Hindawi Publishing Corporation
Boundary Value Problems
Volume 2009, Article ID 685040, 6 pages
doi:10.1155/2009/685040

Research Article

Existence of Positive Solution to Second-Order Three-Point BVPs on Time Scales

Jian-Ping Sun

Department of Applied Mathematics, Lanzhou University of Technology, Lanzhou, Gansu 730050, China

Correspondence should be addressed to Jian-Ping Sun, jpsun@lut.cn

Received 19 April 2009; Accepted 14 September 2009

Recommended by Kanishka Perera

We are concerned with the following nonlinear second-order three-point boundary value problem on time scales $-x^{\Delta\Delta}(t) = f(t, x(t))$, $t \in [a,b]_{\mathbb{T}}$, $x(a) = 0$, $x(\sigma^2(b)) = \delta x(\eta)$, where $a, b \in \mathbb{T}$ with $a < b$, $\eta \in (a,b)_{\mathbb{T}}$ and $0 < \delta < (\sigma^2(b) - a)/(\eta - a)$. A new representation of Green's function for the corresponding linear boundary value problem is obtained and some existence criteria of at least one positive solution for the above nonlinear boundary value problem are established by using the iterative method.

1. Introduction

Let $\mathbb{T}$ be a time scale, that is, $\mathbb{T}$ is an arbitrary nonempty closed subset of $\mathbb{R}$. For each interval $\mathbf{I}$ of $\mathbb{R}$, we define $\mathbf{I}_{\mathbb{T}} = \mathbf{I} \cap \mathbb{T}$. For more details on time scales, one can refer to [1–5]. Recently, three-point boundary value problems (BVPs for short) for second-order dynamic equations on time scales have received much attention. For example, in 2002, Anderson [6] studied the following second-order three-point BVP on time scales:

$$\begin{gathered} u^{\Delta\nabla}(t) + a(t) f(u(t)) = 0, \quad t \in (0,T)_{\mathbb{T}}, \\ u(0) = 0, \qquad u(T) = \alpha u(\eta), \end{gathered} \tag{1.1}$$

where $0, T \in \mathbb{T}$, $\eta \in (0, \rho(T))_{\mathbb{T}}$ and $0 < \alpha < T/\eta$. Some existence results of at least one positive solution and of at least three positive solutions were established by using the well-known Krasnoselskii and Leggett-Williams fixed point theorems. In 2003, Kaufmann [7] applied the Krasnoselskii fixed point theorem to obtain the existence of multiple positive solutions to the BVP (1.1). For some other related results, one can refer to [8–10] and references therein.

In this paper, we are concerned with the existence of at least one positive solution for the following second-order three-point BVP on time scales:

$$\begin{aligned} &-x^{\Delta\Delta}(t)=f(t,x(t)), \quad t\in[a,b]_{\mathbb{T}}, \\ &x(a)=0, \qquad x\left(\sigma^2(b)\right)=\delta x(\eta). \end{aligned} \tag{1.2}$$

Throughout this paper, we always assume that $a,b\in\mathbb{T}$ with $a<b$, $\eta\in(a,b)_{\mathbb{T}}$, and $0<\delta<(\sigma^2(b)-a)/(\eta-a)$.

It is interesting that the method used in this paper is completely different from that in [6, 7, 9, 10], that is, a new representation of Green's function for the corresponding linear BVP is obtained and some existence criteria of at least one positive solution to the BVP (1.2) are established by using the iterative method.

For the function f, we impose the following hypotheses:

(H1) $f:[a,b]_{\mathbb{T}}\times\mathbb{R}^+\to\mathbb{R}^+$ is continuous;

(H2) for fixed $t\in[a,b]_{\mathbb{T}}$, $f(t,u)$ is monotone increasing on u;

(H3) there exists $q\in(0,1)$ such that

$$f(t,ru)\geq r^q f(t,u) \quad \text{for } r\in(0,1),\ (t,u)\in[a,b]_{\mathbb{T}}\times\mathbb{R}^+. \tag{1.3}$$

Remark 1.1. If (H3) is satisfied, then

$$f(t,\lambda u)\leq \lambda^q f(t,u) \quad \text{for } \lambda\in(1,+\infty),\ (t,u)\in[a,b]_{\mathbb{T}}\times\mathbb{R}^+. \tag{1.4}$$

