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Research Article

Several Existence Theorems of Multiple Positive Solutions of Nonlinear *m*-Point BVP for an Increasing Homeomorphism and Homomorphism on Time Scales

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By using fixed point theorems in cones, the existence of multiple positive solutions is considered for nonlinear m-point boundary value problem for the following second-order boundary value problem on time scales $(\phi(u^{\Delta}))^{\nabla} + a(t)f(t,u(t)) = 0$, $t \in (0,T)$, $\phi(u^{\Delta}(0)) = \sum_{i=1}^{m-2} a_i\phi(u^{\Delta}(\xi_i))$, $u(T) = \sum_{i=1}^{m-2} b_i u(\xi_i)$, where $\phi : R \to R$ is an increasing homeomorphism and homomorphism and $\phi(0) = 0$. Some new results are obtained for the existence of twin or an arbitrary odd number of positive solutions of the above problem by applying Avery-Henderson and Leggett-Williams fixed point theorems, respectively. In particular, our criteria generalize and improve some known results by Ma and Castaneda (2001). We must point out for readers that there is only the p-Laplacian case for increasing homeomorphism and homomorphism. As an application, one example to demonstrate our results is given.

1. Introduction

In this paper, we will be concerned with the existence of positive solutions for the following boundary value problem on time scales:

$$\left(\phi\left(u^{\Delta}\right)\right)^{\nabla} + a(t)f(t,u(t)) = 0, \quad t \in (0,T), \tag{1.1}$$

$$\phi\left(u^{\Delta}(0)\right) = \sum_{i=1}^{m-2} a_i\phi\left(u^{\Delta}(\xi_i)\right), \qquad u(T) = \sum_{i=1}^{m-2} b_i u(\xi_i), \tag{1.2}$$

where $\phi : R \to R$ is an increasing homeomorphism and homomorphism and $\phi(0) = 0$.

A time scale $\mathbf{T}$ is a nonempty closed subset of R. We make the blanket assumption that $0, T$ are points in $\mathbf{T}$. By an interval $(0, T)$, we always mean the intersection of the real interval $(0, T)$ with the given time scale, that is, $(0, T) \cap \mathbf{T}$.

A projection $\phi : R \to R$ is called an increasing homeomorphism and homomorphism, if the following conditions are satisfied:

(i) if $x \leq y$, then $\phi(x) \leq \phi(y)$, $\forall x, y \in R$;

(ii) ϕ is a continuous bijection and its inverse mapping is also continuous;

(iii) $\phi(xy) = \phi(x)\phi(y)$, $\forall x, y \in R$.

We will assume that the following conditions are satisfied throughout this paper:

(H_1) $0 < \xi_1 < \cdots < \xi_{m-2} < \rho(T)$, a_i, $b_i \in [0, +\infty)$ satisfy $0 < \sum_{i=1}^{m-2} a_i < 1$, and $\sum_{i=1}^{m-2} b_i < 1, T\sum_{i=1}^{m-2} b_i \geq \sum_{i=1}^{m-2} b_i\xi_i$;

(H_2) $a(t) \in C_{ld}((0, T), [0, +\infty))$ and there exists $t_0 \in (\xi_{m-2}, T)$, such that $a(t_0) > 0$;

(H_3) $f \in C([0, T] \times [0, +\infty), [0, +\infty))$. (The Δ-derivative and the ∇-derivative in (1.1), (1.2) and the C_{ld} space in (H_2) are defined in Section 2.)

Recently, there has been much attention paid to the existence of positive solutions for second-order nonlinear boundary value problems on time scales, for examples, see [1–6] and references therein. At the same time, multipoint nonlinear boundary value problems with p-Laplacian operators on time scales have also been studied extensively in the literature, for details, see [4, 5, 7–13] and the references therein. But to the best of our knowledge, few people considered the second-order dynamic equations of increasing homeomorphism and positive homomorphism on time scales.

For the existence problems of positive solutions of boundary value problems on time scales, some authors have obtained many results in the recent years, especially [6, 7, 9, 10, 14, 15] and the references therein. To date few papers have appeared in the literature concerning multipoint boundary value problems for an increasing homeomorphism and homomorphism on time scales.

In [16], Liang and Zhang studied the existence of countably many positive solutions for nonlinear singular boundary value problems:

$$\begin{gathered} \left(\varphi\left(u'\right)\right)' + a(t)f(u(t)) = 0, \quad t \in (0, 1), \\ u(0) = \sum_{i=1}^{m-2}\alpha_i u(\xi_i), \qquad \varphi\left(u'(1)\right) = \sum_{i=1}^{m-2}\beta_i\varphi\left(u'(\xi_i)\right), \end{gathered} \tag{1.3}$$

where $\varphi : R \to R$ is an increasing homeomorphism and positive homomorphism and $\varphi(0) = 0$. By using the fixed point index theory and a new fixedpoint theorem in cones, they obtained countably many positive solutions for problem (1.3).

Very recently, Sang et al. [6] investigated the nonlinear m-point BVP on time scales (1.1) and (1.2).

Let

$$M = \phi^{-1}\left(\int_0^T a(\tau)\nabla\tau + \frac{\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} a(\tau)\nabla\tau}{1-\sum_{i=1}^{m-2} a_i}\right) \times \frac{T-\sum_{i=1}^{m-2} b_i\xi_i}{1-\sum_{i=1}^{m-2} b_i},$$

$$N = \phi^{-1}\left(\frac{\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} a(\tau)\nabla\tau}{1-\sum_{i=1}^{m-2} a_i}\right)T$$
$$+\frac{\sum_{i=1}^{m-2} b_i \phi^{-1}\left(\int_0^{\xi_i} a(\tau)\nabla\tau + \sum_{i=1}^{m-2} a_i \int_0^{\xi_i} a(\tau)\nabla\tau / \left(1-\sum_{i=1}^{m-2} a_i\right)\right)(T-\xi_i)}{1-\sum_{i=1}^{m-2} b_i}. \tag{1.4}$$

They mainly obtained the following results.

Theorem 1.1. *Assume that* (H_1), (H_2), *and* (H_3) *hold, there exist* $c,\ b,\ d > 0$, *such that* $0 < d/\gamma < c < \gamma b < b$, *and suppose that* f *satisfies the following additional conditions:*

(H_4) $f(t,u) \geq 0,\ (t,u) \in [0,T] \times [d,b]$;

(H_5) $f(t,u) < \phi(c/M),\ (t,u) \in [0,T] \times [0,c]$;

(H_6) $f(t,u) > \phi(b/N),\ (t,u) \in [0,T] \times [\gamma b,b]$.

