

Research Article

Existence and Location Results for Fully Nonlinear Boundary Value Problem of n th-Order Nonlinear System

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By appropriate bounding function pair and modified functions, using the theory of differential inequalities, this paper presents the existence and location criteria of solutions for the system of general n th-order differential equations with nonlinear boundary conditions. We give an example showing that the results are sharp. Our results extend many existing results.

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1. Introduction

In this paper, we are concerned with the following boundary value problem (BVP) for an n th-order nonlinear system:

$$\begin{aligned}y^{(n)} &= f(t, y, y', \dots, y^{(n-1)}), \\P_i(y(a), y'(a), \dots, y^{(n-1)}(a)) &= 0, \quad i = 1, \dots, n-1, \\P_n(y(b), y'(b), \dots, y^{(n-1)}(b)) &= 0,\end{aligned}\tag{1.1}$$

where $t \in I = [a, b]$, $y = (y_1, y_2, \dots, y_m) \in \mathbb{R}^m$, $f(t, \xi_0, \xi_1, \dots, \xi_{n-1}) \in C(I \times \mathbb{R}^{mn}, \mathbb{R}^m)$, $P_i(\eta_0, \eta_1, \dots, \eta_{n-1}) \in C(\mathbb{R}^{mn}, \mathbb{R}^m)$, $P_n(\zeta_0, \zeta_1, \dots, \zeta_{n-1}) \in C(\mathbb{R}^{mn}, \mathbb{R}^m)$.

Boundary value problems for ordinary differential equations play a very important role in both theory and applications. They are used to describe a large number of physical, biological, and chemical phenomena. There have been many accomplishments on the study

of the existence of solutions for BVPs of nonlinear differential equations using the theory of differential inequalities (cf. [1–31]). Although the method has been particularly fruitful for low-order ordinary differential equations (cf. [3–8, 14, 18, 31, 32]), Kelley [33] and Klaasen [3] did obtain early applications to higher-order ODEs. For more information, we refer the readers to [2, 3, 9, 10, 12–16, 19–21, 26, 28, 33–35] and the references therein. For the case of differential systems, the results are few (cf. [2, 4–6, 8, 11, 29–32]). On the other hand, there are many papers, see [17, 18, 22, 23, 25, 36–39] and the references therein, concerning the existence of solutions for BVPs by using other approaches (e.g., the shooting method, many kinds of fixed-point theorems, many kinds of degree theories, etc.). However, there are very few results on the study of the existence of solutions for the general nonlinear system with the general nonlinear boundary conditions. To fill the gap, we will investigate BVP (1.1).

The aim of this paper is to generalize or complement the existing results. In order to do so, as a sequel of [28, 29], following the thoughts and methods of Fabry and Habets [1], the authors considered the nonlinear BVP (1.1) for differential equation and even the more general BVP (4.1) with the full nonlinear boundary conditions and obtained some results [40–43]. In this paper we consider the nonlinear BVP (1.1) for differential systems and even the more general BVP (4.1) with the full nonlinear boundary conditions. To the best of our knowledge, the general cases of BVP (1.1) and BVP (4.1) have not been studied in the available reference materials. By appropriate bounding function pair and modified functions, using the theory of differential inequalities, we establish some sufficient conditions which guarantee the existence of at least one solution for these BVPs. We give an example showing that our results are sharp.

A novel feature of our work is that we present a new definition of bounding function pair for BVP. It is well known that bounding function pair (i.e., upper-lower solutions in many references) is very important to study the existence of solutions for BVPs. Because of the complexity of the vector case (cf. [2, chapters 2 and 7]), how to give an *appropriate* definition of bounding function pair for the full nonlinear BVP of differential system is the difficulty in our work.

The method of this paper, which may be called *simultaneous modification*, is distinctive. It is not only modifying the nonlinear function in the original equations, but also transforming the original nonlinear boundary conditions into some new boundary conditions which are easy to discuss. Thus, we get the new BVP which will be discussed in the first place, then the judgement of the existence of solutions for the original BVP will be attained naturally. This technique dealing with the nonlinear problem is simpler and clearer compared with the method of shooting.

Throughout the paper, the comparison between the two vectors will be viewed as the same comparisons according to their components, and the operations between the two vectors will be viewed as the same operations according to their components.

The rest of this paper is organized as follows. In Section 2, we first give two basic definitions, that is, bounding function pair and the Nagumo condition, and then we study the modified boundary value problem of BVP (1.1), that is, BVP (2.17). Following the preparative theorem in Section 2, in Section 3, we state and prove the main result, that is, the sufficient criterion of the existence of solutions for BVP (1.1). In Section 4, a more general boundary value problem (4.1) is investigated. Moreover in Section 5, an example is illustrated to show that our results are sharp. Finally, in Section 6, some remarks are given.

