

## Research Article

# An Existence Result for Nonlinear Fractional Differential Equations on Banach Spaces

Mouffak Benchohra,<sup>1</sup> Alberto Cabada,<sup>2</sup> and Djamil Seba<sup>3</sup>

<sup>1</sup> *Laboratoire de Mathématiques, Université de Sidi Bel-Abbès, BP 89, 22000 Sidi Bel-Abbès, Algeria*

<sup>2</sup> *Departamento de Analisis Matematico, Facultad de Matematicas, Universidad de Santiago de Compostela, 15782, Santiago de Compostela, Spain*

<sup>3</sup> *Département de Mathématiques, Université de Boumerdès, Avenue de l'Indépendance, 35000 Boumerdès, Algeria*

Correspondence should be addressed to Mouffak Benchohra, benchohra@univ-sba.dz

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The aim of this paper is to investigate a class of boundary value problem for fractional differential equations involving nonlinear integral conditions. The main tool used in our considerations is the technique associated with measures of noncompactness.

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## 1. Introduction

The theory of fractional differential equations has been emerging as an important area of investigation in recent years. Let us mention that this theory has many applications in describing numerous events and problems of the real world. For example, fractional differential equations are often applicable in engineering, physics, chemistry, and biology. See Hilfer [1], Glockle and Nonnenmacher [2], Metzler et al. [3], Podlubny [4], Gaul et al. [5], among others. Fractional differential equations are also often an object of mathematical investigations; see the papers of Agarwal et al. [6], Ahmad and Nieto [7], Ahmad and Otero-Espinar [8], Belarbi et al. [9], Belmekki et al [10], Benchohra et al. [11–13], Chang and Nieto [14], Daftardar-Gejji and Bhalekar [15], Figueiredo Camargo et al. [16], and the monographs of Kilbas et al. [17] and Podlubny [4].

Applied problems require definitions of fractional derivatives allowing the utilization of physically interpretable initial conditions, which contain  $y(0)$ ,  $y'(0)$ , and so forth. the same requirements of boundary conditions. Caputo's fractional derivative satisfies these demands. For more details on the geometric and physical interpretation for fractional derivatives of both the Riemann-Liouville and Caputo types, see [18, 19].

In this paper we investigate the existence of solutions for boundary value problems with fractional order differential equations and nonlinear integral conditions of the form

$$\begin{aligned} {}^c D^r y(t) &= f(t, y(t)), \quad \text{for each } t \in J = [0, T], \\ y(0) - y'(0) &= \int_0^T g(s, y(s)) ds, \\ y(T) + y'(T) &= \int_0^T h(s, y(s)) ds, \end{aligned} \tag{1.1}$$

where  ${}^c D^r$ ,  $1 < r \leq 2$  is the Caputo fractional derivative,  $f$ ,  $g$ , and  $h : J \times E \rightarrow E$  are given functions satisfying some assumptions that will be specified later, and  $E$  is a Banach space with norm  $\|\cdot\|$ .

Boundary value problems with integral boundary conditions constitute a very interesting and important class of problems. They include two, three, multipoint, and nonlocal boundary value problems as special cases. Integral boundary conditions are often encountered in various applications; it is worthwhile mentioning the applications of those conditions in the study of population dynamics [20] and cellular systems [21].

Moreover, boundary value problems with integral boundary conditions have been studied by a number of authors such as, for instance, Arara and Benchohra [22], Benchohra et al. [23, 24], Infante [25], Peculyte et al. [26], and the references therein.

In our investigation we apply the method associated with the technique of measures of noncompactness and the fixed point theorem of Mönch type. This technique was mainly initiated in the monograph of Bana and Goebel [27] and subsequently developed and used in many papers; see, for example, Bana and Sadarangoni [28], Guo et al. [29], Lakshmikantham and Leela [30], Mönch [31], and Szufła [32].

## 2. Preliminaries

In this section, we present some definitions and auxiliary results which will be needed in the sequel.