Then (1.1) *and* (1.2) *has at least two positive solutions* u_1 *and* u_2.

Motivated by the above papers, the purpose of our paper is to show the existence of twin or an arbitrary odd number of positive solutions to the BVP (1.1), (1.2). The most important is that the authors would like to point out that there is only the p-Laplacian case for increasing homeomorphism and homomorphism, this point was proposed by professor Jeff Webb. This is the main motivation for us to write down the present paper. We also point out that when $\mathbf{T} = R,\ \phi(u) = u$, (1.1) and (1.2) becomes a boundary value problem of differential equations and just is the problem considered in [15]. Our main results extend and include the main results of [5, 15, 16].

The rest of the paper is arranged as follows. We state some basic time scale definitions and prove several preliminary results in Section 2. Sections 3, 4, and 5 are devoted to the existence of positive solutions of (1.1) and (1.2), with the main tool being the Avery-Henderson and Leggett-Williams fixed point theorems. Finally, in Section 6, we give an example to illustrate our main results.

2. Preliminaries and Some Lemmas

For convenience, we list the following definitions which can be found in [2, 17–19].

Definition 2.1. A time scale $\mathbf{T}$ is a nonempty closed subset of real numbers R. For $t < \sup \mathbf{T}$ and $r > \inf \mathbf{T}$, define the forward jump operator σ and backward jump operator ρ, respectively, by

$$\sigma(t) = \inf\{\tau \in \mathbf{T} \mid \tau > t\} \in \mathbf{T},$$
$$\rho(r) = \sup\{\tau \in \mathbf{T}\tau < r\} \in \mathbf{T}, \tag{2.1}$$

for all $t, r \in \mathbf{T}$. If $\sigma(t) > t$, t is said to be right scattered, and if $\rho(r) < r$, r is said to be left scattered; if $\sigma(t) = t$, t is said to be right dense, and if $\rho(r) = r$, r is said to be left dense. If $\mathbf{T}$ has a right scattered minimum m, define $\mathbf{T}_k = \mathbf{T} - \{m\}$; otherwise set $\mathbf{T}_k = \mathbf{T}$. If $\mathbf{T}$ has a left scattered maximum M, define $\mathbf{T}^k = \mathbf{T} - \{M\}$; otherwise set $\mathbf{T}^k = \mathbf{T}$.

Definition 2.2. For $f : \mathbf{T} \to R$ and $t \in \mathbf{T}^k$, the delta derivative of f at the point t is defined to be the number $f^{\Delta}(t)$ (provided it exists) with the property that for each $\epsilon > 0$, there is a neighborhood U of t such that

$$\left| f(\sigma(t)) - f(s) - f^{\Delta}(t)(\sigma(t) - s) \right| \le \epsilon |\sigma(t) - s|, \tag{2.2}$$

for all $s \in U$.

For $f : \mathbf{T} \to R$ and $t \in \mathbf{T}_k$, the nabla derivative of f at t is the number $f^{\nabla}(t)$ (provided it exists) with the property that for each $\epsilon > 0$, there is a neighborhood U of t such that

$$\left| f(\rho(t)) - f(s) - f^{\nabla}(t)(\rho(t) - s) \right| \le \epsilon \left| \rho(t) - s \right|, \tag{2.3}$$

for all $s \in U$.

Definition 2.3. A function f is left-dense continuous (i.e., *ld*-continuous), if f is continuous at each left-dense point in $\mathbf{T}$ and its right-sided limit exists at each right-dense point in $\mathbf{T}$.

Definition 2.4. If $G^{\Delta}(t) = f(t)$, then we define the delta integral by

$$\int_a^b f(t)\Delta t = G(b) - G(a). \tag{2.4}$$

If $F^{\nabla}(t) = f(t)$, then we define the nabla integral by

$$\int_a^b f(t)\nabla t = F(b) - F(a). \tag{2.5}$$

To prove the main results in this paper, we will employ several lemmas. These lemmas are based on the linear BVP

$$\left(\phi\left(u^{\Delta}\right)\right)^{\nabla} + h(t) = 0, \quad t \in (0, T), \tag{2.6}$$

$$\phi\left(u^{\Delta}(0)\right) = \sum_{i=1}^{m-2} a_i \phi\left(u^{\Delta}(\xi_i)\right), \qquad u(T) = \sum_{i=1}^{m-2} b_i u(\xi_i). \tag{2.7}$$

Lemma 2.5. *For $h \in C_{ld}[0, T]$ the BVP* (2.6) *and* (2.7) *has the unique solution*

$$u(t) = -\int_0^t \phi^{-1}\left(\int_0^s h(\tau)\nabla\tau - \ddot{A}\right)\Delta s + B, \tag{2.8}$$

where

$$\ddot{A} = -\frac{\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} h(\tau)\nabla\tau}{1-\sum_{i=1}^{m-2} a_i},$$
$$B = \frac{\int_0^T \phi^{-1}\left(\int_0^s h(\tau)\nabla\tau - \ddot{A}\right)\Delta s - \sum_{i=1}^{m-2} b_i \int_0^{\xi_i} \phi^{-1}\left(\int_0^s h(\tau)\nabla\tau - \ddot{A}\right)\Delta s}{1-\sum_{i=1}^{m-2} b_i}. \tag{2.9}$$

Proof. Let u be as in (2.8). By [18, Theorem 2.10 (iii)], taking the delta derivative of (2.8), we have

$$u^{\Delta}(t) = -\phi^{-1}\left(\int_0^t h(\tau)\nabla\tau - \ddot{A}\right), \tag{2.10}$$

moreover, we get

$$\phi\left(u^{\Delta}(t)\right) = -\left(\int_0^t h(\tau)\nabla\tau - \ddot{A}\right), \tag{2.11}$$

taking the nabla derivative of this expression yields $(\phi(u^{\Delta}))^{\nabla} = -h(t)$. And routine calculations verify that u satisfies the boundary value conditions in (2.7), so that u given in (2.8) is a solution of (2.6) and (2.7).