2. Preparative Theorem

2.1. Basic Concepts

We first define a function

$$\delta(r, x, s) \equiv \begin{cases} r & \text{if } x < r, \\ x & \text{if } r \leq x \leq s, \\ s & \text{if } s < x, \end{cases} \quad (2.1)$$

where $r, x, s \in \mathbb{R}$, $r \leq s$. Moreover, if

$$r = (r_1, \dots, r_m), \quad x = (x_1, \dots, x_m), \quad s = (s_1, \dots, s_m), \quad (2.2)$$

then, we define

$$\delta(r, x, s) \equiv (\delta(r_1, x_1, s_1), \dots, \delta(r_m, x_m, s_m)). \quad (2.3)$$

Definition 2.1. Assume that $\alpha, \beta \in C^n(I, \mathbb{R}^m)$. The pair of vector-valued functions $(\alpha(t), \beta(t))$ is called a bounding function pair (or simply, a bounding pair) of BVP(1.1) in case that for some positive constant N depending on (f, α, β) , and for all $u \in C^n(I, \mathbb{R}^m)$,

$$(i) \quad \alpha^{(i)}(t) \leq \beta^{(i)}(t), \quad t \in I, \quad i = 0, 1, \dots, n-2,$$

(ii)

$$\begin{aligned} \alpha_j^{(n)}(t) &\geq f_j(t, \bar{u}(t), \bar{u}'(t), \dots, \bar{u}^{(n-3)}(t), \bar{u}_{\alpha_j}^{(n-2)}(t), \bar{u}_{\alpha_j}^{(n-1)}(t)), \\ \beta_j^{(n)}(t) &\leq f_j(t, \bar{u}(t), \bar{u}'(t), \dots, \bar{u}^{(n-3)}(t), \bar{u}_{\beta_j}^{(n-2)}(t), \bar{u}_{\beta_j}^{(n-1)}(t)), \quad j = 1, \dots, m, \end{aligned} \quad (2.4)$$

where

$$\begin{aligned} \bar{u}^{(i)}(t) &= \delta(\alpha^{(i)}(t), u^{(i)}(t), \beta^{(i)}(t)), \quad i = 0, 1, \dots, n-3, \\ \bar{u}_{\alpha_j}^{(k)}(t) &= (\bar{u}_1^{(k)}(t), \dots, \bar{u}_{j-1}^{(k)}(t), \alpha_j^{(k)}(t), \bar{u}_{j+1}^{(k)}(t), \dots, \bar{u}_m^{(k)}(t)), \\ \bar{u}_{\beta_j}^{(k)}(t) &= (\bar{u}_1^{(k)}(t), \dots, \bar{u}_{j-1}^{(k)}(t), \beta_j^{(k)}(t), \bar{u}_{j+1}^{(k)}(t), \dots, \bar{u}_m^{(k)}(t)), \quad k = n-2, n-1, \\ \bar{u}_l^{(n-2)}(t) &= \delta(\alpha_l^{(n-2)}(t), u_l^{(n-2)}(t), \beta_l^{(n-2)}(t)), \\ \bar{u}_l^{(n-1)} &= \delta(-N, u_l^{(n-1)}, N), \quad l = 1, 2, \dots, j-1, j+1, \dots, m; \end{aligned} \quad (2.5)$$

(iii)

$$\begin{aligned}
& P_{i,j}(\bar{u}(a), \dots, \bar{u}^{(i-2)}(a), \bar{u}_{\alpha_j}^{(i-1)}(a), \bar{u}_{\alpha_j}^{(i)}(a), \bar{u}^{(i+1)}(a), \dots, \bar{u}^{(n-1)}(a)) \\
& \leq 0 \leq P_{i,j}(\bar{u}(a), \dots, \bar{u}^{(i-2)}(a), \bar{u}_{\beta_j}^{(i-1)}(a), \bar{u}_{\beta_j}^{(i)}(a), \bar{u}^{(i+1)}(a), \dots, \bar{u}^{(n-1)}(a)), \\
& P_{n,j}(\bar{u}(b), \dots, \bar{u}^{(n-3)}(b), \bar{u}_{\alpha_j}^{(n-2)}(b), \bar{u}_{\alpha_j}^{(n-1)}(b)) \\
& \leq 0 \leq P_{n,j}(\bar{u}(b), \dots, \bar{u}^{(n-3)}(b), \bar{u}_{\beta_j}^{(n-2)}(b), \bar{u}_{\beta_j}^{(n-1)}(b)),
\end{aligned} \tag{2.6}$$

where $i = 1, 2, \dots, n-1$, $j = 1, 2, \dots, m$, and

$$\begin{aligned}
\bar{u}_{\alpha_j}^{(k)}(t) &= (\bar{u}_1^{(k)}(t), \dots, \bar{u}_{j-1}^{(k)}(t), \alpha_j^{(k)}(t), \bar{u}_{j+1}^{(k)}(t), \dots, \bar{u}_m^{(k)}(t)), \\
\bar{u}_{\beta_j}^{(k)}(t) &= (\bar{u}_1^{(k)}(t), \dots, \bar{u}_{j-1}^{(k)}(t), \beta_j^{(k)}(t), \bar{u}_{j+1}^{(k)}(t), \dots, \bar{u}_m^{(k)}(t)), \quad k = 0, 1, 2, \dots, n-1, \\
\bar{u}^{(n-2)}(t) &= \delta(\alpha^{(n-2)}(t), u^{(n-2)}(t), \beta^{(n-2)}(t)), \\
\bar{u}^{(n-1)}(a) &= \delta(-\bar{N}, u^{(n-1)}(a), \bar{N}), \quad \bar{N} = (N, \dots, N).
\end{aligned} \tag{2.7}$$

