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Research Article

Topological Optimization with the p-Laplacian Operator and an Application in Image Processing

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We focus in this paper on the theoretical and numerical aspect os image processing. We consider a non linear boundary value problem (the p-Laplacian) from which we will derive the asymptotic expansion of the Mumford-Shah functional. We give a theoretical expression of the topological gradient as well as a numerical confirmation of the result in the restoration and segmentation of images.

1. Introduction

The goal of the topological optimization problem is to find an optimal design with an a priori poor information on the optimal shape of the structure. The shape optimization problem consists in minimizing a functional $j(\Omega) = J(\Omega, u_\Omega)$ where the function u_Ω is defined, for example, on a variable open and bounded subset Ω of $\mathbb{R}^N$. For $\varepsilon > 0$, let $\Omega_\varepsilon = \Omega \setminus \overline{(x_0 + \varepsilon\omega)}$ be the set obtained by removing a small part $x_0 + \varepsilon\omega$ from Ω, where $x_0 \in \Omega$ and $\omega \subset \mathbb{R}^N$ is a fixed open and bounded subset containing the origin. Then, using general adjoint method, an asymptotic expansion of the function will be obtained in the following form:

$$\begin{gathered} j(\Omega_\varepsilon) = j(\Omega) + f(\varepsilon)g(x_0) + o\big(f(\varepsilon)\big), \\ \lim_{\varepsilon \to 0} f(\varepsilon) = 0, \quad f(\varepsilon) > 0. \end{gathered} \tag{1.1}$$

The topological sensitivity $g(x_0)$ provides information when creating a small hole located at x_0. Hence, the function g will be used as descent direction in the optimization process.

In this paper, we study essentially a topological optimization problem with a nonlinear operator. There are many works in literature concerning topological optimization. However, many of these authors study linear operators. We notice that Amstutz in [1] established some results in topological optimization with a semilinear operator of the form $-\Delta u + \phi(u) = 0$ in a domain Ω with some hypothesis in ϕ.

In this paper, we will limit in the p-Laplacian operator and we reline the theoretical result obtained with an application in image processing.

The paper is organized as follows: in Section 2, we recall image processing models and the Mumford-Shah functional which are widely studied in literature. In Section 3, we present the general adjoint method. Section 4 is devoted to the topological optimization problem and the main result of the paper which is proved inSection 5. InSection 6, the topological optimization algorithm and numerical applications in image processing are presented.

2. Formulation of the Problem

2.1. A Model of Image Processing

Many models and algorithms [2] have been proposed for the study of image processing.

In [3], Koenderink noticed that the convolution of signal with Gaussian noise at each scale is equivalent to the solution of the heat equation with the signal as initial datum. Denoting by u_0 this datum, the "scale space" analysis associated with u_0 consists in solving the system

$$\begin{aligned} \frac{\partial u(x,t)}{\partial t} &= \Delta u(x,t) \\ u(x,0) &= u_0(x) \end{aligned} \quad \text{in } \mathbb{R}^N. \tag{2.1}$$

The solution of this equation with an initial datum is $u(x,t) = G_t * u_0$, where $G_\sigma = (1/4\pi\sigma)\exp(-\|x\|^2/4\sigma)$ is the Gauss function, and $\|x\|$ the euclidian norm of $x \in \mathbb{R}^N$.

In [4], Malik and Perona in their theory introduced a filter in (2.1) for the detection of edges. They proposed to replace the heat equation by a nonlinear equation:

$$\begin{aligned} \frac{\partial u}{\partial t} &= \operatorname{div}\big(f(|\nabla u|)\nabla u\big) \\ u(x,0) &= u_0(x) \end{aligned} \quad \text{in } \mathbb{R}^N. \tag{2.2}$$

In this equation, f is a smooth and nonincreasing function with $f(0) = 1$, $f(s) \geq 0$, and $f(s)$ tending to zero at infinity. The idea is that the smoothing process obtained by the equation should be conditional: if $|\nabla u(x)|$ is large, then the diffusion will be low and, therefore, the exact location of the edges will be kept. If $|\nabla u(x)|$ is small, then the diffusion will tend to be smooth still more around x. Notice that the new model reduces to the heat equation when $f(s) = 1$.

Nordström [5] introduced a new term in (2.2) which forces $u(x,t)$ to remain close to x. Because of the forcing term $u - v$, the new equation

$$\frac{\partial u}{\partial t} - \operatorname{div}\big(f(|\nabla u|)\nabla u\big) = v - u \quad \text{in } \Omega \times (0,T]$$
$$\frac{\partial u}{\partial \nu} = 0 \quad \text{in } \partial\Omega \times (0,T] \tag{2.3}$$
$$u(x,0) = u_0(x) \quad \text{in } \Omega \times \{t=0\}$$

has the advantage to have a nontrivial steady state, eliminating, therefore, the problem of choosing a stoping time.

2.2. The Mumford-Shah Functional

One of the most widely studied mathematical models in image processing and computer vision addresses both goals simultaneously, namely, Mumford and Shah [6] who presented the variational problem of minimizing a functional involving a piecewise smooth representation of an image. The Mumford-Shah model defines the segmentation problem as a joint smoothing/edge detection problem: given an image $v(x)$, one seeks simultaneously a "piecewise smoothed image" $u(x)$ with a set K of abrupt discontinuities, the edges of v. Then the "best" segmentation of a given image is obtained by minimizing the functional

$$E(u,K) = \int_{\Omega\setminus K} \Big(\alpha|\nabla u|^2 + \beta(u-v)^2\Big)dx + \mathcal{H}^{N-1}(K), \tag{2.4}$$

where $\mathcal{H}^{N-1}(K)$ is the $(N-1)$-dimensional Hausdorff measure of $K$and α and β are positive constants .

The first term imposes that u is smooth outside the edges, the second that the piecewise smooth image $u(x)$ indeed approximates $v(x)$, and the third that the discontinuity set K has minimal length (and, therefore, in particular, it is as smooth as possible).

The existence of minimums of the Mumford-Shah has been proved in some sense, we refer to [7]. However, we are not aware that the existence problem of the solution for this problem is closed.

Before beginning the study of the topological optimization method, let us recall additional information about the other techniques.

We say that an image can be viewed as a piecewise smooth function and edges can be considered as a set of singularities.

We recall that a classical way to restore an image u from its noisy version v defined in a domain included in $\mathbb{R}^2$ is to solve the following PDE problem:

$$u - \operatorname{div}(c\nabla u) = v \quad \text{in } \Omega,$$
$$\frac{\partial u}{\partial \nu} = 0 \quad \text{in } \partial\Omega, \tag{2.5}$$

where c is a small positive constant. This method is well known to give poor results: it blurs important structures like edges. In order to improve this method, nonlinear isotropic and anisotropic methods were introduced, we can cite here the work of Malik and Perona, Catté et al., and, more recently, Weickert and Aubert.