It is easy to see that the BVP $(\phi(u^{\Delta}))^{\nabla} = 0, \phi(u^{\Delta}(0)) = \sum_{i=1}^{m-2} a_i\phi(u^{\Delta}(\xi_i)), u(T) = \sum_{i=1}^{m-2} b_i u(\xi_i)$ has only the trivial solution. Thus u in (2.8) is the unique solution of (2.6) and (2.7). The proof is complete. □

Lemma 2.6. *Assume that* (H_1) *holds, for* $h \in C_{ld}[0,T]$ *and* $h \geq 0$*, then the unique solution* u *of* (2.6) *and* (2.7) *satisfies*

$$u(t) \geq 0, \quad \text{for } t \in [0,T]. \tag{2.12}$$

Proof. Let

$$\varphi_0(s) = \phi^{-1}\left(\int_0^s h(\tau)\nabla\tau - \ddot{A}\right). \tag{2.13}$$

Since

$$\begin{aligned}\int_0^s h(\tau)\nabla\tau - \ddot{A} &= \int_0^s h(\tau)\nabla\tau + \frac{\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} h(\tau)\nabla\tau}{1-\sum_{i=1}^{m-2} a_i} \\ &\geq 0,\end{aligned} \tag{2.14}$$

then $\varphi_0(s) \geq 0$.

According to Lemma 2.5, we get

$$
\begin{aligned}
u(0) &= B = \frac{\int_0^T \varphi_0(s)\Delta s - \sum_{i=1}^{m-2} b_i \int_0^{\xi_i} \varphi_0(s)\Delta s}{1-\sum_{i=1}^{m-2} b_i} \\
&= \frac{\int_0^T \varphi_0(s)\Delta s - \sum_{i=1}^{m-2} b_i \left(\int_0^T \varphi_0(s)\Delta s - \int_{\xi_i}^T \varphi_0(s)\Delta s\right)}{1-\sum_{i=1}^{m-2} b_i} \\
&= \int_0^T \varphi_0(s)\Delta s + \frac{\sum_{i=1}^{m-2} b_i \int_{\xi_i}^T \varphi_0(s)\Delta s}{1-\sum_{i=1}^{m-2} b_i} \\
&\geq 0, \\
u(T) &= -\int_0^T \varphi_0(s)\Delta s + B \\
&= -\int_0^T \varphi_0(s)\Delta s + \frac{\int_0^T \varphi_0(s)\Delta s - \sum_{i=1}^{m-2} b_i \int_0^{\xi_i} \varphi_0(s)\Delta s}{1-\sum_{i=1}^{m-2} b_i} \\
&= \frac{\sum_{i=1}^{m-2} b_i \int_{\xi_i}^T \varphi_0(s)\Delta s}{1-\sum_{i=1}^{m-2} b_i} \\
&\geq 0.
\end{aligned}
\tag{2.15}
$$

If $t \in (0,T)$, we have

$$
\begin{aligned}
u(t) &= -\int_0^t \varphi_0(s)\Delta s + \frac{1}{1-\sum_{i=1}^{m-2} b_i}\left[\int_0^T \varphi_0(s)\Delta s - \sum_{i=1}^{m-2} b_i \int_0^{\xi_i} \varphi_0(s)\Delta s\right] \\
&\geq -\int_0^T \varphi_0(s)\Delta s + \frac{1}{1-\sum_{i=1}^{m-2} b_i}\left[\int_0^T \varphi_0(s)\Delta s - \sum_{i=1}^{m-2} b_i \int_0^{\xi_i} \varphi_0(s)\Delta s\right] \\
&= \frac{1}{1-\sum_{i=1}^{m-2} b_i}\left[-\left(1-\sum_{i=1}^{m-2} b_i\right)\int_0^T \varphi_0(s)\Delta s + \int_0^T \varphi_0(s)\Delta s - \sum_{i=1}^{m-2} b_i \int_0^{\xi_i} \varphi_0(s)\Delta s\right] \\
&= \frac{1}{1-\sum_{i=1}^{m-2} b_i}\sum_{i=1}^{m-2} b_i \int_{\xi_i}^T \varphi_0(s)\Delta s \geq 0.
\end{aligned}
\tag{2.16}
$$

So $u(t) \geq 0, t \in [0,T]$.

Let the norm on $C_{ld}[0,T]$ be the maximum norm. Then the $C_{ld}[0,T]$ is a Banach space. Choose the cone $P \subset C_{ld}[0,T]$ defined by

$$
P = \left\{u \in C_{ld}[0,T]: \ u(t) \geq 0, \ \text{for } t \in [0,T], \ u^{\Delta\nabla}(t) \leq 0, \ u^{\Delta}(t) \leq 0, \ \text{for } t \in (0,T)\right\}. \tag{2.17}
$$

Clearly, $\|u\| = u(0)$ for $u \in P$. Define the operator $A : P \to C_{ld}[0,T]$ by

$$(Au)(t) = -\int_0^t \phi^{-1}\left(\int_0^s a(\tau)f(\tau,u(\tau))\nabla\tau - \widetilde{A}\right)\Delta s + \widetilde{B}, \tag{2.18}$$

where

$$\widetilde{A} = -\frac{\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} a(\tau)f(\tau,u(\tau))\nabla\tau}{1-\sum_{i=1}^{m-2} a_i},$$
$$\widetilde{B} = \frac{\int_0^T \phi^{-1}\left(\int_0^s a(\tau)f(\tau,u(\tau))\nabla\tau - \widetilde{A}\right)\Delta s - \sum_{i=1}^{m-2} b_i \int_0^{\xi_i} \phi^{-1}\left(\int_0^s a(\tau)f(\tau,u(\tau))\nabla\tau - \widetilde{A}\right)\Delta s}{1-\sum_{i=1}^{m-2} b_i}. \tag{2.19}$$

It is obvious from Lemma 2.6 that, $Au(t) \geq 0$ for $t \in [0,T]$.

From the definition of A, we claim that for each $u \in P$, $Au \in P$and $Au(t)$ satisfies (1.2) and $Au(0)$ is the maximum value of $Au(t)$ on $[0,T]$.

In fact, let

$$\varphi(s) = \phi^{-1}\left(\int_0^s a(\tau)f(\tau,u(\tau))\nabla\tau - \widetilde{A}\right). \tag{2.20}$$

Then it holds

$$(Au)^{\Delta}(t) = -\varphi(t). \tag{2.21}$$

Since

$$\begin{aligned}\int_0^t a(\tau)f(\tau,u(\tau))\nabla\tau - \widetilde{A} &= \int_0^t a(\tau)f(\tau,u(\tau))\nabla\tau + \frac{\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} a(\tau)f(\tau,u(\tau))\nabla\tau}{1-\sum_{i=1}^{m-2} a_i} \\ &\geq 0,\end{aligned} \tag{2.22}$$

then $\varphi(t) \geq 0$. So $(Au)^{\Delta}(t) \leq 0, t \in (0,T)$.