2. Main Results

Lemma 2.1. *The BVP* (1.2) *is equivalent to the integral equation*

$$x(t)=\int_a^{\sigma(b)}K(t,s)f(s,x(s))\Delta s, \quad t\in\left[a,\sigma^2(b)\right]_{\mathbb{T}}, \tag{2.1}$$

where

$$K(t,s)=G(t,s)+\frac{\delta G(\eta,s)}{\sigma^2(b)-a-\delta(\eta-a)}(t-a) \tag{2.2}$$

is called the Green's function for the corresponding linear BVP, here

$$G(t,s)=\frac{1}{\sigma^2(b)-a}\begin{cases}(t-a)\left(\sigma^2(b)-\sigma(s)\right), & t\leq s,\\ (\sigma(s)-a)\left(\sigma^2(b)-t\right), & t\geq\sigma(s)\end{cases} \tag{2.3}$$

is the Green's function for the BVP:

$$
\begin{gathered}
-x^{\Delta\Delta}(t) = 0, \quad t \in [a,b]_{\mathbb{T}}, \\
x(a) = x\left(\sigma^2(b)\right) = 0.
\end{gathered}
\tag{2.4}
$$

Proof. Let x^* be a solution of the BVP:

$$
\begin{gathered}
-x^{\Delta\Delta}(t) = f(t,x(t)), \quad t \in [a,b]_{\mathbb{T}}, \\
x(a) = x\left(\sigma^2(b)\right) = 0.
\end{gathered}
\tag{2.5}
$$

Then, it is easy to know that

$$
\begin{gathered}
x^*(t) = \int_a^{\sigma(b)} G(t,s) f(s,x(s))\Delta s, \quad t \in \left[a,\sigma^2(b)\right]_{\mathbb{T}}, \\
x^*(a) = x^*\left(\sigma^2(b)\right) = 0.
\end{gathered}
\tag{2.6}
$$

Now, if x is a solution of the BVP (1.2), then it can be expressed by

$$
x(t) = C_1 + C_2 t + x^*(t),
\tag{2.7}
$$

which together with the boundary conditions in (1.2) and (2.6) implies that

$$
\begin{aligned}
x(t) &= \frac{\delta x^*(\eta)}{\sigma^2(b) - a - \delta(\eta - a)}(t-a) + x^*(t) \\
&= \int_a^{\sigma(b)} K(t,s) f(s,x(s))\Delta s, \quad t \in \left[a,\sigma^2(b)\right]_{\mathbb{T}}.
\end{aligned}
\tag{2.8}
$$

On the other hand, if x satisfies (2.1), then it is easy to verify that x is a solution of the BVP (1.2). □

Lemma 2.2. *For any* $(t,s) \in [a,\sigma^2(b)]_{\mathbb{T}} \times [a,\sigma(b)]_{\mathbb{T}}$, *one has*

$$
\frac{\delta G(\eta,s)}{\sigma^2(b) - a - \delta(\eta - a)}(t-a) \le K(t,s) \le \left[1 + \frac{\delta G(\eta,s)}{\sigma^2(b) - a - \delta(\eta - a)}\right](t-a).
\tag{2.9}
$$

Proof. Since it is obvious from the expression of $G(t,s)$ that

$$
0 \le G(t,s) \le \frac{1}{\sigma^2(b) - a}(t-a)\left(\sigma^2(b) - t\right), \quad (t,s) \in \left[a,\sigma^2(b)\right]_{\mathbb{T}} \times [a,\sigma(b)]_{\mathbb{T}},
\tag{2.10}
$$

we know that (2.9) is fulfilled. □

Our main result is the following theorem.

Theorem 2.3. *Assume that (H1)–(H3) are satisfied. Then, the BVP* (1.2) *has at least one positive solution w. Furthermore, there exist $M \geq m > 0$ such that*

$$m(t-a) \leq w(t) \leq M(t-a), \quad t \in \left[a, \sigma^2(b)\right]_{\mathbb{T}}. \tag{2.11}$$

Proof. Let

$$\begin{aligned}
E &= \left\{x \mid x : \left[a, \sigma^2(b)\right]_{\mathbb{T}} \longrightarrow \mathbb{R} \text{ is continuous}\right\}, \\
D &= \left\{x \in E \mid \text{there exist } M_x \geq m_x > 0 \text{ such that } m_x(t-a) \leq x(t) \leq M_x(t-a) \text{ for } t \in \left[a, \sigma^2(b)\right]_{\mathbb{T}}\right\}, \\
P &= \left\{x \in E \mid x(t) \geq 0 \text{ for } t \in \left[a, \sigma^2(b)\right]_{\mathbb{T}}\right\}.
\end{aligned} \tag{2.12}$$

Define an operator $F : D \to P$:

$$(Fx)(t) = \int_a^{\sigma(b)} K(t,s) f(s, x(s)) \Delta s, \quad t \in \left[a, \sigma^2(b)\right]_{\mathbb{T}}. \tag{2.13}$$

Then it is obvious that fixed points of the operator F in D are positive solutions of the BVP (1.2).