Denote by  $C(J, E)$  the Banach space of continuous functions  $J \rightarrow E$ , with the usual supremum norm

$$\|y\|_\infty = \sup \{\|y(t)\|, t \in J\}. \tag{2.1}$$

Let  $L^1(J, E)$  be the Banach space of measurable functions  $y : J \rightarrow E$  which are Bochner integrable, equipped with the norm

$$\|y\|_{L^1} = \int_0^T \|y(s)\| ds. \tag{2.2}$$

Let  $L^\infty(J, E)$  be the Banach space of measurable functions  $y : J \rightarrow E$  which are bounded, equipped with the norm

$$\|y\|_{L^\infty} = \inf \{c > 0 : \|y(t)\| \leq c, \text{ a.e. } t \in J\}. \quad (2.3)$$

Let  $AC^1(J, E)$  be the space of functions  $y : J \rightarrow E$ , whose first derivative is absolutely continuous.

Moreover, for a given set  $V$  of functions  $v : J \rightarrow E$  let us denote by

$$\begin{aligned} V(t) &= \{v(t), v \in V\}, \quad t \in J, \\ V(J) &= \{v(t) : v \in V\}, \quad t \in J. \end{aligned} \quad (2.4)$$

Now let us recall some fundamental facts of the notion of Kuratowski measure of noncompactness.

*Definition 2.1* (see [27]). Let  $E$  be a Banach space and  $\Omega_E$  the bounded subsets of  $E$ . The Kuratowski measure of noncompactness is the map  $\alpha : \Omega_E \rightarrow [0, \infty]$  defined by

$$\alpha(B) = \inf \left\{ \varepsilon > 0 : B \subseteq \bigcup_{i=1}^n B_i \text{ and } \text{diam}(B_i) \leq \varepsilon \right\}; \quad \text{here } B \in \Omega_E. \quad (2.5)$$

### Properties

The Kuratowski measure of noncompactness satisfies some properties (for more details see [27]).

- (a)  $\alpha(B) = 0 \Leftrightarrow \overline{B}$  is compact ( $B$  is relatively compact).
- (b)  $\alpha(B) = \alpha(\overline{B})$ .
- (c)  $A \subseteq B \Rightarrow \alpha(A) \leq \alpha(B)$ .
- (d)  $\alpha(A + B) \leq \alpha(A) + \alpha(B)$ .
- (e)  $\alpha(cB) = |c|\alpha(B)$ ;  $c \in \mathbb{R}$ .
- (f)  $\alpha(\text{co}B) = \alpha(B)$ .

Here  $\overline{B}$  and  $\text{co}B$  denote the closure and the convex hull of the bounded set  $B$ , respectively.

For completeness we recall the definition of Caputo derivative of fractional order.

*Definition 2.2* (see [17]). The fractional order integral of the function  $h \in L^1([a, b])$  of order  $r \in \mathbb{R}_+$ ; is defined by

$$I_a^r h(t) = \frac{1}{\Gamma(r)} \int_a^t \frac{h(s)}{(t-s)^{1-r}} dt, \quad (2.6)$$

where  $\Gamma$  is the gamma function. When  $a = 0$ , we write  $I^r h(t) = [h * \varphi_r](t)$ , where

$$\varphi_r(t) = \frac{t^{r-1}}{\Gamma(r)} \quad \text{for } t > 0, \quad (2.7)$$

$\varphi_r(t) = 0$  for  $t \leq 0$ , and  $\varphi_r \rightarrow \delta(t)$  as  $r \rightarrow 0$ .

Here  $\delta$  is the delta function.

*Definition 2.3* (see [17]). For a function  $h$  given on the interval  $[a, b]$ , the Caputo fractional-order derivative of  $h$ , of order  $r > 0$ , is defined by

$${}^c D_{a^+}^r h(t) = \frac{1}{\Gamma(n-r)} \int_a^t \frac{h^{(n)}(s) ds}{(t-s)^{1-n+r}}. \quad (2.8)$$

Here  $n = [r] + 1$  and  $[r]$  denotes the integer part of  $r$ .