Definition 2.2. A continuous function $f(t, \xi_0, \dots, \xi_{n-1})$ is said to satisfy the Nagumo condition with respect to variable ξ_{n-1} on the set

$$\begin{aligned}
\mathfrak{D} &= \{(t, \xi_0, \dots, \xi_{n-1}) \mid t \in I; |\xi_{j,k}| \leq r_j, j = 0, 1, \dots, n-2, \\
& k = 1, \dots, m, r_j \text{ is a positive constant; } \xi_{n-1} \in \mathbb{R}^m\},
\end{aligned} \tag{2.8}$$

in case there exist functions $\Phi_i \in C([0, +\infty), (0, +\infty))$, $i = 1, \dots, m$, such that

$$\begin{aligned}
|f_i(t, \xi_0, \dots, \xi_{n-1})| &\leq \Phi_i(|\xi_{n-1,i}|), \\
\int^{+\infty} \frac{s \, ds}{\Phi_i(s)} &= +\infty.
\end{aligned} \tag{2.9}$$

2.2. The Modified Problem

Assume that there are two vector-valued functions $\alpha(t)$, $\beta(t)$ satisfying

$$\alpha^{(j)}(t) \leq \beta^{(j)}(t), \quad j = 0, 1, \dots, n-2. \tag{2.10}$$

We define function $\bar{f}(t, y, y', \dots, y^{(n-1)})$ which components are

$$\bar{f}_i(t, y, y', \dots, y^{(n-1)}) \equiv f_i(t, \bar{y}, \bar{y}', \dots, \bar{y}^{(n-1)}) + h(y_i^{(n-2)}), \tag{2.11}$$

where $\bar{y}^{(j)}(t) = \delta(\alpha^{(j)}(t), \mathbf{y}^{(j)}(t), \beta^{(j)}(t))$ ($j = 0, 1, \dots, n-2$), and $\bar{y}^{(n-1)}(t) = \delta(-\bar{N}, \mathbf{y}^{(n-1)}(t), \bar{N})$, $\bar{N} = (N, \dots, N)$. N is a positive constant such that

$$N > \max_{\substack{i \in I \\ i=1, \dots, m}} \left\{ \frac{2M}{b-a}, |\alpha_i^{(n-1)}(t)|, |\beta_i^{(n-1)}(t)| \right\}, \quad (2.12)$$

$$\int_{2M/(b-a)}^N \frac{s \, ds}{\Phi_i(s)} > 2M, \quad (2.13)$$

in which $M > \max_{t \in I, i=1, \dots, m} \{|\alpha_i^{(n-2)}(t)|, |\beta_i^{(n-2)}(t)|\}$. $h(\mathbf{y}_i^{(n-2)})$ is continuous, bounded, and

$$h(\mathbf{y}_i^{(n-2)}) \begin{cases} < 0 & \text{if } \mathbf{y}_i^{(n-2)} < \alpha_i^{(n-2)}, \\ = 0 & \text{if } \alpha_i^{(n-2)} \leq \mathbf{y}_i^{(n-2)} \leq \beta_i^{(n-2)}, \\ > 0 & \text{if } \mathbf{y}_i^{(n-2)} > \beta_i^{(n-2)}. \end{cases} \quad (2.14)$$

Such function $h(\cdot)$ is easy to obtain, for example, let

$$h(\mathbf{y}_i^{(n-2)}) \equiv \frac{\mathbf{y}_i^{(n-2)} - \bar{\mathbf{y}}_i^{(n-2)}}{1 + |\mathbf{y}_i^{(n-2)} - \bar{\mathbf{y}}_i^{(n-2)}|}. \quad (2.15)$$

In addition, we define

$$\begin{aligned} \bar{P}_i(\mathbf{y}(t), \mathbf{y}'(t), \dots, \mathbf{y}^{(n-1)}(t)) \\ \equiv \delta(\alpha^{(i-1)}(t), \mathbf{y}^{(i-1)}(t) - P_i(\mathbf{y}(t), \mathbf{y}'(t), \dots, \mathbf{y}^{(n-1)}(t)), \beta^{(i-1)}(t)), \quad i = 1, 2, \dots, n-1, \\ \bar{P}_n(\mathbf{y}(t), \mathbf{y}'(t), \dots, \mathbf{y}^{(n-1)}(t)) \\ \equiv \delta(\alpha^{(n-2)}(t), \mathbf{y}^{(n-2)}(t) - P_n(\mathbf{y}(t), \mathbf{y}'(t), \dots, \mathbf{y}^{(n-1)}(t)), \beta^{(n-2)}(t)). \end{aligned} \quad (2.16)$$

Then, we consider the following modified problem:

$$\begin{aligned} \mathbf{y}^{(n)} &= \bar{f}(t, \mathbf{y}, \mathbf{y}', \dots, \mathbf{y}^{(n-1)}), \\ \mathbf{y}^{(i-1)}(a) &= \bar{P}_i(\mathbf{y}(a), \mathbf{y}'(a), \dots, \mathbf{y}^{(n-1)}(a)), \quad i = 1, \dots, n-1, \\ \mathbf{y}^{(n-2)}(b) &= \bar{P}_n(\mathbf{y}(b), \mathbf{y}'(b), \dots, \mathbf{y}^{(n-1)}(b)). \end{aligned} \quad (2.17)$$

2.3. Preparative Theorem

Lemma 2.3. Assume that

(A1) BVP (1.1) has a bounding pair $(\alpha(t), \beta(t))$ on the interval I by Definition 2.1;

(A2) the function $f(t, \xi_0, \dots, \xi_{n-1})$ in BVP (1.1) satisfies the Nagumo condition with respect to ξ_{n-1} by Definition 2.2.