In topological gradient approach, c takes only two values: c_0, for example, $c_0 = 1$, in the smooth part of the image and a small value ϵ on edges. For this reason, classical nonlinear diffusive approaches, where c takes all the values of the interval $[\epsilon, c_0]$, could be seen as a relaxation of the topological optimization method. Many classification models have been studied and tested on synthetic and real images in image processing literature, and results are more or less comparative taking in to account the complexity of algorithms suggested and/or the cost of operations defined. We can cite here some models enough used like the structural approach by regions growth, the stochastic approaches, and the variational approaches which are based on various strategies like level set formulations, the Mumford-Shah functional, active contours and geodesic active contour methods, or wavelet transforms.

The segmentation problem consists in splitting an image into its constituent parts. Many approaches have been studied. We can cite here some variational approaches such as the use of the Mumford-Shah functional, or active contours and snakes. Our approach consists in using the restoration algorithm in order to find the edges of the image, and we will give a numerical result.

To end this section, let us sum up our aim about the numerical aspects.

Considering the Mumford Shah functional, our objective in the numerical point of view is to apply topological gradient approach to images. We are going to show that it is possible to solve these image processing problems using topological optimization tools for the detection of edges (the topological gradient method is able to denoise an image and preserve features such as edges). Then, the restoration becomes straightforward, and in most applications, a satisfying approximation of the optimal solution is reached at the first iteration or the second iteration of the optimization process.

We refer the reader, for additional details, to the work in [8–17].

3. General Adjoint Method

In this section, we give an adaptation of the adjoint method introduced in [18] to a nonlinear problem. Let $\mathcal{V}$ be a Hilbert space and let

$$a_\varepsilon(u, v) = l_\varepsilon(v), \quad v \in \mathcal{V}, \tag{3.1}$$

be a variational formulation associated to a partial differential equation. We suppose that there exist forms $\delta a(u, v), \delta l$, and a function $f(\varepsilon) > 0$ which goes to zero when ε goes to zero. Let u_ε (resp., u_0) be the solution of (3.1) for $\varepsilon > 0$ (resp., for $\varepsilon = 0$).

We suppose that the following hypothesis hold:

(H-1) $\|u_\varepsilon - u_0\|_{\mathcal{V}} = o(f(\varepsilon))$,

(H-2) $\|a_\varepsilon - a_0 - f(\varepsilon)\delta a\|_{\mathcal{V}} = O(f(\varepsilon))$,

(H-3) $\|l_\varepsilon - l_0 - f(\varepsilon)\delta l\|_{\mathcal{V}} = O(f(\varepsilon))$.

Let $j(\varepsilon) = J_\varepsilon(u_\varepsilon)$ be the cost function. We suppose that for $\varepsilon = 0$, $J_0(u_0) = J(u_0)$ is differentiable with respect to u_0 and we denote by $DJ(u_0)$ the derivative of J at u_0.

(H-4) We suppose that there exists a function δJ such that

$$J_\varepsilon(u_\varepsilon) - J_0(u_0) = DJ(u_0)(u_\varepsilon - u_0) + f(\varepsilon)\delta J(u_0) + O(f(\varepsilon)). \tag{3.2}$$

Under the aforementioned hypothesis, we have the following theorem.

Theorem 3.1. *Let $j(\varepsilon) = J_\varepsilon(u_\varepsilon)$ be the cost function, then j has the following asymptotic expansion:*

$$j(\varepsilon) - j(0) = (\delta a(u_0, w_0) + \delta J(u_0) - \delta l(w_0)) f(\varepsilon) + o(f(\varepsilon)), \tag{3.3}$$

where w_0 is the unique solution of the adjoint problem: find w_0 such that

$$a_0(\phi, w_0) = -DJ(u_0)\phi, \quad \forall \phi \in \mathcal{V}. \tag{3.4}$$

The expression $g(x_0) = \delta a(u(x_0), \eta(x_0)) + \delta J(u(x_0)) - \delta l(\eta(x_0))$ is called the topological gradient and will be used as descent direction in the optimization process.

The fundamental property of an adjoint technique is to provide the variation of a function with respect to a parameter by using a solution u_Ω and adjoint state v_Ω which do not depend on the chosen parameter. Numerically, it means that only two systems must be solved to obtain the discrete approximation of $g(x)$ for all $x \in \Omega$.

Proof of Theorem 3.1. Let $\mathcal{L}(u, \eta) = a(u, \eta) + J(u) - l(\eta)$ be the Lagrangian of the system as introduced by Lions, in [19] and applied by Cea in optimal design problems in [18]. We use the fact that the variation of the Lagrangian is equal to the variation of the cost function:

$$\begin{aligned} j(\varepsilon) - j(0) &= \mathcal{L}_\varepsilon(u_\varepsilon, \eta) - \mathcal{L}_0(u_0, \eta) \\ &= (a_\varepsilon(u_\varepsilon, \eta) - l_\varepsilon(\eta)) - (a_0(u_0, \eta) - l_0(\eta)) + (J_\varepsilon(u_\varepsilon) - J(u_0)) \\ &= \underbrace{a_\varepsilon(u_\varepsilon, \eta) - a_0(u, \eta)}_{\text{(i)}} - \underbrace{l_\varepsilon(\eta) + l_0(\eta)}_{\text{(ii)}} + \underbrace{J_\varepsilon(u_\varepsilon) - J(u_0)}_{\text{(iii)}}. \end{aligned} \tag{3.5}$$

It follows from hypothesis (H-2) that (i) is equal to

$$a_\varepsilon(u_\varepsilon, \eta) - a_0(u_0, \eta) = f(\varepsilon)\delta a(u_0, \eta) + O(f(\varepsilon)), \tag{3.6}$$

(H-3) implies that (ii) is equal to

$$l_\varepsilon(\eta) - l_0(\eta) = f(\varepsilon)\delta l(\eta) + O(f(\varepsilon)), \tag{3.7}$$

and (H-4) gives an equivalent expression of (iii)

$$J_\varepsilon(u_\varepsilon) - J_0(u_0) = DJ(u_0)(u_\varepsilon - u_0) + f(\varepsilon)\delta J(u_0) + O(f(\varepsilon)). \tag{3.8}$$

Further (3.5) becomes

$$j(\varepsilon) - j(0) = f(\varepsilon)\left[\delta a\left(u_0, \eta\right) + \delta J(u_0) - \delta l\left(\eta\right)\right] + DJ(u_0)(u_\varepsilon - u_0) + O\left(f(\varepsilon)\right). \tag{3.9}$$

Let η_0 be the solution of the adjoint problem: find $\eta_0 \in \mathcal{V}$ such that

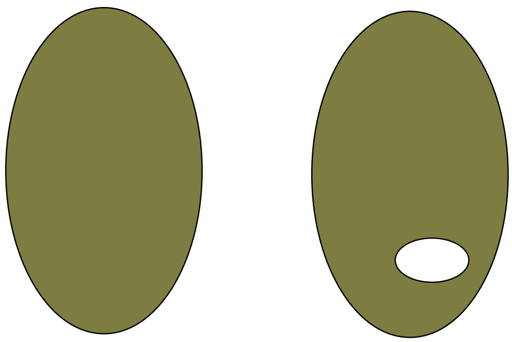
$$a_0\left(\phi, \eta_0\right) = -DJ(u_0)\phi, \quad \forall \phi \in \mathcal{V}. \tag{3.10}$$

It follows from the hypothesis (H-1)–(H-4) that

$$j(\varepsilon) = j(0) + f(\varepsilon)g(x_0) + O\left(f(\varepsilon)\right), \quad \forall x_0 \in \Omega, \tag{3.11}$$

which finishes the proof of the theorem. □

4. Position of the Problem and Topological Sensitivity



Initial domain Perturbed domain

For $\varepsilon > 0$, let $\Omega_\varepsilon = \Omega \setminus \overline{\omega_\varepsilon}$, where $\omega_\varepsilon = x_0 + \varepsilon\omega_0,\ x_0 \in \Omega$, and $\omega_0 \subset \mathbb{R}^N$ is a reference domain.

