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Research Article

Uniqueness and Parameter Dependence of Positive Solution of Fourth-Order Nonhomogeneous BVPs

Jian-Ping Sun and Xiao-Yun Wang

Department of Applied Mathematics, Lanzhou University of Technology, Lanzhou, Gansu 730050, China

Correspondence should be addressed to Jian-Ping Sun, jpsun@lut.cn

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We investigate the following fourth-order four-point nonhomogeneous Sturm-Liouville boundary value problem: $u^{(4)} = f(t, u), t \in [0, 1]$, $\alpha u(0) - \beta u'(0) = \lambda_1, \gamma u(1) + \delta u'(1) = \lambda_2$, $au''(\xi_1) - bu'''(\xi_1) = -\lambda_3, cu''(\xi_2) + du'''(\xi_2) = -\lambda_4$, where $0 \le \xi_1 < \xi_2 \le 1$ and $\lambda_i (i = 1, 2, 3, 4)$ are nonnegative parameters. Some sufficient conditions are given for the existence and uniqueness of a positive solution. The dependence of the solution on the parameters $\lambda_i (i = 1, 2, 3, 4)$ is also studied.

1. Introduction

Boundary value problems (BVPs for short) consisting of fourth-order differential equation and four-point homogeneous boundary conditions have received much attention due to their striking applications. For example, Chen et al. [1] studied the fourth-order nonlinear differential equation

$$u^{(4)} = f(t, u), \quad t \in (0, 1), \tag{1.1}$$

with the four-point homogeneous boundary conditions

$$u(0) = u(1) = 0, \tag{1.2}$$

$$au''(\xi_1) - bu'''(\xi_1) = 0, \qquad cu''(\xi_2) + du'''(\xi_2) = 0, \tag{1.3}$$

where $0 \le \xi_1 < \xi_2 \le 1$. By means of the upper and lower solution method and Schauder fixed point theorem, some criteria on the existence of positive solutions to the BVP (1.1)–(1.3) were established. Bai et al. [2] obtained the existence of solutions for the BVP (1.1)–(1.3) by using a nonlinear alternative of Leray-Schauder type. For other related results, one can refer to [3–5] and the references therein.

Recently, nonhomogeneous BVPs have attracted many authors' attention. For instance, Ma [6, 7] and L. Kong and Q. Kong [8–10] studied some second-order multipoint nonhomogeneous BVPs. In particular, L. Kong and Q. Kong [10] considered the following second-order BVP with multipoint nonhomogeneous boundary conditions

$$u'' + a(t)f(u) = 0, \quad t \in (0,1),$$
$$u(0) = \sum_{i=1}^{m} a_i u(t_i) + \lambda, \quad u(1) = \sum_{i=1}^{m} b_i u(t_i) + \mu, \tag{1.4}$$

where λ and μ are nonnegative parameters. They derived some conditions for the above BVP to have a unique solution and then studied the dependence of this solution on the parameters λ and μ. Sun [11] discussed the existence and nonexistence of positive solutions to a class of third-order three-point nonhomogeneous BVP. The authors in [12] studied the multiplicity of positive solutions for some fourth-order two-point nonhomogeneous BVP by using a fixed point theorem of cone expansion/compression type. For more recent results on higher-order BVPs with nonhomogeneous boundary conditions, one can see [13–16].

Inspired greatly by the above-mentioned excellent works, in this paper we are concerned with the following Sturm-Liouville BVP consisting of the fourth-order differential equation:

$$u^{(4)} = f(t,u), \quad t \in [0,1] \tag{1.5}$$

and the four-point nonhomogeneous boundary conditions

$$\alpha u(0) - \beta u'(0) = \lambda_1, \qquad \gamma u(1) + \delta u'(1) = \lambda_2, \tag{1.6}$$

$$a u''(\xi_1) - b u'''(\xi_1) = -\lambda_3, \qquad c u''(\xi_2) + d u'''(\xi_2) = -\lambda_4, \tag{1.7}$$

where $0 \le \xi_1 < \xi_2 \le 1$ and $\lambda_i \ (i = 1,2,3,4)$ are nonnegative parameters. Under the following assumptions:

(A1) $\alpha, \beta, \gamma, \delta, a, b, c$, and d are nonnegative constants with $\beta > 0$, $\delta > 0$, $\rho_1 := \alpha\gamma + \alpha\delta + \gamma\beta > 0$, $\rho_2 := ad + bc + ac(\xi_2 - \xi_1) > 0$, $-a\xi_1 + b > 0$, and $c(\xi_2 - 1) + d > 0$;

(A2) $f(t,u) : [0,1] \times [0,+\infty) \to [0,+\infty)$ is continuous and monotone increasing in u for every $t \in [0,1]$;

(A3) there exists $0 \le \theta < 1$ such that

$$f(t,ku) \ge k^{\theta} f(t,u) \quad \text{for any } t \in [0,1],\ k \in (0,1),\ u \in [0,+\infty), \tag{1.8}$$

we prove the uniqueness of positive solution for the BVP (1.5)–(1.7) and study the dependence of this solution on the parameters λ_i $(i = 1, 2, 3, 4)$.

2. Preliminary Lemmas

First, we recall some fundamental definitions.

Definition 2.1. Let X be a Banach space with norm $\|\cdot\|$. Then

(1) a nonempty closed convex set $P \subseteq X$ is said to be a cone if $mP \subseteq P$ for all $m \geq 0$ and $P \cap (-P) = \{\mathbf{0}\}$, where $\mathbf{0}$ is the zero element of X;

(2) every cone P in X defines a partial ordering in X by $u \leq v \Leftrightarrow v - u \in P$;

(3) a cone P is said to be normal if there exists $M > 0$ such that $\mathbf{0} \leq u \leq v$ implies that $\|u\| \leq M\|v\|$;

(4) a cone P is said to be solid if the interior $\overset{\circ}{P}$ of P is nonempty.

Definition 2.2. Let P be a solid cone in a real Banach space X, $T : \overset{\circ}{P} \to \overset{\circ}{P}$ an operator, and $0 \leq \theta < 1$. Then T is called a θ-concave operator if

$$T(ku) \geq k^{\theta} Tu \quad \text{for any } k \in (0,1),\ u \in \overset{\circ}{P}. \tag{2.1}$$

Next, we state a fixed point theorem, which is our main tool.

Lemma 2.3 (see [17]). *Assume that P is a normal solid cone in a real Banach space X, $0 \leq \theta < 1$, and $T : \overset{\circ}{P} \to \overset{\circ}{P}$ is a θ-concave increasing operator. Then T has a unique fixed point in $\overset{\circ}{P}$.*

The following two lemmas are crucial to our main results.

Lemma 2.4. *Assume that ρ_1 and ρ_2 are defined as in (A1) and $\rho_1\rho_2 \neq 0$. Then for any $h \in C[0,1]$, the BVP consisting of the equation*

$$u^{(4)}(t) = h(t), \quad t \in [0,1] \tag{2.2}$$

and the boundary conditions (1.6) *and* (1.7) *has a unique solution*

$$u(t) = \int_0^1 G_1(t,s) \int_{\xi_1}^{\xi_2} G_2(s,\tau) h(\tau) d\tau\, ds + \sum_{i=1}^{4} \lambda_i \phi_i(t), \quad t \in [0,1], \tag{2.3}$$

where

$$G_1(t,s) = \frac{1}{\rho_1}\begin{cases}(\alpha s+\beta)(\gamma+\delta-\gamma t), & 0 \le s \le t \le 1,\\ (\alpha t+\beta)(\gamma+\delta-\gamma s), & 0 \le t \le s \le 1,\end{cases}$$

$$G_2(t,s) = \frac{1}{\rho_2}\begin{cases}(a(s-\xi_1)+b)(c(\xi_2-t)+d), & s \le t,\ \xi_1 \le s \le \xi_2,\\ (a(t-\xi_1)+b)(c(\xi_2-s)+d), & t \le s,\ \xi_1 \le s \le \xi_2,\end{cases}$$