Moreover, ϕ^{-1} is a monotone increasing and continuous function and

$$\left(\int_0^t a(\tau)f(\tau,u(\tau))\nabla\tau - \widetilde{A}\right)^{\nabla} = -a(t)f(t,u(t)) \leq 0, \tag{2.23}$$

then we obtain $(Au)^{\Delta\nabla}(t) \leq 0$, so, $A : P \to P$. So by applying Arzela-Ascoli theorem on time scales [20], we can obtain that $A(P)$ is relatively compact. In view of Lebesgue's dominated convergence theorem on time scales [21], it is easy to prove that A is continuous. Hence, $A : P \to P$ is completely continuous. □

Lemma 2.7. *If $u \in P$, then $u(t) \geq ((T-t)/T)\|u\|$ for $t \in [0,T]$.*

Proof. Since $u^{\Delta\nabla}(t) \leq 0$, it follows that $u^{\Delta}(t)$ is nonincreasing. Thus, for $0 < t < T$,

$$\begin{aligned} u(t) - u(0) &= \int_0^t u^{\Delta}(s)\Delta s \geq t u^{\Delta}(t), \\ u(T) - u(t) &= \int_t^T u^{\Delta}(s)\Delta s \leq (T-t)u^{\Delta}(t), \end{aligned} \tag{2.24}$$

from which we have

$$u(t) \geq \frac{tu(T) + (T-t)u(0)}{T} \geq \frac{T-t}{T}u(0) = \frac{T-t}{T}\|u\|. \tag{2.25}$$

The proof is complete. □

In the rest of this section, we provide some background material from the theory of cones in Banach spaces, and we then state several fixed point theorems which we will use later.

Let E be a Banach space and $\ddot{E}$ a cone in E. A map $\psi : \ddot{E} \to [0,+\infty)$ is said to be a nonnegative, continuous, and increasing functional provided that ψ is nonnegative, continuous and satisfies $\psi(x) \leq \psi(y)$ for all $x, y \in \ddot{E}$ and $x \leq y$.

Given a nonnegative continuous functional ψ on a cone $\ddot{E}$ of a real Banach space E, we define, for each $d > 0$, the set $\ddot{E}(\psi, d) = \{x \in \ddot{E} : \ \psi(x) < d\}$.

Lemma 2.8 (see [22]). *Let $\ddot{E}$ be a cone in a real Banach space E. Let α and γ be increasing, nonnegative continuous functionals on $\ddot{E}$, and let θ be a nonnegative continuous functional on $\ddot{E}$ with $\theta(0) = 0$ such that, for some $c > 0$ and $H > 0$,*

$$\gamma(x) \leq \theta(x) \leq \alpha(x), \qquad \|x\| \leq H\gamma(x), \tag{2.26}$$

for all $x \in \overline{\ddot{E}(\gamma, c)}$. Suppose that there exists a completely continuous operator $A : \overline{\ddot{E}(\gamma, c)} \to \ddot{E}$ and $0 < a < b < c$ such that

$$\theta(\lambda x) \leq \lambda\theta(x) \quad \text{for } 0 \leq \lambda \leq 1, \quad x \in \partial\ddot{E}(\theta, b), \tag{2.27}$$

and

(i) *$\gamma(Ax) > c$ for all $x \in \partial\ddot{E}(\gamma, c)$;*

(ii) *$\theta(Ax) < b$ for all $x \in \partial\ddot{E}(\theta, b)$;*

(iii) *$\ddot{E}(\alpha, a) \neq \emptyset$ and $\alpha(Ax) > a$ for $x \in \partial\ddot{E}(\alpha, a)$.*

Then, A has at least two fixed points, x_1 and x_2 belonging to $\overline{\ddot{E}(\gamma, c)}$ satisfying

$$a < \alpha(x_1) \quad \text{with } \theta(x_1) < b, \qquad b < \theta(x_2) \quad \text{with } \gamma(x_2) < c. \tag{2.28}$$

The following lemma is similar to Lemma 2.8.

Lemma 2.9 (see [23]). *Let $\ddot{E}$ be a cone in a real Banach space E. Let α and γ be increasing, nonnegative continuous functionals on $\ddot{E}$, and let θ be a nonnegative continuous functional on $\ddot{E}$ with $\theta(0) = 0$ such that, for some $c > 0$ and $H > 0$,*

$$\gamma(x) \leq \theta(x) \leq \alpha(x), \qquad \|x\| \leq H\gamma(x) \tag{2.29}$$

for all $x \in \overline{\ddot{E}(\gamma, c)}$. Suppose that there exists a completely continuous operator $A : \overline{\ddot{E}(\gamma, c)} \to \ddot{E}$ and $0 < a < b < c$ such that

$$\theta(\lambda x) \leq \lambda\theta(x) \quad \text{for } 0 \leq \lambda \leq 1,\ x \in \partial\ddot{E}(\theta, b), \tag{2.30}$$

and

(i) *$\gamma(Ax) < c$ for all $x \in \partial\ddot{E}(\gamma, c)$;*

(ii) *$\theta(Ax) > b$ for all $x \in \partial\ddot{E}(\theta, b)$;*

(iii) *$\ddot{E}(\alpha, a) \neq \emptyset$ and $\alpha(Ax) < a$ for $x \in \partial\ddot{E}(\alpha, a)$.*

Then, A has at least two fixed points, x_1 and x_2 belonging to $\overline{\ddot{E}(\gamma, c)}$ satisfying

$$a < \alpha(x_1) \quad \text{with } \theta(x_1) < b, \qquad b < \theta(x_2) \quad \text{with } \gamma(x_2) < c. \tag{2.31}$$

Let $0 < a < b$ be given and let α be a nonnegative continuous concave functional on the cone $\ddot{E}$. Define the convex sets $\ddot{E}_a$, $\ddot{E}(\alpha, a, b)$ by

$$\ddot{E}_a = \{x \in \ddot{E} : \|x\| < a\}, \qquad \ddot{E}(\alpha, a, b) = \{x \in \ddot{E} : a \leq \alpha(x), \|x\| < b\}. \tag{2.32}$$

Finally we state the Leggett-Williams fixed point theorem [3].

