $\widetilde{\psi}_0 \in Z_\Omega$.

Acknowledgments

The authors are thankful to the referee for the useful comments and suggestions. Their work was supported in part by the NNSF of China Grants no. 90511009, no. 10801017 and National Basic Research Program of China (973 Program) no. 2007CB814800.

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