First, in view of (H2), it is easy to know that $F : D \to P$ is increasing.

Next, we may assert that $F : D \to D$, which implies that for any $x \in D$, there exist positive constants l and L such that

$$(Fx)(t) \leq Lx(t), \qquad (Fx)(t) \geq lx(t) \quad \text{for } x \in \left[a, \sigma^2(b)\right]_{\mathbb{T}}. \tag{2.14}$$

In fact, for any $x \in D$, there exist $0 < m_x < 1 < M_x$ such that

$$m_x(t-a) \leq x(t) \leq M_x(t-a) \quad \text{for } t \in \left[a, \sigma^2(b)\right]_{\mathbb{T}}, \tag{2.15}$$

which together with (H2), (H3), and Remark 1.1 implies that

$$(m_x)^q f(t, t-a) \leq f(t, x(t)) \leq (M_x)^q f(t, t-a) \quad \text{for } t \in [a, b]_{\mathbb{T}}. \tag{2.16}$$

By Lemma 2.2 and (2.16), for any $t \in [a, \sigma^2(b)]_{\mathbb{T}}$, we have

$$\begin{aligned}
(Fx)(t) &\leq (M_x)^q \int_a^{\sigma(b)} \left[1 + \frac{\delta G(\eta, s)}{\sigma^2(b) - a - \delta(\eta - a)}\right] f(s, s-a) \Delta s (t-a), \\
(Fx)(t) &\geq (m_x)^q \int_a^{\sigma(b)} \frac{\delta G(\eta, s)}{\sigma^2(b) - a - \delta(\eta - a)} f(s, s-a) \Delta s (t-a).
\end{aligned} \tag{2.17}$$

If we let

$$
\begin{aligned}
M_0 &= (M_x)^q \int_a^{\sigma(b)} \left[1 + \frac{\delta G(\eta, s)}{\sigma^2(b) - a - \delta(\eta - a)}\right] f(s, s-a)\Delta s, \\
m_0 &= (m_x)^q \int_a^{\sigma(b)} \frac{\delta G(\eta, s)}{\sigma^2(b) - a - \delta(\eta - a)} f(s, s-a)\Delta s,
\end{aligned}
\tag{2.18}
$$

then it follows from (2.17) and (2.18) that

$$
m_0(t-a) \le (Fx)(t) \le M_0(t-a) \quad \text{for } t \in \left[a, \sigma^2(b)\right]_{\mathbb{T}}, \tag{2.19}
$$

which shows that $Fx \in D$.

Now, for any fixed $h_0 \in D$, we denote

$$
l_{h_0} = \sup\left\{l > 0 \mid (Fh_0)(t) \ge l h_0(t),\ t \in \left[a, \sigma^2(b)\right]_{\mathbb{T}}\right\} \tag{2.20}
$$

$$
L_{h_0} = \inf\left\{L > 0 \mid (Fh_0)(t) \le L h_0(t),\ t \in \left[a, \sigma^2(b)\right]_{\mathbb{T}}\right\} \tag{2.21}
$$

$$
m = \min\left\{\frac{1}{2}, (l_{h_0})^{1/(1-q)}\right\}, \qquad M = \max\left\{2, (L_{h_0})^{1/(1-q)}\right\} \tag{2.22}
$$

and let

$$
u_n(t) = (Fu_{n-1})(t), \qquad v_n(t) = (Fv_{n-1})(t), \quad t \in \left[a, \sigma^2(b)\right]_{\mathbb{T}},\ n = 1, 2, \ldots, \tag{2.23}
$$

where

$$
u_0(t) = m h_0(t), \qquad v_0(t) = M h_0(t), \quad t \in \left[a, \sigma^2(b)\right]_{\mathbb{T}}. \tag{2.24}
$$