Lemma 2.10 (see [3]). *Let $\ddot{E}$ be a cone in a real Banach space E, $A : \overline{\ddot{E}_c} \to \overline{\ddot{E}_c}$ completely continuous, and α a nonnegative continuous concave functional on $\ddot{E}$ with $\alpha(x) \leq \|x\|$ for all $x \in \overline{\ddot{E}_c}$. Suppose that there exist $0 < d < a < b \leq c$ such that*

(i) *$\{x \in \ddot{E}(\alpha, a, b) : \alpha(x) > a\} \neq \emptyset$ and $\alpha(Ax) > a$ for $x \in \ddot{E}(\alpha, a, b)$;*

(ii) *$\|Ax\| < d$ for $\|x\| \leq d$;*

(iii) *$\alpha(Ax) > a$ for $x \in \ddot{E}(\alpha, a, c)$ with $\|Ax\| > b$.*

Then, A has at least three fixed points, x_1, x_2, x_3 satisfying

$$\|x_1\| < d, \qquad a < \alpha(x_2), \qquad \|x_3\| > d, \qquad \alpha(x_3) < a. \tag{2.33}$$

Now, for the convenience, we introduce the following notations. Let $l = \max\{t \in \mathbf{T} : 0 \leq t \leq \mathbf{T}/2\}$ and fixed $c \in T$ such that $0 < c < l$, denote

$$M = \frac{T-l}{T}\int_0^l \phi^{-1}\left(\int_0^s a(\tau)\nabla\tau\right)\Delta s,$$
$$N = \frac{1}{1-\sum_{i=1}^{m-2} b_i}\int_0^T \phi^{-1}\left(\int_0^s a(\tau)\nabla\tau + \frac{\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} a(\tau)\nabla\tau}{1-\sum_{i=1}^{m-2} a_i}\right)\Delta s, \tag{2.34}$$
$$L = \frac{T-c}{T}\int_0^c \phi^{-1}\left(\int_0^s a(\tau)\nabla\tau\right)\Delta s.$$

Define the nonnegative, increasing, and continuous functionals γ, θ, and α on P by

$$\gamma(u) = \min_{t\in[c,l]} u(t) = u(l), \qquad \theta(u) = \min_{t\in[0,l]} u(t) = u(l),$$
$$\alpha(u) = \min_{t\in[0,c]} u(t) = u(c). \tag{2.35}$$

We observe that, for each $u \in P$, $\gamma(u) = \theta(u) \leq \alpha(u)$.

In addition, for each $u \in P$, $\gamma(u) = u(l) \geq ((T-l)/T)\|u\|$. Thus $\|u\| \leq (T/(T-l))\gamma(u)$, $u \in P$.

Finally, we also note that $\theta(\lambda u) = \lambda\theta(u)$, $0 \leq \lambda \leq 1$, and $u \in \partial P(\theta, b')$.

3. Existence Theorems of Twin Positive Solutions

Theorem 3.1. *Assume that there are positive numbers $a' < b' < c'$ such that*

$$0 < a' < \frac{L}{N}b' < \frac{(T-l)L}{TN}c'. \tag{3.1}$$

Assume further that $f(t,u)$ satisfies the following conditions:

(i) $f(t,u) > \phi(c'/M)$, $(t,u) \in [0,l] \times [c', (T/(T-l))c']$,

(ii) $f(t,u) < \phi(b'/N)$, $(t,u) \in [0,T] \times [0, (T/(T-l))b']$,

(iii) $f(t,u) > \phi(a'/L)$, $(t,u) \in [0,c] \times [a', (T/(T-c))a']$.

Then (1.1) *and* (1.2) *has at least two positive solutions u_1 and u_2 such that*

$$a' < \min_{t\in[0,c]} u_1(t) \quad \textit{with} \ \min_{t\in[0,l]} u_1(t) < b', \qquad b' < \min_{t\in[0,l]} u_2(t) \quad \textit{with} \ \min_{t\in[c,l]} u_2(t) < c'. \tag{3.2}$$

Proof. By the definition of the operator A and its properties, it suffices to show that the conditions of Lemma 2.8 hold with respect to A.

We first show that if $u \in \partial P(\gamma, c')$ then $\gamma(Au) > c'$. Indeed, if $u \in \partial P(\gamma, c')$, then $\gamma(u) = \min_{t\in[c,l]} u(t) = u(l) = c'$. Since $u \in P$, $\|u\| \leq (T/(T-l))\gamma(u) = (T/(T-l))c'$, we have $c' \leq u(t) \leq$

$(T/(T-l))c'$, $t \in [0,l]$. As a consequence of (i), $f(t,u) > \phi(c'/M)$, $t \in [0,l]$. Also, $Au \in P$ implies that

$$\begin{aligned}
\gamma(Au) &= Au(l) \geq \frac{T-l}{T}Au(0) = \frac{T-l}{T}\widetilde{B} = \frac{T-l}{T} \\
&\quad\cdot \frac{\int_0^T \phi^{-1}\left(\int_0^s a(\tau)f(\tau,u(\tau))\nabla\tau - \widetilde{A}\right)\Delta s - \sum_{i=1}^{m-2} b_i \int_0^{\xi_i}\phi^{-1}\left(\int_0^s a(\tau)f(\tau,u(\tau))\nabla\tau - \widetilde{A}\right)\Delta s}{1-\sum_{i=1}^{m-2} b_i} \\
&\geq \frac{T-l}{T}\int_0^T \phi^{-1}\left(\int_0^s a(\tau)f(\tau,u(\tau))\nabla\tau - \widetilde{A}\right)\Delta s \\
&= \frac{T-l}{T}\int_0^T \phi^{-1}\left(\int_0^s a(\tau)f(\tau,u(\tau))\nabla\tau + \frac{\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} a(\tau)f(\tau,u(\tau))\nabla\tau}{1-\sum_{i=1}^{m-2} a_i}\right)\Delta s \\
&\geq \frac{T-l}{T}\int_0^T \phi^{-1}\left(\int_0^s a(\tau)f(\tau,u(\tau))\nabla\tau\right)\Delta s \geq \frac{T-l}{T}\int_0^l \phi^{-1}\left(\int_0^s a(\tau)f(\tau,u(\tau))\nabla\tau\right)\Delta s \\
&> \frac{T-l}{T}\cdot\frac{c'}{M}\int_0^l \phi^{-1}\left(\int_0^s a(\tau)\nabla\tau\right)\Delta s = c'.
\end{aligned} \tag{3.3}$$

Next, we verify that $\theta(Au) < b'$ for $u \in \partial P(\theta, b')$.