The topological optimization problem consists of determining the asymptotic expansion of the N-dimensional part of the Mumford-Shah energy, and applying it to image processing. For $v \in L^2(\Omega)$, let us consider the functional

$$J_{\Omega_\varepsilon}(u_\varepsilon) = \alpha \int_{\Omega_\varepsilon} |\nabla u_\varepsilon|^2 dx + \beta \int_{\Omega_\varepsilon} (u_\varepsilon - v)^2 dx, \tag{4.1}$$

where u_ε is the solution of the stationary model of (2.3) with $f(s) = |s|^{p-2},\ p > 1$, that is, u_ε satisfies

$$\begin{gathered} u_\varepsilon - \operatorname{div}\left(|\nabla u_\varepsilon|^{p-2} \nabla u_\varepsilon\right) = v \quad \text{in } \Omega_\varepsilon, \\ \frac{\partial u_\varepsilon}{\partial \nu} = 0 \quad \text{on } \partial\Omega_\varepsilon \setminus \partial\omega_\varepsilon, \\ u_\varepsilon = 0 \quad \text{on } \partial\omega_\varepsilon, \end{gathered} \tag{4.2}$$

α, β are positive constants and for $\varepsilon = 0$, $u_0 = u$ is the solution of

$$u - \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) = v \quad \text{in } \Omega, \qquad \in L^2(\Omega)$$
$$\frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega \in L^2(\Omega). \tag{4.3}$$

Before going on, let us underline that interesting works were done by Auroux and Masmoudi [10], Auroux et al. [12], and Auroux [11] by considering the Laplace operator, that is, ($p = 2$) and the first term of the Mumford Shah functional, that is,

$$J_\Omega(u) = \int_\Omega |\nabla u|^2 d, \tag{4.4}$$

as criteria to be optimized.

Contrarily to the Dirichlet case, we do not get a natural prolongment for the Neumann condition to the boundary $\partial\Omega$. For this reason, all the domains which will be used are supposed to satisfy the following uniform extension property ($\mathcal{P}$). Let D be an open subset of $\mathbb{R}^N$:

$$(\mathcal{P})\begin{cases} \text{(i)} \quad \forall \omega \subset D, \text{ there exist a continuous and linear operator} \\ \quad\quad \text{of prolongment } P_\omega : W^{1,p}(\omega) \longrightarrow W^{1,p}(D) \text{ and apositive constant} \\ \quad\quad c_\omega \text{ such that } \|P_\omega(u)\|_{W^{1,p}(D)} \le c_\omega \|u\|_{W^{1,p}(\omega)}, \\ \text{(ii)} \quad \text{there exists aconstant } M > 0 \text{ such that } \forall \omega \subset D,\ \|P_\omega\| \le M. \end{cases} \tag{4.5}$$

Lemma 4.1. *Let $v \in L^2(\Omega)$ problem* (4.3) *(resp.,* (4.2)*) has a unique solution. Moreover, one has*

$$\|u_\varepsilon - u_0\|_{\mathcal{V}} = O\left(f(\varepsilon)\right), \tag{4.6}$$

where

$$\|u\|_{\mathcal{V}} = \|u\|_{L^2(\Omega)} + \|\nabla u\|_{L^p(\Omega)} = \left(\int_\Omega |u|^2 dx\right)^{1/2} + \left(\int_\Omega |\nabla u|^p dx\right)^{1/p}. \tag{4.7}$$

In order to prove the lemma, we need the following result which in [20, Theorem 2.1]. Let Ω be a bounded open domain of $\mathbb{R}^N$ (no smoothness is assumed on $\partial\Omega$) and p, p' be real numbers such that

$$1 < p,\ p' < +\infty, \quad \frac{1}{p} + \frac{1}{p'} = 1. \tag{4.8}$$

Consider the operator A defined on $W^{1,p}(\Omega)$ by

$$A(u) = -\operatorname{div}(a(x, u(x), \nabla u(x))), \tag{4.9}$$

where $a : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a Caratheodory function satisfying the classical Laray-Lions hypothesis in the sense of [20], described in what follows:

$$|a(x,s,\zeta)| \le c(x) + k_1|s|^{p-1} + k_2|\zeta|^{p-1}, \tag{4.10}$$

$$[a(x,s,\zeta) - a(x,s,\zeta^*)](\zeta - \zeta^*) > 0\ , \tag{4.11}$$

$$\frac{a(x,s,\zeta)\zeta}{|\zeta| + |\zeta|^{p-1}} \underset{|\zeta|\to+\infty}{\longrightarrow} +\infty\ , \tag{4.12}$$

almost every $x \in \Omega$, for all $s \in \mathbb{R}, \zeta, \zeta^* \in \mathbb{R}^N$, $\zeta \neq \zeta^*$.

Consider the nonlinear elliptic equation

$$-\operatorname{div}(a(x,u_n,\nabla u_n)) = f_n + g_n \quad \text{in } \mathfrak{D}'(\Omega). \tag{4.13}$$

We assume that

$$u_n \text{ converges to } u \text{ on } W^{1,p}(\Omega) \text{ weakly and strongly in } L^p_{\text{loc}}(\Omega), \tag{4.14}$$

$$f_n \longrightarrow f \quad \text{strongly in } W^{-1,p'}(\Omega), \text{ and almost everywhere in } \Omega. \tag{4.15}$$

The hypotheses (4.13)–(4.15) imply that $g_n \in W^{-1,p'}(\Omega)$ and is bounded on this set. We suppose either g_n is bounded in $\mathcal{M}_b(\Omega)$ (space on Radon measures), that is,

$$\left|\langle g_n, \varphi\rangle\right| \le c_K \|\varphi\|_{L^\infty(K)}, \quad \forall\varphi \in \mathfrak{D}(K), \text{ with } \operatorname{supp}(\varphi) \subset K, \tag{4.16}$$

c_K is a constant which depends only on the compact K.

Theorem 4.2. *If the hypotheses* (4.10)–(4.15) *are satisfied, then*

$$\nabla u_n \longrightarrow \nabla u \quad \textit{strongly in } \left(L^m_{\text{loc}}(\Omega)\right)^N, \quad \forall m < p, \tag{4.17}$$

where u is the solution of

$$-\operatorname{div}(a(x,u,\nabla u)) = f + g \quad \textit{in } \mathfrak{D}'(\Omega). \tag{4.18}$$

For the proof of the theorem, we refer to [20].