*Definition 2.4.* A map  $f : J \times E \rightarrow E$  is said to be Carathéodory if

- (i)  $t \mapsto f(t, u)$  is measurable for each  $u \in E$ ;
- (ii)  $u \mapsto f(t, u)$  is continuous for almost each  $t \in J$ .

For our purpose we will only need the following fixed point theorem and the important Lemma.

**Theorem 2.5** (see [31, 33]). *Let  $D$  be a bounded, closed and convex subset of a Banach space such that  $0 \in D$ , and let  $N$  be a continuous mapping of  $D$  into itself. If the implication*

$$V = \overline{c_0 N}(V) \quad \text{or} \quad V = N(V) \cup \{0\} \implies \alpha(V) = 0 \quad (2.9)$$

*holds for every subset  $V$  of  $D$ , then  $N$  has a fixed point.*

**Lemma 2.6** (see [32]). *Let  $D$  be a bounded, closed, and convex subset of the Banach space  $C(J, E)$ ,  $G$  a continuous function on  $J \times J$ , and a function  $f : J \times E \rightarrow E$  satisfies the Carathéodory conditions, and there exists  $p \in L^1(J, \mathbb{R}_+)$  such that for each  $t \in J$  and each bounded set  $B \subset E$  one has*

$$\lim_{k \rightarrow 0^+} \alpha(f(J_{t,k} \times B)) \leq p(t) \alpha(B); \quad \text{where } J_{t,k} = [t-k, t] \cap J. \quad (2.10)$$

*If  $V$  is an equicontinuous subset of  $D$ , then*

$$\alpha\left(\left\{\int_J G(s, t) f(s, y(s)) ds : y \in V\right\}\right) \leq \int_J \|G(t, s)\| p(s) \alpha(V(s)) ds. \quad (2.11)$$

### 3. Existence of Solutions

Let us start by defining what we mean by a solution of the problem (1.1).

*Definition 3.1.* A function  $y \in AC^1(J, E)$  is said to be a solution of (1.1) if it satisfies (1.1).

Let  $\sigma, \rho_1, \rho_2 : J \rightarrow E$  be continuous functions and consider the linear boundary value problem

$$\begin{aligned} {}^c D^r y(t) &= \sigma(t), \quad t \in J, \\ y(0) - y'(0) &= \int_0^T \rho_1(s) ds, \\ y(T) + y'(T) &= \int_0^T \rho_2(s) ds. \end{aligned} \tag{3.1}$$

**Lemma 3.2** (see [11]). *Let  $1 < r \leq 2$  and let  $\sigma, \rho_1, \rho_2 : J \rightarrow E$  be continuous. A function  $y$  is a solution of the fractional integral equation*

$$y(t) = P(t) + \int_0^T G(t, s) \sigma(s) ds \tag{3.2}$$

with

$$P(t) = \frac{(T+1-t)}{T+2} \int_0^T \rho_1(s) ds + \frac{(t+1)}{T+2} \int_0^T \rho_2(s) ds, \tag{3.3}$$

$$G(t, s) = \begin{cases} \frac{(t-s)^{r-1}}{\Gamma(r)} - \frac{(1+t)(T-s)^{r-1}}{(T+2)\Gamma(r)} - \frac{(1+t)(T-s)^{r-2}}{(T+2)\Gamma(r-1)}, & 0 \leq s \leq t, \\ -\frac{(1+t)(T-s)^{r-1}}{(T+2)\Gamma(r)} - \frac{(1+t)(T-s)^{r-2}}{(T+2)\Gamma(r-1)}, & t \leq s \leq T, \end{cases} \tag{3.4}$$

if and only if  $y$  is a solution of the fractional boundary value problem (3.1).

*Remark 3.3.* It is clear that the function  $t \mapsto \int_0^T |G(t, s)| ds$  is continuous on  $J$ , and hence is bounded. Let

$$\tilde{G} := \sup \left\{ \int_0^T |G(t, s)| ds, \quad t \in J \right\}. \tag{3.5}$$

For the forthcoming analysis, we introduce the following assumptions

(H1) The functions  $f, g, h : J \times E \rightarrow E$  satisfy the Carathéodory conditions.