Then, BVP (2.17) has a solution $y \in C^n(I, \mathbb{R}^m)$ such that

$$\begin{aligned} \alpha^{(i)}(t) &\leq y^{(i)}(t) \leq \beta^{(i)}(t), \quad i = 0, 1, \dots, n-2, \\ |y_j^{(n-1)}(t)| &\leq N, \quad j = 1, 2, \dots, m, t \in I, \end{aligned} \quad (2.18)$$

where N is defined in \bar{f}_i .

The following three propositions will lead to the proof of Lemma 2.3.

Proposition 2.4. The modified BVP (2.17) has a solution $y(t) \in C^n(I, \mathbb{R}^m)$.

Proof. Noticing that the functions \bar{f} and \bar{P}_i ($i = 1, 2, \dots, n$) are bounded, this proposition immediately follows from the Schauder fixed-point theorem. The details here are omitted. \square

Proposition 2.5. Every solution $y(t)$ of the modified BVP (2.17) satisfies

$$\alpha^{(i)}(t) \leq y^{(i)}(t) \leq \beta^{(i)}(t), \quad t \in I, i = 0, 1, \dots, n-2. \quad (2.19)$$

Proof. First, we show that

$$\alpha^{(n-2)}(t) \leq y^{(n-2)}(t) \leq \beta^{(n-2)}(t), \quad t \in I. \quad (2.20)$$

If $\alpha^{(n-2)}(t) \leq y^{(n-2)}(t)$ is not true, then there exist some i and $\xi \in [a, b]$ such that

$$\max_{i \in I} (\alpha_i^{(n-2)}(t) - y_i^{(n-2)}(t)) = \alpha_i^{(n-2)}(\xi) - y_i^{(n-2)}(\xi) > 0. \quad (2.21)$$

Then, $\xi \neq a, b$ by the boundary conditions of BVP (2.17). Thus,

$$\alpha_i^{(n-1)}(\xi) - y_i^{(n-1)}(\xi) = 0, \quad (2.22)$$

$$\alpha_i^{(n)}(\xi) - y_i^{(n)}(\xi) \leq 0. \quad (2.23)$$

However, on the other hand, from the definition of $\alpha(t)$ and that $y(t)$ is a solution of (2.17), we have

$$\begin{aligned} \alpha_i^{(n)}(\xi) - y_i^{(n)}(\xi) &\geq f_i(\xi, \bar{y}(\xi), \dots, \bar{y}^{(n-3)}(\xi), \bar{y}_{\alpha_i}^{(n-2)}(\xi), \bar{y}_{\alpha_i}^{(n-1)}(\xi)) \\ &\quad - f_i(\xi, \bar{y}(\xi), \dots, \bar{y}^{(n-3)}(\xi), \bar{y}^{(n-2)}(\xi), \bar{y}^{(n-1)}(\xi)) - h(y_i^{(n-2)}(\xi)) \\ &= -h(y_i^{(n-2)}(\xi)) > 0. \end{aligned} \quad (2.24)$$

This contradicts (2.23). Hence,

$$\alpha^{(n-2)}(t) \leq y^{(n-2)}(t), \quad t \in I. \quad (2.25)$$

A similar proof shows that

$$y^{(n-2)}(t) \leq \beta^{(n-2)}(t), \quad t \in I. \quad (2.26)$$

Summing up, (2.20) is true. From (2.20), the function $y^{(n-3)}(t) - \alpha^{(n-3)}(t)$ is increasing in I . Noticing

$$\alpha^{(n-3)}(a) \leq y^{(n-3)}(a), \quad (2.27)$$

we know that $\alpha^{(n-3)}(t) \leq y^{(n-3)}(t)$. A similar proof shows $y^{(n-3)}(t) \leq \beta^{(n-3)}(t)$. Using the same argument, it follows that $\alpha^{(i)}(t) \leq y^{(i)}(t) \leq \beta^{(i)}(t)$, $i = n-4, n-5, \dots, 2, 1$. Thus, the proof of Proposition 2.5 is completed. \square

Proposition 2.6. *Every solution $y(t)$ of the modified BVP (2.17) satisfies*

$$|y_j^{(n-1)}(t)| \leq N, \quad t \in I, \quad j = 1, 2, \dots, m. \quad (2.28)$$

Proof. Suppose that there exist some j and $\tau \in [a, b]$ such that

$$|y_j^{(n-1)}(\tau)| > N. \quad (2.29)$$

Without loss of generality, we assume that $y_j^{(n-1)}(\tau) > N$. There exists $\xi \in (a, b)$ such that

$$y_j^{(n-1)}(\xi) = \frac{y_j^{(n-2)}(b) - y_j^{(n-2)}(a)}{b-a} \leq \frac{2M}{b-a} < N. \quad (2.30)$$

Hence, there exists some subinterval $[c, d]$ (or $[d, c]$) $\subset [a, b]$ such that

$$\begin{aligned} y_j^{(n-1)}(c) &= \frac{2M}{b-a}, & y_j^{(n-1)}(d) &= N, \\ \frac{2M}{b-a} &\leq y_j^{(n-1)}(t) \leq N, & \forall t \in [c, d] \text{ (or } [d, c]). \end{aligned} \quad (2.31)$$

From condition (A2),

$$\begin{aligned} \left| \int_c^d \frac{y_j^{(n-1)}(s)y_j^{(n)}(s)}{\Phi_j(|y_j^{(n-1)}(s)|)} ds \right| &\leq \left| \int_c^d y_j^{(n-1)}(s) ds \right| \\ &= |y_j^{(n-2)}(d) - y_j^{(n-2)}(c)| \\ &\leq 2M. \end{aligned} \quad (2.32)$$