Remark 4.3. Such assumptions are satisfied if $a(x,s,\zeta) = |\zeta|^{p-2}\zeta$, that is, the operator with which we work on in this paper see [20, Remark 2.2].

Proof of Lemma 4.1. We prove at first that (4.3) has a unique solution. The approach is classic: the variational approach.

Let

$$\mathcal{V} = \left\{ u \in L^2(\Omega), \nabla u \in (L^p(\Omega))^N, \text{ and } u \text{ satisfying the property } \mathcal{P} \right\}, \tag{4.19}$$

$$I[u] = \frac{1}{2}\int_\Omega u^2 dx + \frac{1}{p}\int_\Omega |\nabla u|^p dx - \int_\Omega u \cdot v dx. \tag{4.20}$$

If there exists $u \in \mathcal{V}$ such that $I[u] = \min_{\eta \in \mathcal{V}} I[\eta]$, then u satisfies an optimality condition which is the Euler-Lagrange equation $I'[u] = 0$. Let $h \in \mathfrak{D}(\Omega),\ t \in \mathbb{R}$,

$$\begin{aligned} I[u+th] &= \frac{1}{2}\int_\Omega (u+th)^2 dx + \frac{1}{p}\int_\Omega |\nabla(u+th)|^p dx - \int_\Omega v(u+th)dx \\ &= \frac{1}{2}\int_\Omega (u+th)^2 dx + \frac{1}{p}\int_\Omega \left(|\nabla(u+th)|^2\right)^{p/2} dx - \int_\Omega v(u+th)dx \\ &= \frac{1}{2}\int_\Omega (u+th)^2 dx + \frac{1}{p}\int_\Omega \left(|\nabla u|^2 + 2t\nabla u \nabla h + t^2|\nabla h|^2\right)^{p/2} dx \\ &\quad - \int_\Omega v(u+th)dx. \end{aligned} \tag{4.21}$$

Using Taylor expansion, it follows that

$$\begin{aligned} &\int_\Omega \left[|\nabla u|^p + t\nabla h \nabla u |\nabla u|^{p-2} - uv - thv\right] dx + o(t), \\ \langle I'[u], h\rangle = \lim_{t\to 0}\frac{I[u+th]-I[u]}{t} &= 0 \Longleftrightarrow \left\langle u - \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) - v, h \right\rangle = 0, \quad \forall h \in \mathfrak{D}(\Omega). \end{aligned} \tag{4.22}$$

Since $h \in \mathfrak{D}(\Omega)$, we have $u - \operatorname{div}(|\nabla w|^{p-2}\nabla u) = v$ in $\mathfrak{D}'(\Omega)$.

Now, let $u \in W^{1,p}(\Omega)$ be a weak solution of (4.3), then the variational formulation gives

$$\int_\Omega \left(u\varphi + \left(|\nabla u|^{p-2}\nabla u\right)\right)\nabla\varphi - \int_\Gamma |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = \int_\Omega v\varphi. \tag{4.23}$$

Let us introduce $C_c^1(\Omega)$ be the set of functions C^1-regular in Ω and with compact support in Ω. Choosing $\varphi \in C_c^1(\Omega)$, it follows that

$$u - \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) = v \quad \text{in } \mathfrak{D}'\Omega. \tag{4.24}$$

We come back to (4.3) with $\varphi \in C^1(\overline{\Omega})$, and we obtain

$$\int_{\partial\Omega} |\nabla u|^{p-2} \frac{\partial u}{\partial \nu} \varphi = 0, \tag{4.25}$$

as u is not constant almost everywhere, we obtain

$$\frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega. \tag{4.26}$$

Consequently (4.3) admits a unique solution which is obtained by minimizing the functional (4.20). For achieving this part of the proof, it suffices to prove that the problem: find $u \in \mathcal{V}$ such that

$$I[u] = \min_{\eta \in \mathcal{V}} I[\eta] \tag{4.27}$$

admits a unique solution, where $I[\cdot]$ is defined by (4.20).

In fact, $I[w]$ is a strict convex functional and the set $\mathcal{V}$ is convex. To achieve our aim, we prove that $I[\cdot]$ is bounded from what follows: let $\alpha = \inf_{\eta \in \mathcal{V}} I[\eta]$, then $\alpha > -\infty$.

We have

$$\begin{aligned} I[\eta] &\geq \frac{1}{2}\int_{\Omega} \eta^2 dx - \int_{\Omega} \eta \cdot v dx \geq \frac{1}{2}\|\eta\|^2_{L^2(\Omega)} - \|v\|_{L^2(\Omega)} \|\eta\|_{L^2(\Omega)} \\ &\geq E\left(\left\|\frac{v}{2}\right\|_{L^2(\Omega)}\right) > -\infty, \end{aligned} \tag{4.28}$$

where $E(X) = X^2 - \|v\|_{L^2(\Omega)} X$.

This proves that $I[\cdot]$ has a unique minimum and this minimum is the solution of (4.3).

We use the same arguments to prove that (4.2) admits a unique solution and the solution is obtained by minimizing the functional (4.1) in the set

$$\mathcal{V}_\varepsilon = \left\{ u \in L^2(\Omega),\ \nabla u \in (L^p(\Omega))^N,\ u \text{ satisfies } \mathcal{D},\ u = 0 \text{ on } \partial\omega_\varepsilon \right\}. \tag{4.29}$$

To end the proof, we have to prove that $\|u_\varepsilon - u_0\|_{\mathcal{V}} \underset{\varepsilon \to 0}{\longrightarrow} 0$.

On the one hand, we suppose for simplicity that $\omega_\varepsilon = B(x_0, \varepsilon)$. The proof will remain true if ω_ε is regular enough, for example, $\partial\omega_\varepsilon$ has a Lipschitz regularity with a uniform Lipschitz's constant. Let $\varepsilon = 1/n$, here and in the following, we set $u_n = u_\varepsilon = u(\Omega_n)$. That is, u_n is the solution of

$$\begin{gathered} u_n - \operatorname{div}\left(|\nabla u_n|^{p-2} \nabla u_n\right) = v \quad \text{in } \Omega_n, \\ \frac{\partial u_n}{\partial \nu} = 0 \quad \text{on } \partial\Omega = \partial\Omega_n \setminus \partial B\left(x_0, \frac{1}{n}\right), \\ u_n = 0 \quad \text{on } \partial B\left(x_0, \frac{1}{n}\right). \end{gathered} \tag{4.30}$$

Let $\widetilde{u}_n$ be the extension of u_ε in $B(x_0, 1/n)$, that is,

$$\widetilde{u}_n(x) = \begin{cases} u_n(x), & \text{if } x \in \Omega_n, \\ 0, & \text{if } x \in \overline{B\left(x_0, \frac{1}{n}\right)}. \end{cases} \tag{4.31}$$