(H2) There exist  $p_f, p_g, p_h \in L^\infty(J, \mathbb{R}_+)$ , such that

$$\begin{aligned} \|f(t, y)\| &\leq p_f(t) \|y\| \quad \text{for a.e. } t \in J \text{ and each } y \in E, \\ \|g(t, y)\| &\leq p_g(t) \|y\|, \quad \text{for a.e. } t \in J \text{ and each } y \in E, \\ \|h(t, y)\| &\leq p_h(t) \|y\|, \quad \text{for a.e. } t \in J \text{ and each } y \in E. \end{aligned} \quad (3.6)$$

(H3) For almost each  $t \in J$  and each bounded set  $B \subset E$  we have

$$\begin{aligned} \lim_{k \rightarrow 0^+} \alpha(f(J_{t,k} \times B)) &\leq p_f(t) \alpha(B), \\ \lim_{k \rightarrow 0^+} \alpha(g(J_{t,k} \times B)) &\leq p_g(t) \alpha(B), \\ \lim_{k \rightarrow 0^+} \alpha(h(J_{t,k} \times B)) &\leq p_h(t) \alpha(B). \end{aligned} \quad (3.7)$$

**Theorem 3.4.** *Assume that assumptions (H1)–(H3) hold. If*

$$\frac{T(T+1)}{T+2} \left[ \|p_g\|_{L^\infty} + \|p_h\|_{L^\infty} \right] + \tilde{G} \|p_f\|_{L^\infty} < 1, \quad (3.8)$$

*then the boundary value problem (1.1) has at least one solution.*

*Proof.* We transform the problem (1.1) into a fixed point problem by defining an operator  $N : C(J, E) \rightarrow C(J, E)$  as

$$(Ny)(t) = P_y(t) + \int_0^T G(t, s) f(s, y(s)) ds, \quad (3.9)$$

where

$$P_y(t) = \frac{(T+1-t)}{T+2} \int_0^T g(s, y(s)) ds + \frac{(t+1)}{T+2} \int_0^T h(s, y(s)) ds, \quad (3.10)$$

and the function  $G(t, s)$  is given by (3.4). Clearly, the fixed points of the operator  $N$  are solution of the problem (1.1). Let  $R > 0$  and consider the set

$$D_R = \{y \in C(J, E) : \|y\|_\infty \leq R\}. \quad (3.11)$$

Clearly, the subset  $D_R$  is closed, bounded, and convex. We will show that  $N$  satisfies the assumptions of Theorem 2.5. The proof will be given in three steps.

*Step 1.*  $N$  is continuous.

Let  $\{y_n\}$  be a sequence such that  $y_n \rightarrow y$  in  $C(J, E)$ . Then, for each  $t \in J$ ,

$$\begin{aligned} |(Ny_n)(t) - (Ny)(t)| &\leq \frac{T+1}{T+2} \int_0^T \|g(s, y_n(s)) - g(s, y(s))\| ds \\ &\quad + \frac{T+1}{T+2} \int_0^T \|h(s, y_n(s)) - h(s, y(s))\| ds \\ &\quad + \int_0^T |G(t, s)| \|f(s, y_n(s)) - f(s, y(s))\| ds. \end{aligned} \quad (3.12)$$

Let  $\rho > 0$  be such that

$$\|y_n\|_\infty \leq \rho, \quad \|y\|_\infty \leq \rho. \quad (3.13)$$

By (H2) we have

$$\begin{aligned} \|g(s, y_n(s)) - g(s, y(s))\| &\leq 2\rho p_g(s) := \sigma_1(s); \quad \sigma_1 \in L^1(J, \mathbb{R}_+), \\ \|h(s, y_n(s)) - h(s, y(s))\| &\leq 2\rho p_h(s) := \sigma_2(s); \quad \sigma_2 \in L^1(J, \mathbb{R}_+), \\ |G(\cdot, s)| \|f(s, y_n(s)) - f(s, y(s))\| &\leq 2\rho |G(\cdot, s)| p_f(s) := \sigma_3(s); \quad \sigma_3 \in L^1(J, \mathbb{R}_+). \end{aligned} \quad (3.14)$$

Since  $f, g$ , and  $h$  are Carathéodory functions, the Lebesgue dominated convergence theorem implies that

$$\|N(y_n) - N(y)\|_\infty \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (3.15)$$

*Step 2.*  $N$  maps  $D_R$  into itself.