Let us choose $u \in \partial P(\theta, b')$, then $\theta(u) = \min_{t\in[0,l]} u(t) = u(l) = b'$, and $0 \leq u(t) \leq \|u\| \leq (T/(T-l))u(l) = (T/(T-l))b'$, for $t \in [0,T]$. Using (ii),

$$f(t,u(t)) < \phi\left(\frac{b'}{N}\right), \quad t \in [0,T]. \tag{3.4}$$

Also, $Au \in P$ implies that

$$\begin{aligned}
\theta(Au) &= Au(l) \leq Au(0) = \widetilde{B} \\
&\leq \frac{1}{1-\sum_{i=1}^{m-2} b_i}\int_0^T \phi^{-1}\left(\int_0^s a(\tau)f(\tau,u(\tau))\nabla\tau - \widetilde{A}\right)\Delta s \\
&= \frac{1}{1-\sum_{i=1}^{m-2} b_i}\int_0^T \phi^{-1}\left(\int_0^s a(\tau)f(\tau,u(\tau))\nabla\tau + \frac{\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} a(\tau)f(\tau,u(\tau))\nabla\tau}{1-\sum_{i=1}^{m-2} a_i}\right)\Delta s \\
&\leq \frac{b'}{N}\cdot\frac{1}{1-\sum_{i=1}^{m-2} b_i}\int_0^T \phi^{-1}\left(\int_0^s a(\tau)\nabla\tau + \frac{\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} a(\tau)\nabla\tau}{1-\sum_{i=1}^{m-2} a_i}\right)\Delta s \\
&= b'.
\end{aligned} \tag{3.5}$$

Finally, we prove that $P(\alpha, a') \neq \emptyset$ and $\alpha(Au) > a'$ for $u \in \partial P(\alpha, a')$.

In fact, the constant function $a'/2 \in P(\alpha, a')$. Moreover, for $u \in \partial P(\alpha, a')$, we have $\alpha(u) = \min_{t\in[0,c]} u(t) = u(c) = a'$. This implies that $a' \le u(t) \le (T/(T-c))a'$, $t \in [0,c]$. Using assumption (iii), $f(t, u(t)) > \phi(a'/L)$, $t \in [0,c]$. As before, by $Au \in P$, we obtain

$$\begin{aligned}
\alpha(Au) = (Au)(c) &\ge \frac{T-c}{T} Au(0) = \frac{T-c}{T}\widetilde{B} \\
&\ge \frac{T-c}{T}\int_0^T \phi^{-1}\left(\int_0^s a(\tau) f(\tau, u(\tau))\nabla\tau\right)\Delta s \ge \frac{T-c}{T}\int_0^c \phi^{-1}\left(\int_0^s a(\tau) f(\tau, u(\tau))\nabla\tau\right)\Delta s \\
&> \frac{T-c}{T}\cdot\frac{a'}{L}\int_0^c \phi^{-1}\left(\int_0^s a(\tau)\nabla\tau\right)\Delta s = a'.
\end{aligned} \tag{3.6}$$

Thus, by Lemma 2.8, there exist at least two fixed points of A which are positive solutions u_1 and u_2, belonging to $\overline{P(\gamma, c')}$, of the BVP (1.1) and (1.2) such that

$$a' < \alpha(u_1) \quad \text{with } \theta(u_1) < b', \qquad b' < \theta(u_2) \quad \text{with } \gamma(u_2) < c'. \tag{3.7}$$

The proof is complete. □

Theorem 3.2. *Assume that there are positive numbers $a' < b' < c'$ such that*

$$0 < a' < \frac{T-c}{T} b' < \frac{(T-c)M}{TN} c'. \tag{3.8}$$

Assume further that $f(t,u)$ satisfies the following conditions:

(i) $f(t,u) < \phi(c'/N)$, $(t,u) \in [0,T] \times [0, (T/(T-l))c']$,

(ii) $f(t,u) > \phi(b'/M)$, $(t,u) \in [0,l] \times [b', (T/(T-l))b']$,

(iii) $f(t,u) < \phi(a'/N)$, $(t,u) \in [0,c] \times [0, (T/(T-c))a']$.

Then (1.1) *and* (1.2) *has at least two positive solutions u_1 and u_2 such that*

$$a' < \min_{t\in[0,c]} u_1(t) \quad \text{with } \min_{t\in[0,\,l]} u_1(t) < b', \qquad b' < \min_{t\in[0,l]} u_2(t) \quad \text{with } \min_{t\in[c,l]} u_2(t) < c'. \tag{3.9}$$

Using Lemma 2.9, the proof is similar to that of Theorem 3.1 and we omit it here.

4. Existence Theorems of Triple Positive Solutions

In this section, let the nonnegative continuous functional $\psi : P \to [0, +\infty)$ be defined by

$$\psi(u) = \min_{t\in[0,l]} u(t) = u(l), \quad u \in P. \tag{4.1}$$

Note that for $u \in P$, $\psi(u) \le \|u\|$.

Theorem 4.1. *Suppose that there exist positive constants* $0 < d' < a'$ *such that*

(i) $f(t,u) < \phi(d'/N)$, $(t,u) \in [0,T] \times [0,d']$;

(ii) $f(t,u) \geq \phi(a'/M)$, $(t,u) \in [0,l] \times [a',(T/(T-l))a']$;

(iii) *one of the following conditions holds:*

(D_1) $\lim_{u\to\infty}\max_{t\in[0,T]}(f(t,u)/\phi(u)) < \phi(1/N)$;

(D_2) *there exists a number* $c' > (T/(T-l))a'$ *such that* $f(t,u) < \phi(c'/N)$, $(t,u) \in [0,T] \times [0,c']$.

Then (1.1) *and* (1.2) *has at least three positive solutions.*

Proof. By the definition of operator A and its properties, it suffices to show that the conditions of Lemma 2.10 hold with respect to A.

We first show that if (D_1) holds, then there exists a number $l' > (T/(T-l))a'$ such that $A : \overline{P_{l'}} \to P_{l'}$. Suppose that

$$\lim_{u\to\infty}\max_{t\in[0,T]}\frac{f(t,u)}{\phi(u)} < \phi\left(\frac{1}{N}\right) \tag{4.2}$$

holds, then there are $\tau > 0$ and $\delta < 1/N$ such that if $u > \tau$, then $\max_{t\in[0,T]}(f(t,u)/\phi(u)) \leq \phi(\delta)$, that is to say, $f(t,u) \leq \phi(\delta u)$, $(t,u) \in [0,T] \times [\tau,+\infty)$.