The variational form of the problem is

$$\int_\Omega \widetilde{u}_n^2 dx + \int_\Omega |\nabla \widetilde{u}_n|^p dx = \int_\Omega \widetilde{u}_n v dx. \tag{4.32}$$

Since

$$|\nabla \widetilde{u}_n|^p \geq 0, \quad \text{we get} \int_\Omega |\nabla \widetilde{u}_n|^p dx \geq 0. \tag{4.33}$$

This implies that

$$\int_\Omega |\widetilde{u}_n|^2 dx \leq \int_\Omega |\widetilde{u}_n v| dx. \tag{4.34}$$

By Cauchy-Schwarz inequality, we have

$$\|\widetilde{u}_n\|^2_{L^2(\Omega)} \leq \|\widetilde{u}_n\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)}, \tag{4.35}$$

and then

$$\|\widetilde{u}_n\|_{L^2(\Omega)} \leq \|v\|_{L^2(\Omega)} \leq c_1. \tag{4.36}$$

By the same arguments, and estimation (4.36), we prove that

$$\|\nabla \widetilde{u}_n\|_{L^p(\Omega)} \leq \|v\|_{L^2(\Omega)} \leq c_2. \tag{4.37}$$

Since Ω is bounded, $\|\widetilde{u}_n\|_{L^2(\Omega)}$ and $\|\nabla \widetilde{u}_n\|_{L^p(\Omega)}$ are bounded, and thus there exist $u^* \in L^2(\Omega)$ and $T \in (L^p(\Omega))^N$ such that

$$\widetilde{u}_n \underset{\sigma(L^2, L^2)}{\rightharpoonup} u^*, \qquad \nabla \widetilde{u}_n \underset{\sigma(L^p, L^{p'})}{\rightharpoonup} T, \tag{4.38}$$

we can prove now that $T = \nabla u^*$ almost everywhere in Ω:

$$\begin{gathered} \frac{\partial \widetilde{u}_n}{\partial x_i} \underset{\sigma(L^p, L^{p'})}{\rightharpoonup} T_i \quad (i = 1, \ldots, N) \\ \Longleftrightarrow \forall f \in L^{p'}(\Omega), \quad \left\langle \frac{\partial \widetilde{u}_n}{\partial x_i}, f \right\rangle \longrightarrow \langle T, f \rangle, \end{gathered} \tag{4.39}$$

$\mathfrak{D}(\Omega) \subset L^{p'}(\Omega)$, thus for all $\varphi \in \mathfrak{D}(\Omega)$,

$$\langle T_i, \varphi \rangle \longleftarrow \left\langle \frac{\partial \widetilde{u}_n}{\partial x_i}, \varphi \right\rangle = -\left\langle \widetilde{u}_n, \frac{\partial \varphi}{\partial x_i} \right\rangle \longrightarrow -\left\langle u^*, \frac{\partial \varphi}{\partial x_i} \right\rangle. \tag{4.40}$$

Thus,

$$\begin{aligned} \langle T, \varphi \rangle &= -\left\langle u^*, \frac{\partial \varphi}{\partial x_i} \right\rangle, \quad \forall \varphi \in \mathfrak{D}(\Omega), \\ \Longrightarrow T_i &= \frac{\partial u^*}{\partial x_i} \quad \text{in } \mathfrak{D}'(\Omega) \Longrightarrow T_i = \frac{\partial u^*}{\partial x_i} \text{ almost everywhere in } \Omega. \end{aligned} \tag{4.41}$$

It follows from (4.38) that $\widetilde{u}_n \underset{\sigma(L^2,L^2)}{\rightharpoonup} u^*$ and $\varphi \in \mathfrak{D}(\Omega)$ thus $\partial\varphi/\partial x_i \in \mathfrak{D}(\Omega)$. This implies that

$$\begin{aligned} -\left\langle \widetilde{u}_n, \frac{\partial \varphi}{\partial x_i} \right\rangle &\longrightarrow -\left\langle u^*, \frac{\partial \varphi}{\partial x_i} \right\rangle = \left\langle \frac{\partial u^*}{\partial x_i}, \varphi \right\rangle, \\ \left\langle \frac{\partial u^*}{\partial x_i}, \varphi \right\rangle &= \langle T, \varphi \rangle, \quad \forall \varphi \in \mathfrak{D}(\Omega) \Longrightarrow T_i = \frac{\partial u^*}{\partial x_i} \text{ almost everywhere.} \end{aligned} \tag{4.42}$$

Thus it follows from the Theorem 4.2 that $u^* = u$, and $\nabla u = \nabla u^*$, and we deduce $\|u_n - u^*\|_{\mathcal{V}}$. □

The main result is the following which gives the asymptotic expansion of the cost function.

Theorem 4.4 (Theorem (main result)). *Let $j(\varepsilon) = J_\varepsilon(u_\varepsilon)$ be the Mumford-Shah functional. Then j has the following asymptotic expansion:*

$$j(\varepsilon) - j(0) = f(\varepsilon)\delta j(u_0, \eta_0) + o(f(\varepsilon)), \tag{4.43}$$

where $\delta j(u_0, \eta_0) = \delta a(u_0, \eta_0) + \delta J(u_0) - \delta l(\eta_0)$ and η_0 is the unique solution of the so-called adjoint problem: find η_0 such that

$$a_0(\psi, \eta_0) = -DJ(u_0)\psi \quad \forall \psi \in \mathcal{V}. \tag{4.44}$$

5. Proof of the Main Result

A sufficient condition to prove the main result is to show at first that the hypothesis (H-1), (H-2), (H-3), and (H-4) (cf. Section 3) are satisfied. Then we apply in the second step of the proof the Theorem 3.1 to get the desired result.

The Lemma 4.1 gives the hypothesis ((H-3)).

The variational formulation associated to (3.1) is

$$\int_{\Omega_\varepsilon} u_\varepsilon \phi dx + \int_{\Omega_\varepsilon} |\nabla u_\varepsilon|^{p-1} \nabla u_\varepsilon \nabla \phi dx = \int_{\Omega_\varepsilon} v \phi dx, \quad \forall \phi \in \mathcal{V}_\varepsilon. \tag{5.1}$$

We set

$$a_\varepsilon(u_\varepsilon,\phi)=\int_{\Omega_\varepsilon}u_\varepsilon\phi dx+\int_{\Omega_\varepsilon}|\nabla u_\varepsilon|^{p-1}\nabla u_\varepsilon\nabla\phi dx,\quad \forall\phi\in\mathcal{V}_\varepsilon,$$
$$l_\varepsilon(\phi)=\int_{\Omega_\varepsilon}v\phi dx,\quad \forall\phi\in\mathcal{V}_\varepsilon. \tag{5.2}$$

5.1. Variation of $a_\varepsilon - a_0$

Proposition 5.1. *The asymptotic expansion of a_ε is given by*

$$a_\varepsilon(u_\varepsilon,\eta)-a_0(u,\eta)-f(\varepsilon)\delta a(u,\eta)=O(f(\varepsilon)), \tag{5.3}$$

where

$$f(\varepsilon)\delta a(u,\eta)=-\left[\int_{\omega_\varepsilon}\left(u\eta+|\nabla u|^{p-2}\nabla u\nabla\eta\right)dx\right],\quad \forall\eta\in\mathcal{V}_\varepsilon. \tag{5.4}$$

For the proof, we need the following result:

Theorem 5.2 (Theorem (Egoroff's theorem)). *Let μ be a measure on $\mathbb{R}^N$ and suppose $f_k:\mathbb{R}^N\to\mathbb{R}^m(k=1,2,\ldots)$ are μ-measurable. Assume also $A\subset\mathbb{R}^N$ is μ-measurable, with $\mu(A)<\infty$, and $f_k\to g\mu$ a.e on A. Then for each $\epsilon>0$ there exists a μ-measurable set $B\subset A$ such that*

(i) $\mu(A-B)<\epsilon$,

(ii) $f_k\to g$ *uniformly on B.*

For the proof of Egoroff's theorem we refer to [21].