For each  $y \in D_R$ , by (H2) and (3.8) we have for each  $t \in J$

$$\begin{aligned} \|N(y)(t)\| &\leq \frac{T+1}{T+2} \int_0^T \|g(s, y(s))\| ds + \frac{T+1}{T+2} \int_0^T \|h(s, y(s))\| ds \\ &\quad + \int_0^T |G(t, s)| \|f(s, y(s))\| ds \\ &\leq R \left[ \frac{T(T+1)}{T+2} \|p_g\|_{L^\infty} + \frac{T(T+1)}{T+2} \|p_h\|_{L^\infty} + \tilde{G} \|p_f\|_{L^\infty} \right] \\ &< R. \end{aligned} \quad (3.16)$$

*Step 3.*  $N(D_R)$  is bounded and equicontinuous.

By Step 2, it is obvious that  $N(D_R) \subset C(J, E)$  is bounded.

For the equicontinuity of  $N(D_R)$ . Let  $t_1, t_2 \in J, t_1 < t_2$  and  $y \in D_R$ . Then

$$\begin{aligned} \|(Ny)(t_2) - (Ny)(t_1)\| &= \left\| -\frac{t_2 - t_1}{T+2} \int_0^T g(s, y(s)) ds + \frac{t_2 - t_1}{T+2} \int_0^T h(s, y(s)) ds \right. \\ &\quad \left. + \int_0^T [G(t_2, s) - G(t_1, s)] f(s, y(s)) ds \right\| \\ &\leq \frac{t_2 - t_1}{T+2} TR \left[ \|p_g\|_{L^\infty} + \|p_h\|_{L^\infty} \right] \\ &\quad + R \|p_f\|_{L^\infty} \int_0^T |G(t_2, s) - G(t_1, s)| ds. \end{aligned} \quad (3.17)$$

As  $t_1 \rightarrow t_2$ , the right-hand side of the above inequality tends to zero.

Now let  $V$  be a subset of  $D_R$  such that  $V \subset \overline{\text{co}}(N(V) \cup \{0\})$ .

$V$  is bounded and equicontinuous, and therefore the function  $v \rightarrow v(t) = \alpha(V(t))$  is continuous on  $J$ . By (H3), Lemma 2.6, and the properties of the measure  $\alpha$  we have for each  $t \in J$

$$\begin{aligned} v(t) &\leq \alpha(N(V)(t) \cup \{0\}) \\ &\leq \alpha(N(V)(t)) \\ &\leq \int_0^T \left| \frac{T+1-t}{T+2} \right| p_g(s) \alpha(V(s)) ds + \int_0^T \left| \frac{t+1}{T+2} \right| p_h(s) \alpha(V(s)) ds \\ &\quad + \int_0^T |G(t, s)| p_f(s) \alpha(V(s)) ds \\ &\leq \frac{T(T+1)}{T+2} \|p_g\|_{L^\infty} v(s) + \frac{T(T+1)}{T+2} \|p_h\|_{L^\infty} v(s) + \tilde{G} \|p_f\|_{L^\infty} v(s) \\ &\leq \|v\|_\infty \left[ \frac{T(T+1)}{T+2} \left[ \|p_g\|_{L^\infty} + \|p_h\|_{L^\infty} \right] + \tilde{G} \|p_f\|_{L^\infty} \right]. \end{aligned} \quad (3.18)$$

This means that

$$\|v\|_\infty \left( 1 - \left[ \frac{T(T+1)}{T+2} \left[ \|p_g\|_{L^\infty} + \|p_h\|_{L^\infty} \right] + \tilde{G} \|p_f\|_{L^\infty} \right] \right) \leq 0. \quad (3.19)$$

By (3.8) it follows that  $\|v\|_\infty = 0$ , that is,  $v(t) = 0$  for each  $t \in J$ , and then  $V(t)$  is relatively compact in  $E$ . In view of the Ascoli-Arzelà theorem,  $V$  is relatively compact in  $D_R$ . Applying now Theorem 2.5 we conclude that  $N$  has a fixed point which is a solution of the problem (1.1).