On the other hand, from (2.13) we know that

$$\begin{aligned} \left| \int_c^d \frac{y_j^{(n-1)}(s)y_j^{(n)}(s)}{\Phi_j(|y_j^{(n-1)}(s)|)} ds \right| &= \left| \int_{2M/(b-a)}^N \frac{r dr}{\Phi_j(r)} \right| \\ &= \int_{2M/(b-a)}^N \frac{r dr}{\Phi_j(r)} \\ &> 2M. \end{aligned} \quad (2.33)$$

This inequality contradicts the above one and Proposition 2.6 holds. \square

The proof of Lemma 2.3 is now a simple consequence of Propositions 2.4, 2.5, and 2.6.

3. Main Theorem

Now, the main result of this paper is given in the following theorem.

Theorem 3.1. *Let conditions (A1) and (A2) in Lemma 2.3 hold and assume that*

(A3) *the functions $P_{i,j}(\eta_0, \dots, \eta_{n-1})$ ($i = 1, 2, \dots, n-1$, $j = 1, 2, \dots, m$) are decreasing in η_i , while $P_{n,j}(\eta_0, \dots, \eta_{n-1})$ ($j = 1, 2, \dots, m$) are increasing in η_{n-1} .*

Then, BVP (1.1) has a solution $y \in C^n(I, \mathbb{R}^m)$ such that

$$\begin{aligned} \alpha^{(i)}(t) &\leq y^{(i)}(t) \leq \beta^{(i)}(t), \quad i = 0, 1, \dots, n-2, \\ |y_j^{(n-1)}(t)| &\leq N, \quad j = 1, 2, \dots, m, t \in I, \end{aligned} \quad (3.1)$$

where N is defined in \bar{f}_i .

Proof. From Lemma 2.3 and the definition of \bar{f} , the solution $y(t)$ of the modified BVP (2.17) satisfies (1.1). Obviously, if it is proved that $y(t)$ satisfies the boundary conditions of (1.1) under condition (A3), we may conclude that $y(t)$ is just the solution of BVP (1.1).

First, we prove that

$$P_i(y(a), \dots, y^{(n-1)}(a)) = 0, \quad i = 1, 2, \dots, n-1. \quad (3.2)$$

Case 1. Suppose that

$$\alpha^{(i-1)}(a) \leq y^{(i-1)}(a) - P_i(y(a), \dots, y^{(n-1)}(a)) \leq \beta^{(i-1)}(a). \quad (3.3)$$

Then

$$\begin{aligned} y^{(i-1)}(a) &= \overline{P}_i(y(a), \dots, y^{(n-1)}(a)) \\ &= y^{(i-1)}(a) - P_i(y(a), \dots, y^{(n-1)}(a)). \end{aligned} \quad (3.4)$$

Thus,

$$P_i(y(a), \dots, y^{(n-1)}(a)) = 0. \quad (3.5)$$

Case 2. Suppose that there exist some $i \in \{1, 2, \dots, n-1\}$ and some $j \in \{1, 2, \dots, m\}$ such that

$$\alpha_j^{(i-1)}(a) > y_j^{(i-1)}(a) - P_{i,j}(y(a), y'(a), \dots, y^{(n-1)}(a)). \quad (3.6)$$

Then,

$$y_j^{(i-1)}(a) = \overline{P}_{i,j}(y(a), y'(a), \dots, y^{(n-1)}(a)) = \alpha_j^{(i-1)}(a). \quad (3.7)$$

Hence,

$$P_{i,j}(y(a), y'(a), \dots, y^{(n-1)}(a)) > 0. \quad (3.8)$$

From Proposition 2.5,

$$\overline{y}^{(k)}(a) = \delta(\alpha^{(k)}(a), y^{(k)}(a), \beta^{(k)}(a)) = y^{(k)}(a), \quad k = 0, 1, \dots, n-2. \quad (3.9)$$

From Proposition 2.6,

$$\overline{y}^{(n-1)}(a) = \delta(-\overline{N}, y^{(n-1)}(a), \overline{N}) = y^{(n-1)}(a). \quad (3.10)$$

When $i = 1, 2, \dots, n-2$, Proposition 2.5 implies

$$\overline{y}_{\alpha_j}^{(i)}(a) \leq y^{(i)}(a), \quad j = 1, 2, \dots, m. \quad (3.11)$$

When $i = n-1$, formula (3.7) implies

$$\overline{y}_j^{(n-2)}(a) = \alpha_j^{(n-2)}(a), \quad j = 1, 2, \dots, m. \quad (3.12)$$

Recalling that $\bar{y}_j^{(n-2)}(t) \geq \alpha_j^{(n-2)}(t)$, $t \in [a, b]$, we get $\bar{y}_j^{(n-1)}(a) \geq \alpha_j^{(n-1)}(a)$. Thus,

$$\bar{y}_{\alpha_j}^{(n-1)}(a) \leq y^{(n-1)}(a), \quad j = 1, 2, \dots, m. \quad (3.13)$$