Proof of Proposition 5.1.

$$\begin{aligned}
a_\varepsilon(u_\varepsilon,\eta)-a_0(u,\eta)&=\int_{\Omega_\varepsilon}u_\varepsilon\eta dx+\int_{\Omega_\varepsilon}|\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon\nabla\eta dx\\
&\quad-\int_{\Omega}u\eta dx-\int_{\Omega}|\nabla u|^{p-2}\nabla u\nabla\eta dx\\
&=\underbrace{\int_{\Omega_\varepsilon}(u_\varepsilon-u)\eta dx}_{\mathrm{I}}+\underbrace{\int_{\Omega_\varepsilon}(|\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon-|\nabla u|^{p-2}\nabla u)\nabla\eta dx}_{\mathrm{II}}\\
&\quad-\int_{\omega_\varepsilon}\left(u\eta+|\nabla u|^{p-2}\nabla u\nabla\eta\right)dx
\end{aligned} \tag{5.5}$$

We have now to prove that I and II tend to zero with ε.

Estimation of I:

$$\begin{aligned}\left|\int_{\Omega_\varepsilon}(u_\varepsilon-u)\eta dx\right| &\le \left(\int_{\Omega_\varepsilon}\eta^2 dx\right)^{1/2}\left(\int_{\Omega_\varepsilon}|u_\varepsilon-u|^2 dx\right)^{1/2}\\ &\le \left(\int_{\Omega}\eta^2 dx\right)^{1/2}\left(\int_{\Omega}|u_\varepsilon-u|^2 dx\right)^{1/2} \quad \text{because } \Omega_\varepsilon\subset\Omega\\ &= \|u_\varepsilon-u\|_{L^2(\Omega)}\|\eta\|_{L^2(\Omega)}.\end{aligned} \tag{5.6}$$

As $\|u_\varepsilon-u\|_{L^2(\Omega)}\le\|u_\varepsilon-u\|_{\mathcal{V}}=O(f(\varepsilon))$ (cf. Lemma 4.1), it follows that $\int_{\Omega_\varepsilon}(u_\varepsilon-u)\eta dx\to 0$ with $f(\varepsilon)$.

Estimation of II:

$$\left|\int_{\Omega_\varepsilon}\left(|\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon-|\nabla u|^{p-2}\nabla u\right)\nabla w dx\right|\le\int_{\Omega}\left|\left(|\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon-|\nabla u|^{p-2}\nabla u\right)\nabla w\right|dx. \tag{5.7}$$

Let $\varepsilon=1/n$ and $g_n=(|\nabla u_n|^{p-2}\nabla u_n-|\nabla u|^{p-2}\nabla u)\nabla w$, then g_n converges almost every where to zero, by using Lemma 4.1. Writing

$$\int_\Omega g_n dx=\int_{A_0}g_n dx+\int_{A_0^c}g_n dx\le\lim_{n\to+\infty}\sup_{x\in A_0}|g_n|+\int_{A_0^c=\Omega\setminus A_0}g_n dx. \tag{5.8}$$

if $\Omega\setminus A_0$ is negligible, then $\int_{\Omega\setminus A_0}g_n dx=0$, and by using Egoroff's theorem, we obtain that II goes to zero.

Otherwise, we iterate the operation until k_0, such that $\Omega\setminus(\bigcup_{i=0}^{k_0}A_i)$ be a set of zero measure. Then we get

$$\int_\Omega g_n dx=\int_{\cup_{i=0}^{k_0}A_i}g_n dx+\int_{\Omega\setminus(\cup_{i=0}^{k_0}A_i)}g_n dx. \tag{5.9}$$

As $\Omega\setminus(\bigcup_{i=0}^{k_0}A_i)$ is a set of zero measure, the second part of this equation is equal to zero, and we conclude by using Egoroff's theorem that $\int_\Omega g_n dx$ goes to zero, because $\bigcup_{i=0}^{k_0}A_i$ is measurable. It follows that II goes to zero. We set now

$$f(\varepsilon)\delta a(u,\eta)=-\int_{\omega_\varepsilon}\left(u\eta+|\nabla u|^{p-1}\nabla u\nabla\eta\right)dx \tag{5.10}$$

and we obtain the desired estimation. □

5.2. Variation of the Linear Form

Proposition 5.3. *The variation of the linear form l_ε is given by*

$$l_\varepsilon(\eta) - l_0(\eta) - f(\varepsilon)\delta l(\eta) = 0, \tag{5.11}$$

where

$$f(\varepsilon)\delta l(u,\eta) = -\int_{\omega_\varepsilon} v\eta dx, \quad \forall \eta \in \mathcal{V}_\varepsilon. \tag{5.12}$$

5.3. Variation of the Cost Function

In this section, we will prove that the cost function satisfies the necessary condition for applying the Theorem 3.1 (hypothesis (H-4)).

Proposition 5.4. *Let $J_0(u)$ be the Mumford-Shah functional*

$$J(u) = \alpha\int_\Omega |\nabla u|^2 dx + \beta\int_\Omega |u - v|^2 dx, \tag{5.13}$$

J has the following asymptotic expansion:

$$J_\varepsilon(u_\varepsilon) - J_0(u_0) = DJ_\varepsilon(u_0)(u_\varepsilon - u_0) + f(\varepsilon)\delta J(u_0) + o\big(f(\varepsilon)\big). \tag{5.14}$$

Proof. It holds that

$$\begin{aligned} J_\varepsilon(u_\varepsilon) - J(u_0) &= \int_{\Omega_\varepsilon} \alpha|\nabla u_\varepsilon|^2 dx + \int_{\Omega_\varepsilon} \beta|u_\varepsilon - v|^2 dx \\ &\quad - \left(\int_\Omega \alpha|u_0|^2 dx + \int_\Omega \beta|u_0 - v|^2 dx\right) \\ &= \int_{\Omega_\varepsilon} \alpha\Big(|\nabla u_\varepsilon|^2 - |\nabla u_0|^2\Big) dx + \int_{\Omega_\varepsilon} \beta\Big(|u_\varepsilon - v|^2 - |u_0 - v|^2\Big) dx \\ &\quad - \left(\int_{\omega_\varepsilon} \alpha|\nabla u_0|^2 dx + \int_{\omega_\varepsilon} \beta|u_0 - v|^2 dx\right), \end{aligned} \tag{5.15}$$

$$\int_{\Omega_\varepsilon} \alpha\Big(|\nabla u_\varepsilon|^2 - |\nabla u_0|^2\Big) dx = \int_{\Omega_\varepsilon} \alpha\Big(|\nabla (u_\varepsilon - u_0)|^2\Big) dx + 2\alpha\int_{\Omega_\varepsilon} \nabla u_0 \cdot \nabla (u_\varepsilon - u_0) dx.$$