Then, it is easy to know from (2.20), (2.21), (2.22), (2.23), (2.24), (H3), and Remark 1.1 that

$$
u_0(t) \le u_1(t) \le \cdots \le u_n(t) \le \cdots \le v_n(t) \le \cdots \le v_1(t) \le v_0(t), \quad t \in \left[a, \sigma^2(b)\right]_{\mathbb{T}}. \tag{2.25}
$$

Moreover, if we let $t_0 = m/M$, then it follows from (2.22), (2.23), (2.24), and (H3) by induction that

$$
u_n(t) \ge (t_0)^{q^n} v_n(t), \quad t \in \left[a, \sigma^2(b)\right]_{\mathbb{T}},\ n = 0, 1, 2, \ldots, \tag{2.26}
$$

which together with (2.25) implies that for any positive integers n and p,

$$
0 \le u_{n+p}(t) - u_n(t) \le \left[1 - (t_0)^{q^n}\right] M h_0(t), \quad t \in \left[a, \sigma^2(b)\right]_{\mathbb{T}}. \tag{2.27}
$$

Therefore, there exists a $w \in D$ such that $\{u_n(t)\}_{n=0}^{\infty}$ and $\{v_n(t)\}_{n=0}^{\infty}$ converge uniformly to w on $[a, \sigma^2(b)]_{\mathbb{T}}$ and

$$u_n(t) \leq w(t) \leq v_n(t), \quad t \in \left[a, \sigma^2(b)\right]_{\mathbb{T}}, \ n = 0, 1, 2, \ldots. \tag{2.28}$$

Since F is increasing, in view of (2.28), we have

$$u_{n+1}(t) = (Fu_n)(t) \leq (Fw)(t) \leq (Fv_n)(t) = v_{n+1}(t), \quad t \in \left[a, \sigma^2(b)\right]_{\mathbb{T}}, \ n = 0, 1, 2, \ldots. \tag{2.29}$$

So,

$$(Fw)(t) = w(t), \quad t \in \left[a, \sigma^2(b)\right]_{\mathbb{T}}, \tag{2.30}$$

which shows that w is a positive solution of the BVP (1.2). Furthermore, since $w \in D$, there exist $M \geq m > 0$ such that

$$m(t-a) \leq w(t) \leq M(t-a), \quad t \in \left[a, \sigma^2(b)\right]_{\mathbb{T}}. \tag{2.31}$$

□

Acknowledgment

This work is supported by the National Natural Science Foundation of China (10801068).

References

[1] R. P. Agarwal and M. Bohner, "Basic calculus on time scales and some of its applications," *Results in Mathematics*, vol. 35, no. 1-2, pp. 3–22, 1999.

[2] M. Bohner and A. Peterson, *Dynamic Equations on Time Scales: An Introduction with Application*, Birkhäuser, Boston, Mass, USA, 2001.

[3] M. Bohner and A. Peterson, *Advances in Dynamic Equations on Time Scales*, Birkhäuser, Boston, Mass, USA, 2003.

[4] S. Hilger, "Analysis on measure chains—a unified approach to continuous and discrete calculus," *Results in Mathematics*, vol. 18, no. 1-2, pp. 18–56, 1990.

[5] V. Lakshmikantham, S. Sivasundaram, and B. Kaymakcalan, *Dynamic Systems on Measure Chains*, vol. 370 of *Mathematics and Its Applications*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1996.

[6] D. R. Anderson, "Solutions to second-order three-point problems on time scales," *Journal of Difference Equations and Applications*, vol. 8, no. 8, pp. 673–688, 2002.

[7] E. R. Kaufmann, "Positive solutions of a three-point boundary-value problem on a time scale," *Electronic Journal of Differential Equations*, vol. 2003, no. 82, 11 pages, 2003.

[8] R. A. Khan, J. J. Nieto, and V. Otero-Espinar, "Existence and approximation of solution of three-point boundary value problems on time scales," *Journal of Difference Equations and Applications*, vol. 14, no. 7, pp. 723–736, 2008.

[9] H. Luo and Q. Ma, "Positive solutions to a generalized second-order three-point boundary-value problem on time scales," *Electronic Journal of Differential Equations*, vol. 2005, no. 17, 14 pages, 2005.

[10] H.-R. Sun and W.-T. Li, "Positive solutions for nonlinear three-point boundary value problems on time scales," *Journal of Mathematical Analysis and Applications*, vol. 299, no. 2, pp. 508–524, 2004.