It follows from formulas (3.7)–(3.13) and condition (A3) that, for $i = 1, 2, \dots, n - 1$,

$$\begin{aligned} & P_{i,j}(\bar{y}(a), \dots, \bar{y}^{(i-2)}(a), \bar{y}_{\alpha_j}^{(i-1)}(a), \bar{y}_{\alpha_j}^{(i)}(a), \bar{y}^{(i+1)}(a), \dots, \bar{y}^{(n-1)}(a)) \\ &= P_{i,j}(\bar{y}(a), \dots, \bar{y}^{(i-2)}(a), y^{(i-1)}(a), \bar{y}_{\alpha_j}^{(i)}(a), \bar{y}^{(i+1)}(a), \dots, \bar{y}^{(n-1)}(a)) \\ &\geq P_{i,j}(\bar{y}(a), \dots, \bar{y}^{(i-2)}(a), y^{(i-1)}(a), y^{(i)}(a), \bar{y}^{(i+1)}(a), \dots, \bar{y}^{(n-1)}(a)) > 0. \end{aligned} \quad (3.14)$$

It is easy to see that the last inequality contradicts (iii) of Definition 2.1. Therefore, Case 2 is not true.

Case 3. Suppose that there exist some $i \in \{1, 2, \dots, n - 1\}$ and some $j \in \{1, 2, \dots, m\}$ such that

$$y_j^{(i-1)}(a) - P_{i,j}(y(a), y'(a), \dots, y^{(n-1)}(a)) > \beta_j^{(i-1)}(a). \quad (3.15)$$

Then,

$$y_j^{(i-1)}(a) = \bar{P}_{i,j}(y(a), y'(a), \dots, y^{(n-1)}(a)) = \beta_j^{(i-1)}(a). \quad (3.16)$$

Hence,

$$P_{i,j}(y(a), y'(a), \dots, y^{(n-1)}(a)) < 0. \quad (3.17)$$

Similar to the argument of Case 2, we have

$$\begin{aligned} & P_{i,j}(\bar{y}(a), \dots, \bar{y}^{(i-2)}(a), \bar{y}_{\beta_j}^{(i-1)}(a), \bar{y}_{\beta_j}^{(i)}(a), \bar{y}^{(i+1)}(a), \dots, \bar{y}^{(n-1)}(a)) \\ &= P_{i,j}(\bar{y}(a), \dots, \bar{y}^{(i-2)}(a), y^{(i-1)}(a), \bar{y}_{\beta_j}^{(i)}(a), \bar{y}^{(i+1)}(a), \dots, \bar{y}^{(n-1)}(a)) \\ &\leq P_{i,j}(\bar{y}(a), \dots, \bar{y}^{(i-2)}(a), y^{(i-1)}(a), y^{(i)}(a), \bar{y}^{(i+1)}(a), \dots, \bar{y}^{(n-1)}(a)) < 0. \end{aligned} \quad (3.18)$$

Obviously, the last inequality contradicts (iii) of Definition 2.1. Therefore, this case cannot hold. Summing up, (3.2) holds.

A similar proof shows that

$$P_n(y(b), y'(b), \dots, y^{(n-1)}(b)) = 0. \quad (3.19)$$

Consequently, the proof is completed. \square

Remark 3.2. From (iii) of Definition 2.1 and (A3) of Theorem 3.1, it is easy to see that the functions $P_{i,j}(\eta_0, \dots, \eta_{n-1})$ ($i = 1, 2, \dots, n-1, j = 1, 2, \dots, m$) should be increasing in η_{i-1} .

4. A Generalized Problem

Now, we consider the following boundary value problem with more generalized boundary conditions:

$$\begin{aligned} y^{(n)} &= f(t, y, \dots, y^{(n-1)}), \\ P_i(y(a), \dots, y^{(n-1)}(a), y(b), \dots, y^{(n-1)}(b)) &= 0, \end{aligned} \quad (4.1)$$

where $t \in I, i = 1, 2, \dots, n, y \in \mathbb{R}^m, f$ and P_i are continuous m -dimensional vector-valued functions.

Similar to Definition 2.1, we give the following.

Definition 4.1. Assume $\alpha, \beta \in C^n(I, \mathbb{R}^m)$, The pair of vector-valued functions $(\alpha(t), \beta(t))$ is called a bounding function pair of BVP (4.1) in case that

- (i) same as (i) of Definition 2.1;
- (ii) same as (ii) of Definition 2.1;
- (iii)'

$$\begin{aligned} P_{i,j}(\bar{u}(a), \dots, \bar{u}_{\alpha_j}^{(i-1)}(a), \bar{u}_{\alpha_j}^{(i)}(a), \dots, \bar{u}^{(n-1)}(a), \bar{u}(b), \dots, \bar{u}^{(n-1)}(b)) \\ \leq 0 \leq P_{i,j}(\bar{u}(a), \dots, \bar{u}_{\beta_j}^{(i-1)}(a), \bar{u}_{\beta_j}^{(i)}(a), \dots, \bar{u}^{(n-1)}(a), \bar{u}(b), \dots, \bar{u}^{(n-1)}(b)), \\ P_{n,j}(\bar{u}(a), \dots, \bar{u}^{(n-1)}(a), \bar{u}(b), \dots, \bar{u}^{(n-3)}(b), \bar{u}_{\alpha_j}^{(n-2)}(b), \bar{u}_{\alpha_j}^{(n-1)}(b)) \\ \leq 0 \leq P_{n,j}(\bar{u}(a), \dots, \bar{u}^{(n-1)}(a), \bar{u}(b), \dots, \bar{u}^{(n-3)}(b), \bar{u}_{\beta_j}^{(n-2)}(b), \bar{u}_{\beta_j}^{(n-1)}(b)), \end{aligned} \quad (4.2)$$

where $i = 1, 2, \dots, n-1; j = 1, 2, \dots, m$.