Due to the fact that $u_\varepsilon = u_0|_{\Omega_\varepsilon}$, it suffices to evaluate the difference $J_\varepsilon - J_0$ in D_ε where $D_\varepsilon = B(x_0, R) \setminus \omega_\varepsilon$, and $\omega_\varepsilon \subset B(x_0, R) \subset \Omega$. Taking Lemma 4.1, we have $\|u_\varepsilon - u_0\|_{\mathcal{V}} = o(f(\varepsilon))$, one only needs to prove that

$$\begin{aligned}
&\int_{D_\varepsilon} |\nabla(u_\varepsilon - u_0)|^2 dx = o(f(\varepsilon)), \\
&\int_{D_\varepsilon} |\nabla(u_\varepsilon - u_0)|^2 dx \le \int_{B(x_0,R)} |\nabla(u_\varepsilon - u_0)|^2 dx \\
&\qquad \le (\operatorname{meas}(B(x_0,R)))^{(p-2)/p} \|\nabla(u_\varepsilon - u_0)\|^2_{L^p(\Omega)} \\
&\qquad \le (\operatorname{meas}(B(x_0,R)))^{(p-2)/p} \|(u_\varepsilon - u_0)\|^2_{\mathcal{V}}.
\end{aligned} \tag{5.16}$$

The same arguments as above yield

$$\int_{D_\varepsilon} \beta\left(|u_\varepsilon - v|^2 - |u_0 - v|^2\right) dx = \int_{D_\varepsilon} \beta |u_\varepsilon - u_0|^2 dx + \int_{D_\varepsilon} 2\beta(v - u_0)(u_\varepsilon - u_0) dx, \tag{5.17}$$

$$\int_{D_\varepsilon} \beta |u_\varepsilon - u_0|^2 dx \le \int_\Omega \beta |u_\varepsilon - u_0|^2 dx \le \beta \operatorname{meas}(\Omega) \|u_\varepsilon - u_0\|_{\mathcal{V}}. \tag{5.18}$$

Lemma 4.1 proves that $\|u_\varepsilon - u_0\|_{\mathcal{V}} = o(f(\varepsilon))$, this implies that

$$\int_{D_\varepsilon} \beta |u_\varepsilon - u_0|^2 dx = o(f(\varepsilon)). \tag{5.19}$$

Let us set

$$DJ(u_0)\eta = \int_{D_\varepsilon} 2\beta(v - u_0)\eta dx + 2\alpha \int_{D_\varepsilon} \nabla u_0 \cdot \nabla \eta dx; \quad \forall \eta \in \mathcal{V}_\varepsilon, \tag{5.20}$$

then we obtain

$$J_\varepsilon(u_\varepsilon) - J_0(u_0) = DJ(u_0)(u_\varepsilon - u_0) + f(\varepsilon)\delta J(u_0) + o(f(\varepsilon)) \tag{5.21}$$

with

$$f(\varepsilon)\delta J(u_0) = -\left[\int_{\omega_\varepsilon} \alpha |\nabla u_0|^2 dx + \int_{\omega_\varepsilon} \beta |u_0 - v|^2 dx\right]. \tag{5.22}$$

□

Hence, the hypothesis ((H-1))–((H-4)) are satisfied, and it follows from the Theorem 3.1 that

$$j(\varepsilon) = j(0) + f(\varepsilon) g(x_0) + o(f(\varepsilon)), \tag{5.23}$$

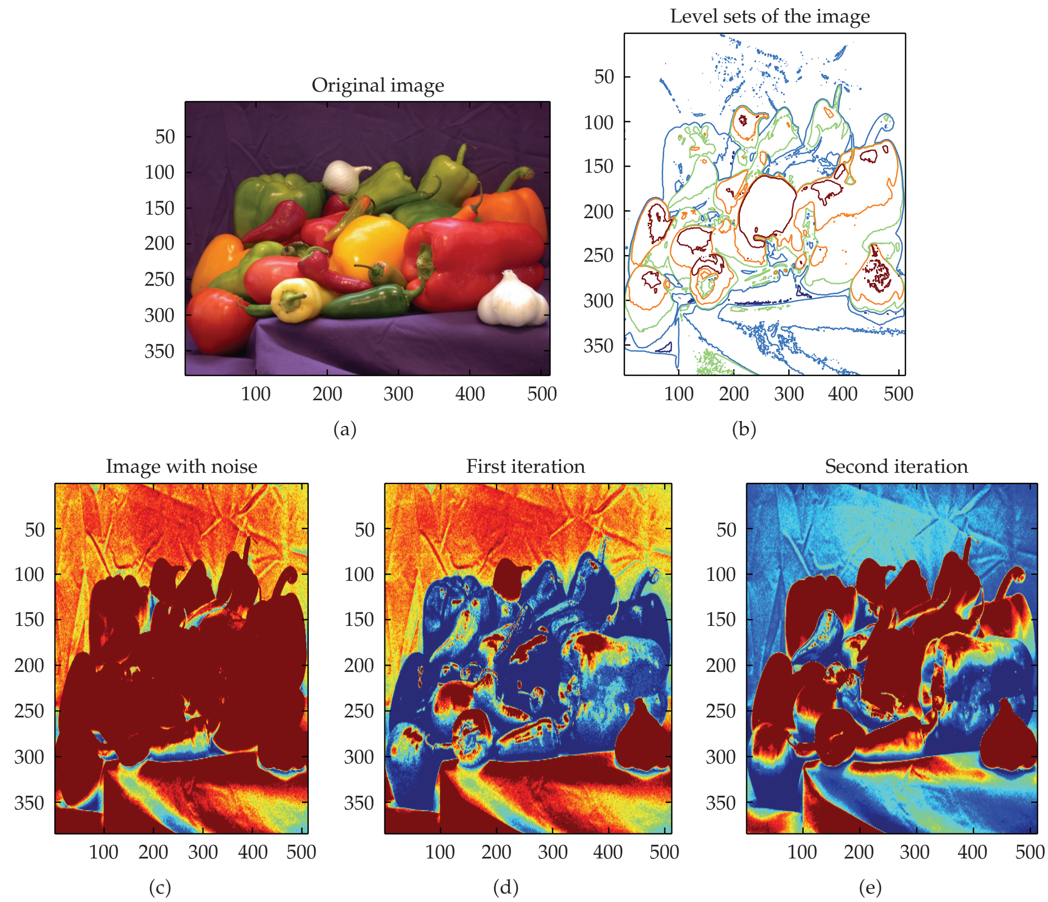


Figure 1: First example.

where

$$g(x_0) = \delta a(u_0, \eta_0) + \delta J(u_0) - \delta l(\eta_0), \tag{5.24}$$

which finishes the proof of Theorem 4.2.