Set $\lambda = \max\{f(t,u) : (t,u) \in [0,T] \times [0,\tau]\}$, then

$$f(t,u) \leq \lambda + \phi(\delta u), \quad (t,u) \in [0,T] \times [0,+\infty). \tag{4.3}$$

Set

$$l' > \max\left\{\frac{T}{T-l}a', \phi^{-1}\left(\frac{\lambda\phi(N)}{1-\phi(\delta N)}\right)\right\}. \tag{4.4}$$

If $u \in \overline{P_{l'}}$, then by (2.18), (4.3), (4.4), we obtain

$$\begin{aligned}
\|Au\| &= Au(0) = \widetilde{B} \\
&\leq \frac{1}{1-\sum_{i=1}^{m-2}b_i}\int_0^T \phi^{-1}\left(\int_0^s a(\tau)f(\tau,u(\tau))\nabla\tau - \widetilde{A}\right)\Delta s \\
&= \frac{1}{1-\sum_{i=1}^{m-2}b_i}\int_0^T \phi^{-1}\left(\int_0^s a(\tau)f(\tau,u(\tau))\nabla\tau + \frac{\sum_{i=1}^{m-2}a_i\int_0^{\xi_i}a(\tau)f(\tau,u(\tau))\nabla\tau}{1-\sum_{i=1}^{m-2}a_i}\right)\Delta s \\
&\leq \phi^{-1}(\lambda+\phi(\delta l'))\frac{1}{1-\sum_{i=1}^{m-2}b_i}\int_0^T \phi^{-1}\left(\int_0^s a(\tau)\nabla\tau + \frac{\sum_{i=1}^{m-2}a_i\int_0^{\xi_i}a(\tau)\nabla\tau}{1-\sum_{i=1}^{m-2}a_i}\right)\Delta s \\
&= \phi^{-1}(\lambda+\phi(\delta l'))N < l'.
\end{aligned} \tag{4.5}$$

Here we used the inequality

$$\lambda + \phi(\delta l') < \phi\left(\frac{l'}{N}\right), \tag{4.6}$$

For this, by (4.4), we have

$$l' > \phi^{-1}\left(\frac{\lambda\phi(N)}{1-\phi(\delta N)}\right), \tag{4.7}$$

so

$$\frac{\lambda\phi(N)}{1-\phi(\delta N)} < \phi(l'), \tag{4.8}$$

by using the property of ϕ and easy computation, we have

$$(\lambda + \phi(\delta l'))\phi(N) < \phi(l'), \tag{4.9}$$

and (4.6) is obtained.

Next we verify that if there is a positive number r' such that if $f(t,u) < \phi(r'/N)$, for $(t,u) \in [0,T] \times [0,r']$, then $A : \overline{P_{r'}} \to P_{r'}$.

Indeed, if $u \in \overline{P_{r'}}$, then

$$\begin{aligned}
\|Au\| &= Au(0) = \widetilde{B} \\
&\le \frac{1}{1-\sum_{i=1}^{m-2} b_i}\int_0^T \phi^{-1}\left(\int_0^s a(\tau)f(\tau,u(\tau))\nabla\tau - \widetilde{A}\right)\Delta s \\
&= \frac{1}{1-\sum_{i=1}^{m-2} b_i}\int_0^T \phi^{-1}\left(\int_0^s a(\tau)f(\tau,u(\tau))\nabla\tau + \frac{\sum_{i=1}^{m-2} a_i\int_0^{\xi_i} a(\tau)f(\tau,u(\tau))\nabla\tau}{1-\sum_{i=1}^{m-2} a_i}\right)\Delta s \\
&< \frac{r'}{N}\cdot\frac{1}{1-\sum_{i=1}^{m-2} b_i}\int_0^T \phi^{-1}\left(\int_0^s a(\tau)\nabla\tau + \frac{\sum_{i=1}^{m-2} a_i\int_0^{\xi_i} a(\tau)\nabla\tau}{1-\sum_{i=1}^{m-2} a_i}\right)\Delta s \\
&= r',
\end{aligned} \tag{4.10}$$

thus, $Au \in P_{r'}$.

Hence, we have shown that either (D_1) or (D_2) holds, then there exists a number c' with $c' > (T/(T-l))a'$ such that $A : \overline{P_{c'}} \to P_{c'}$. Also note that from (i) we have that $A : \overline{P_{d'}} \to P_{d'}$.

Now, we show that $\{u \in P(\psi, a', (T/(T-l))a') : \psi(u) > a'\} \neq \emptyset$ and $\psi(Au) > a'$ for all $u \in P(\psi, a', (T/(T-l))a')$.

In fact,

$$u = \frac{(l+T)a'}{2l} \in \left\{ u \in P\left(\psi, a', \frac{T}{T-l}a'\right) : \psi(u) > a' \right\}. \tag{4.11}$$

For $u \in P(\psi, a', T/(T-l)a')$, we have

$$a' \le \min_{t\in[0,l]} u(t) = u(l) \le u(t) \le \frac{T}{T-l}a', \tag{4.12}$$

for all $t \in [0,l]$. Then, in view of (ii), we know that

$$\begin{aligned}
\psi(Au) &= \min_{t\in[0,l]} Au(t) = Au(l) \ge \frac{T-l}{T}Au(0) = \frac{T-l}{T}\widetilde{B} = \frac{T-l}{T} \\
&\quad \cdot \frac{\int_0^T \phi^{-1}\left(\int_0^s a(\tau)f(\tau,u(\tau))\nabla\tau - \widetilde{A}\right)\Delta s - \sum_{i=1}^{m-2} b_i \int_0^{\xi_i} \phi^{-1}\left(\int_0^s a(\tau)f(\tau,u(\tau))\nabla\tau - \widetilde{A}\right)\Delta s}{1-\sum_{i=1}^{m-2} b_i} \\
&\ge \frac{T-l}{T}\int_0^T \phi^{-1}\left(\int_0^s a(\tau)f(\tau,u(\tau))\nabla\tau - \widetilde{A}\right)\Delta s \\
&= \frac{T-l}{T}\int_0^T \phi^{-1}\left(\int_0^s a(\tau)f(\tau,u(\tau))\nabla\tau + \frac{\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} a(\tau)f(\tau,u(\tau))\nabla\tau}{1-\sum_{i=1}^{m-2} a_i}\right)\Delta s \\
&\ge \frac{T-l}{T}\int_0^T \phi^{-1}\left(\int_0^s a(\tau)f(\tau,u(\tau))\nabla\tau\right)\Delta s \ge \frac{T-l}{T}\int_0^l \phi^{-1}\left(\int_0^s a(\tau)f(\tau,u(\tau))\nabla\tau\right)\Delta s \\
&> \frac{T-l}{T}\cdot\frac{a'}{M}\int_0^l \phi^{-1}\left(\int_0^s a(\tau)\nabla\tau\right)\Delta s = a'.
\end{aligned} \tag{4.13}$$

Finally, we assert that if $u \in P(\psi, a', c')$ and $\|Au\| > (T/(T-l))a'$, then $\psi(Au) > a'$.