□



#### 4. An Example

In this section we give an example to illustrate the usefulness of our main results. Let us consider the following fractional boundary value problem:

$$\begin{aligned} {}^c D^r y(t) &= \frac{2}{19+e^t} |y(t)|, \quad t \in J := [0, 1], \quad 1 < r \leq 2, \\ y(0) - y'(0) &= \int_0^1 \frac{1}{5+e^{5s}} |y(s)| ds, \\ y(1) + y'(1) &= \int_0^1 \frac{1}{3+e^{3s}} |y(s)| ds. \end{aligned} \quad (4.1)$$

Set

$$\begin{aligned} f(t, x) &= \frac{2}{19+e^t} x, \quad (t, x) \in J \times [0, \infty), \\ g(t, x) &= \frac{1}{5+e^{5t}} x, \quad (t, x) \in [0, 1] \times [0, \infty), \\ h(t, x) &= \frac{1}{3+e^{3t}} x, \quad (t, x) \in [0, 1] \times [0, \infty). \end{aligned} \quad (4.2)$$

Clearly, conditions (H1),(H2) hold with

$$p_f(t) = \frac{2}{19+e^t}, \quad p_g(t) = \frac{1}{5+e^{5t}}, \quad p_h(t) = \frac{1}{3+e^{3t}}. \quad (4.3)$$

From (3.4) the function  $G$  is given by

$$G(t, s) = \begin{cases} \frac{(t-s)^{r-1}}{\Gamma(r)} - \frac{(1+t)(1-s)^{r-1}}{3\Gamma(r)} - \frac{(1+t)(1-s)^{r-2}}{3\Gamma(r-1)}, & 0 \leq s \leq t, \\ -\frac{(1+t)(1-s)^{r-1}}{3\Gamma(r)} - \frac{(1+t)(1-s)^{r-2}}{3\Gamma(r-1)}, & t \leq s \leq 1. \end{cases} \quad (4.4)$$

From (4.4), we have

$$\begin{aligned} \int_0^1 G(t, s) ds &= \int_0^t G(t, s) ds + \int_t^1 G(t, s) ds \\ &= \frac{t^r}{\Gamma(r+1)} + \frac{(1+t)(1-t)^r}{3\Gamma(r+1)} - \frac{(1+t)}{3\Gamma(r+1)} \\ &\quad + \frac{(1+t)(1-t)^{r-1}}{3\Gamma(r)} - \frac{(1+t)}{3\Gamma(r)} \\ &\quad - \frac{(1+t)(1-t)^r}{3\Gamma(r+1)} - \frac{(1+t)(1-t)^{r-1}}{3\Gamma(r)}. \end{aligned} \quad (4.5)$$

A simple computation gives

$$G^* < \frac{3}{\Gamma(r+1)} + \frac{2}{\Gamma(r)}. \quad (4.6)$$

Condition (3.8) is satisfied with  $T = 1$ . Indeed

$$\begin{aligned} \frac{T(T+1)}{T+2} [\|p_g\|_{L^\infty} + \|p_h\|_{L^\infty}] + \tilde{G}\|p_f\|_{L^\infty} &< \frac{2}{3} \left[ \frac{1}{6} + \frac{1}{4} \right] + \frac{3}{10\Gamma(r+1)} + \frac{2}{10\Gamma(r)} \\ &= \frac{5}{18} + \frac{3}{10\Gamma(r+1)} + \frac{1}{5\Gamma(r)} < 1, \end{aligned} \quad (4.7)$$

which is satisfied for each  $r \in (1, 2]$ . Then by Theorem 3.4 the problem (4.1) has a solution on  $[0, 1]$ .

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