6. Numerical Results

Remark 6.1. In the particular case where $p = 2$ and $\omega = B(x_0, 1)$, the problem becomes

$$\min J(u), \quad J(u) = \int_{\Omega} \alpha |\nabla u|^2 dx + \int_{\Omega} |u - v|^2 dx, \tag{6.1}$$

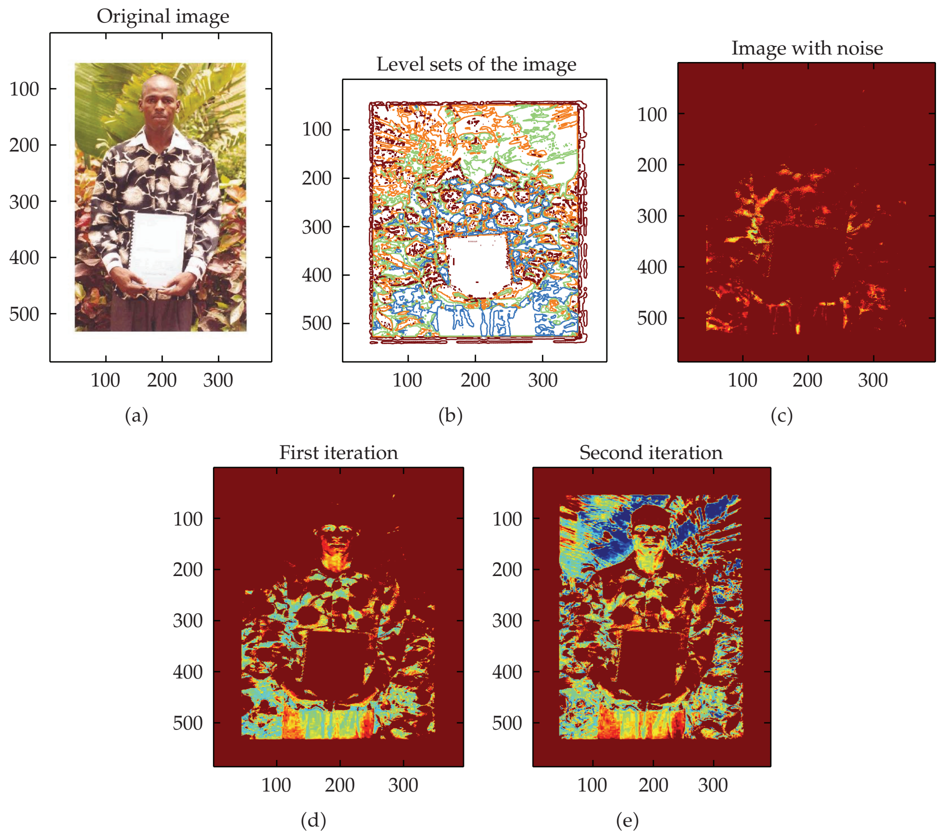


Figure 2: Second example.

where u is the solution of

$$\begin{aligned} u - \Delta u &= v \quad \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{6.2}$$

The topological gradient is given by

$$g(x_0) = -2\pi\left(\alpha|\nabla u(x_0)|^2 + \alpha \nabla u(x_0) \cdot \nabla \eta(x_0) + \beta|u(x_0) - v(x_0)|^2\right), \tag{6.3}$$

where η is the solution of the adjoint problem

$$\begin{aligned} \eta - \Delta \eta &= -DJ(u) \quad \text{in } \Omega, \\ \frac{\partial \eta}{\partial \nu} &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{6.4}$$

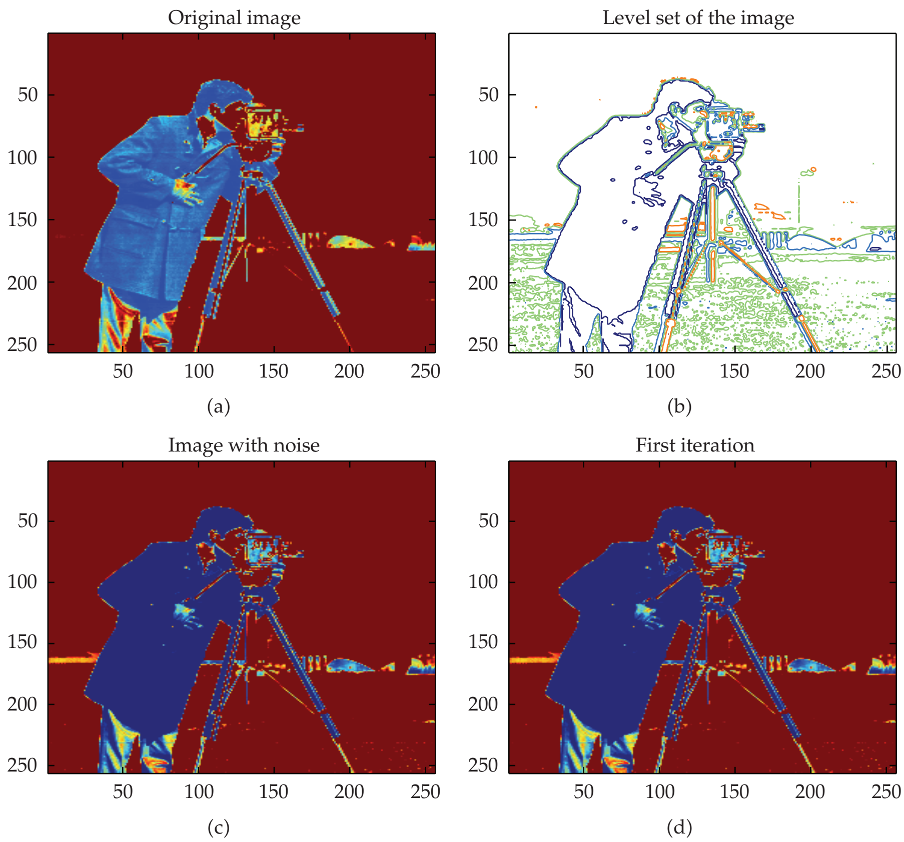


Figure 3: Third example.

6.1. Algorithm

(1) input: image with noise;

(2) compute u and w (direct state and adjoint state) by finite element method;

(3) compute the topological gradient g;

(4) drill a hole where g is "most" negative;

(5) define $u = \varepsilon$ in the hole;

(6) if "stopping criteria" is not satisfied, goto 2 else stop.

6.2. Numerical Examples

In the numerical application, we set $\alpha = \beta = 1$, and we add the noise in the image as follows: let X an $n \times m$ random matrix, where elements are in $[0, 10]$, and $n \times m$ is the dimension of the matrix. The noisy image is obtained by adding X to the original image by adequate MATLAB functions.

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