Suppose $u \in P(\psi, a', c')$ and $\|Au\| > (T/(T-l))a'$, then

$$\psi(Au) = \min_{t\in[0,l]} Au(t) = Au(l) \ge \frac{T-l}{T}Au(0) = \frac{T-l}{T}\|Au\| > a'. \tag{4.14}$$

To sum up, the hypotheses of Lemma 2.10 are satisfied, hence (1.1) and (1.2) has at least three positive solutions u_1, u_2, u_3 such that

$$\|u_1\| < d', \qquad a' < \min_{t\in[0,l]} u_2(t), \qquad \|u_3\| > d' \quad \text{with } \min_{t\in[0,l]} u_3(t) < a'. \tag{4.15}$$

The proof is complete. □

5. Existence Theorems of $2n-1$ Positive Solutions

From Theorem 4.1, we see that, when assumptions like (i), (ii), and (iii) are imposed appropriately on f, we can establish the existence of an arbitrary odd number of positive solutions of (1.1) and (1.2).

Theorem 5.1. *Suppose that there exist positive constants*

$$0 < d_1' < a_1' < \frac{T}{T-l}a_1' < d_2' < a_2' < \frac{T}{T-l}a_2' < d_3' < \cdots < d_n', \quad n \in N, \tag{5.1}$$

such that the following conditions are satisfied:

(i) $f(t,u) < \phi(d_i'/N)$, $(t,u) \in [0,T] \times [0,d_i']$;

(ii) $f(t,u) \geq \phi(a_i'/M)$, $(t,u) \in [0,l] \times [a_i',(T/(T-l))a_i']$.

Then (1.1) *and* (1.2) *has at least* $2n-1$ *positive solutions.*

Proof. When $n=1$, it is immediate from condition (i) that $A : \overline{P_{d_1'}} \to P_{d_1'} \subset \overline{P_{d_1'}}$, which means that A has at least one fixed point $u_1 \in \overline{P_{d_1'}}$ by the Schauder fixed point theorem. When $n=2$, it is clear that Theorem 4.1 holds (with $c_1 = d_2'$). Then we can obtain at least three positive solutions u_1, u_2, u_3 satisfying

$$\|u_1\| < d_1', \qquad a_1' < \min_{t\in[0,l]} u_2(t), \qquad \|u_3\| > d_1' \quad \text{with} \ \min_{t\in[0,l]} u_3(t) < a_1'. \tag{5.2}$$

Following this way, we finish the proof by induction. The proof is complete. □

6. Application

In the section, we will present a simple example of discrete case to explain our results. Concerning the continuous case, differential equation, we refer to [8, 15, 24, 25].

Example 6.1. Let $\mathbf{T} = \{(1/2)^n : n \in N\} \bigcup \{1\}$, $T=1$. Consider the following BVP on time scales

$$\begin{gathered}\left(\phi\left(u^{\Delta}\right)\right)^{\nabla} + f(t,u(t)) = 0, \quad t \in (0,T), \\ \phi\left(u^{\Delta}(0)\right) = \frac{1}{3}\phi\left(u^{\Delta}\left(\frac{1}{2}\right)\right), \qquad u(T) = \frac{1}{4}u\left(\frac{1}{2}\right),\end{gathered} \tag{6.1}$$

where

$$\phi(u) = u, \qquad f(t,u) = f(u) := \frac{4000u^2}{u^2+5000}, \quad (t,u) \in [0,1] \times [0,+\infty). \tag{6.2}$$

It is easy to check that $f : [0,1] \times [0,+\infty) \to [0,+\infty)$ is continuous. In this case, $a(t) \equiv 1$, $a_1 = 1/3, b_1 = 1/4, \xi_1 = 1/2, m = 3$, it follows from a direct calculation that

$$
\begin{aligned}
M &= \frac{T-l}{T}\int_0^l \phi^{-1}\left(\int_0^s a(\tau)\nabla\tau\right)\Delta s = \frac{1}{2}\int_0^{l/2} s ds = \frac{1}{16}, \\
N &= \frac{1}{1-\sum_{i=1}^{m-2} b_i}\int_0^T \phi^{-1}\left(\int_0^s a(\tau)\nabla\tau + \frac{\sum_{i=1}^{m-2} a_i\int_0^{\xi_i} a(\tau)\nabla\tau}{1-\sum_{i=1}^{m-2} a_i}\right)\Delta s \\
&= \frac{1}{1-1/4}\int_0^1\left(s + \frac{(1/3)\cdot(1/2)}{1-1/3}\right)ds = 1.
\end{aligned}
\tag{6.3}
$$

Clearly f is always increasing. If we take $d' = 1/5, a' = 23, c' = 5000$, then

$$0 < d' < a' < 2a' = \frac{T}{T-l}a' < c'. \tag{6.4}$$

Now we check that (i), (ii) and (iii) of Theorem 4.1 are satisfied. In view of $f(1/5) = 4000/125025 = 0.0320$, we get

$$f(t,u) = f(u) < \frac{1}{5N} = \frac{1}{5} = 0.2, \quad u \in [0,d'], \tag{6.5}$$

so that (i) of Theorem 4.1 is satisfied. To verify (ii), note that $f(23) = 4000\cdot 23^2/(23^2+5000) \approx 382.7094$, so that

$$f(u) \geq \frac{23}{M} = 368, \quad u \in [a', 2a']. \tag{6.6}$$

Finally, as $\lim_{u\to\infty} f(u) = 4000$,

$$f(u) \leq 4000 < \frac{c'}{N} = 5000, \quad u \in [0,c'], \tag{6.7}$$

and (D_2) of (iii) holds. Thus by Theorem 4.1, the boundary value problem (6.1) has at least three positive solutions u_1, u_2, u_3 such that

$$\|u_1\| < \frac{1}{5}, \qquad 23 < \min_{t\in[0,1/2]} u_2(t), \qquad \|u_3\| > \frac{1}{5} \quad \text{with} \min_{t\in[0,1/2]} u_3(t) < 23. \tag{6.8}$$

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