For BVP (4.1), we have the following existence theorem.

Theorem 4.2. Assume that

- (A1)' BVP (4.1) has a bounding function pair $(\alpha(t), \beta(t))$ in the interval I by Definition 4.1;
- (A2)' the function $f(t, \xi_0, \dots, \xi_{n-1})$ in BVP (4.1) satisfies the Nagumo condition with respect to ξ_{n-1} by Definition 2.2;
- (A3)' the functions $P_{i,j}(\eta_0, \dots, \eta_{n-1}, \zeta_0, \dots, \zeta_{n-1})$ ($i = 1, 2, \dots, n-1, j = 1, 2, \dots, m$) are decreasing in η_i , while $P_{n,j}(\eta_0, \dots, \eta_{n-1}, \zeta_0, \dots, \zeta_{n-1})$ ($j = 1, 2, \dots, m$) are increasing in η_{n-1} .

Then, BVP (4.1) has a solution $y \in C^n(I, \mathbb{R}^m)$ such that

$$\begin{aligned} \alpha^{(i)}(t) &\leq y^{(i)}(t) \leq \beta^{(i)}(t), \quad i = 0, 1, \dots, n-2, \\ |y_j^{(n-1)}(t)| &\leq N, \quad j = 1, 2, \dots, m, t \in I, \end{aligned} \quad (4.3)$$

where N is defined in \bar{f}_i .

Proof. Consider the modified problem

$$\begin{aligned} y^{(n)} &= \bar{f}(t, y, \dots, y^{(n-1)}), \\ y^{(i-1)}(a) &= \bar{P}_i(a), \quad i = 1, 2, \dots, n-1, \\ y^{(n-2)}(b) &= \bar{P}_n(b). \end{aligned} \quad (4.4)$$

The modified function $\bar{f}(t, y, \dots, y^{(n-1)})$ is defined as BVP (2.17), and

$$\begin{aligned} \bar{P}_i(t) &\equiv \bar{P}_i(y(t), \dots, y^{(n-1)}(t), y(b+a-t), \dots, y^{(n-1)}(b+a-t)) \\ &\equiv \delta(\alpha^{(i-1)}(t), y^{(i-1)}(t) - P_i(y(t), \dots, y^{(n-1)}(t), y(b+a-t), \dots, \\ &\quad y^{(n-1)}(b+a-t)), \beta^{(i-1)}(t)), \end{aligned} \quad (4.5)$$

where $i = 1, 2, \dots, n-1$,

$$\begin{aligned} \bar{P}_n(t) &\equiv \bar{P}_n(y(b+a-t), \dots, y^{(n-1)}(b+a-t), y(t), \dots, y^{(n-1)}(t)) \\ &\equiv \delta(\alpha^{(n-2)}(t), y^{(n-2)}(t) - P_n(y(b+a-t), \dots, y^{(n-1)}(b+a-t), y(t), \dots, \\ &\quad y^{(n-1)}(t)), \beta^{(n-2)}(t)). \end{aligned} \quad (4.6)$$

Using the same argument as the proof of Lemma 2.3, it follows from conditions (A1)' and (A2)' that BVP (4.4) has a solution $y(t)$ satisfying the two inequalities in the conclusions of Lemma 2.3. Furthermore, in an analogous way to the proof of Theorem 3.1, it follows that the solution $y(t)$ of BVP(4.4) is just a solution of BVP (4.1). Consequently, the proof of Theorem 4.2 is completed. The details of the proof will be omitted. \square

Remark 4.3. From (iii)' of Definition 4.1 and (A3)' of Theorem 4.2, it is easy to see that the functions $P_{i,j}(\eta_0, \dots, \eta_{n-1}, \zeta_0, \dots, \zeta_{n-1})$ ($i = 1, 2, \dots, n-1$, $j = 1, 2, \dots, m$) should be increasing in η_{i-1} .

5. An Example

In this section, we present an example by making use of Theorems 3.1 and 4.2. With the example, we try to illustrate the applicability of our results and techniques and show that a bounding pair according to Definitions 2.1 or 4.1 can exist naturally.

Example 5.1. Consider the following 4th-order nonlinear system:

$$\begin{aligned} y_1^{(4)} &= \frac{1}{27}y_1^3 + \frac{1}{2}(y_2')^2 + \left(\frac{y_1''}{3}\right)^3 + (y_1')^2 \sin(y_2''')^2 + \sin t + 4, \\ y_2^{(4)} &= y_2^2 + \frac{1}{2}(y_1')^2 + \left(\frac{y_2''}{5}\right)^3 + \frac{1}{144}(y_2''')^2 \sin(y_1''')^2 + t^2, \end{aligned} \quad (5.1)$$

together with the following boundary conditions:

$$\begin{aligned} P_1 : & \begin{cases} y_1(0) - e^{(y_1(0))^3} = 2.6, \\ y_2(0) + \frac{k}{36}(y_2'(1))^2 = 3.5, \end{cases} \\ P_2 : & \begin{cases} 2y_1'(0) - 3(y_1''(0))^5 = 0, \\ y_1(0) + \frac{1}{y_2'(0) - 1} + k(y_1''(1))^2 = 1.6, \end{cases} \\ P_3 : & \begin{cases} y_1''(0) + y_2'(0) = 0, \\ k(y_2(1))^2 - 2(y_2''(0))^4 + ky_2''(1) = -3, \end{cases} \\ P_4 : & \begin{cases} y_1''(1) = 0, \\ ky_1(0) + y_2'(1) + 6y_2''(1) + \frac{1}{12^3}(y_2'''(1))^3 = -2, \end{cases} \end{aligned} \quad (5.2)$$

where $t \in [0, 1]$, and k is a constant.

Let

$$\alpha(t) = \left(\begin{array}{c} t^4 - 2t^3 + 2.6 \\ t^4 - 2t^3 - t^2 + 3 \end{array} \right), \quad \beta(t) = \left(\begin{array}{c} 2.6 \\ 4 \end{array} \right), \quad t \in [0, 1]. \quad (5.3)$$

Then, for the case of $k = 0$ and the case of $k = 1$, by direct calculation, it is easy to check that $(\alpha(t), \beta(t))$ is a bounding pair of BVP (5.1) and all assumptions of Theorems 3.1 and 4.2 are

fulfilled, respectively. Hence, for any of the two cases, BVP (5.1) has at least one solution $y(t)$ satisfying

$$\begin{aligned} \begin{pmatrix} t^4 - 2t^3 + 2.6 \\ t^4 - 2t^3 - t^2 + 3 \end{pmatrix} &\leq y(t) \leq \begin{pmatrix} 2.6 \\ 4 \end{pmatrix}, \\ \begin{pmatrix} 4t^3 - 6t^2 \\ 4t^3 - 6t^2 - 2t \end{pmatrix} &\leq y'(t) \leq \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \\ \begin{pmatrix} 12t^2 - 12t \\ 12t^2 - 12t - t \end{pmatrix} &\leq y''(t) \leq \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \end{aligned} \quad (5.4)$$

for each $t \in [0, 1]$.

6. Remarks

(1) If the directions of the signs of inequalities in condition (iii) of Definition 2.1 are all changed to the opposite, and conditions (i), (ii) of Definition 2.1 hold, then we denote the revised definition by Definition 2.1. We obtain the following theorem similar to Theorem 3.1.

Theorem 6.1. *Assume that*

(A1)" BVP (1.1) has a bounding function pair $(\alpha(t), \beta(t))$ by Definition 6.1;

(A2) same as (A2) of Theorem 3.1;

(A3)" the monotony of $P_{i,j}$ is opposite to that of (A3).

Then, the conclusion of Theorem 3.1 still holds.

In fact, if we replace P_i ($i = 1, 2, \dots, n$) by $-P_i$ in Theorem 3.1, then, it follows from Theorem 3.1 that Theorem 6.1 is true. We may make the analogous argument for BVP (4.1).

(2) The essentiality of the modified function is to modify a general nonlinear continuous function to a continuous bounded function. It was appearing in different forms in references. In this paper, we give out one concise form.

(3) The definitions of scalar bounding functions are a good many. In this paper, the definitions in vector cases given are new and can be regarded as a kind of improvement and generalization. Of course, the conditions of the definitions may be changed by the actual need. For example, we take Definition 2.1 to discuss the following.

If $\bar{u}_{\alpha_i}^{(n-1)}(t)$, $\bar{u}_{\beta_i}^{(n-1)}(t)$ in f_i of condition (ii) are changed to $\bar{u}^{(n-1)}(t)$, respectively, we still may assure that those results hold.

If both $\bar{u}_{\alpha_i}^{(n-2)}$ and $\bar{u}_{\beta_i}^{(n-2)}(t)$ in f_i of condition (ii) are modified to $\bar{u}^{(n-2)}(t)$, we may simplify the depiction and the proof. But, the modified condition becomes stronger.

It should be pointed out that condition (ii) may be weakened as

$$\begin{aligned} \alpha_i^{(n)}(t) &\geq f_i(t, \alpha(t), \alpha'(t), \dots, \alpha^{(n-2)}(t), \bar{u}^{(n-1)}(t)), \\ \beta_i^{(n)}(t) &\leq f_i(t, \beta(t), \beta'(t), \dots, \beta^{(n-2)}(t), \bar{u}^{(n-1)}(t)). \end{aligned} \quad (*)$$

However, when proving Proposition 2.5, we should add one condition “ f_i are all decreasing in $\xi_0, \xi_1, \dots, \xi_{n-2}$ ”. Thus, (*) implies condition (ii) of Definition 2.1. Consequently, condition (ii) about α, β seems weaker, but in fact, the whole requirement becomes stronger in some sense.

We also may discuss condition (iii) of Definition 2.1 in a similar way.

(4) The Nagumo condition in this paper ensures that the integral inequality (2.13) is true and essentially ensures that the derivative functions of solutions of the considered problems are bounded. Indeed, in some references, the integral equality

$$\int^{+\infty} \frac{s ds}{\Phi_i(s)} = +\infty, \quad (6.1)$$

is straightly substituted by inequality (2.13). Moreover, we exhibit some new forms of the integral inequality (see [29]).

(5) From Theorems 3.1, 4.2, and the above remarks, we include or improve the results in [1–43], since our system and boundary conditions are fully nonlinear. Obviously, the results in all the references are not available to our example.

(6) Last but not least, it should be pointed out that although this paper presents the existence and location criteria of solutions for BVPs, the premise is that the bounding function pair is assumed to be existing. It is well known how to get a precise bounding function pair for a given BVP is a very difficult job in the theory of upper-lower solutions and remains unsolved.

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