$$\phi_1(t) = \frac{1}{\rho_1}(\gamma+\delta-\gamma t), \quad t \in [0,1],$$

$$\phi_2(t) = \frac{1}{\rho_1}(\alpha t+\beta), \quad t \in [0,1], \tag{2.4}$$

$$\phi_3(t) = \frac{1}{\rho_2}\int_0^1 (c(\xi_2-s)+d)G_1(t,s)ds, \quad t \in [0,1],$$

$$\phi_4(t) = \frac{1}{\rho_2}\int_0^1 (a(s-\xi_1)+b)G_1(t,s)ds, \quad t \in [0,1].$$

Proof. Let

$$u''(t) = v(t), \quad t \in [0,1]. \tag{2.5}$$

Then

$$v''(t) = h(t), \quad t \in [0,1]. \tag{2.6}$$

By (2.5) and (1.6), we know that

$$u(t) = -\int_0^1 G_1(t,s)v(s)ds + \frac{1}{\rho_1}(\alpha\lambda_2-\gamma\lambda_1)t + \frac{1}{\rho_1}((\gamma+\delta)\lambda_1+\beta\lambda_2), \quad t \in [0,1]. \tag{2.7}$$

On the other hand, in view of (2.5) and (1.7), we have

$$av(\xi_1) - bv'(\xi_1) = -\lambda_3, \qquad cv(\xi_2) + dv'(\xi_2) = -\lambda_4. \tag{2.8}$$

So, it follows from (2.6) and (2.8) that

$$v(t) = -\int_{\xi_1}^{\xi_2} G_2(t,s)h(s)ds + \frac{1}{\rho_2}(c\lambda_3-a\lambda_4)t + \frac{1}{\rho_2}((a\xi_1-b)\lambda_4-(c\xi_2+d)\lambda_3), \quad t \in [0,1], \tag{2.9}$$

which together with (2.7) implies that

$$u(t) = \int_0^1 G_1(t,s) \int_{\xi_1}^{\xi_2} G_2(s,\tau) h(\tau) d\tau\, ds + \sum_{i=1}^{4} \lambda_i \phi_i(t), \quad t \in [0,1]. \tag{2.10}$$

□

Lemma 2.5. *Assume that (A1) holds. Then*

(1) $G_1(t,s) > 0$ *for* $(t,s) \in [0,1] \times [0,1]$;

(2) $G_2(t,s) > 0$ *for* $(t,s) \in [0,1] \times [\xi_1, \xi_2]$;

(3) $\phi_i(t) > 0$ *for* $t \in [0,1]$, $\quad i = 1,2,3,4$.

3. Main Result

For convenience, we denote $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ and $\boldsymbol{\mu} = (\mu_1, \mu_2, \mu_3, \mu_4)$. In the remainder of this paper, the following notations will be used:

(1) $\boldsymbol{\lambda} \to \infty$ if at least one of λ_i $(i = 1,2,3,4)$ approaches ∞;

(2) $\boldsymbol{\lambda} \to \boldsymbol{\mu}$ if $\lambda_i \to \mu_i$ for $i = 1,2,3,4$;

(3) $\boldsymbol{\lambda} > \boldsymbol{\mu}$ if $\lambda_i \geq \mu_i$ for $i = 1,2,3,4$ and at least one of them is strict.

Let $X = C[0,1]$. Then $(X, \|\cdot\|)$ is a Banach space, where $\|\cdot\|$ is defined as usual by the sup norm.

Our main result is the following theorem.

Theorem 3.1. *Assume that (A1)–(A3) hold. Then the BVP* (1.5)–(1.7) *has a unique positive solution* $u_{\boldsymbol{\lambda}}(t)$ *for any* $\boldsymbol{\lambda} > \mathbf{0}$, *where* $\mathbf{0} = (0,0,0,0)$. *Furthermore, such a solution* $u_{\boldsymbol{\lambda}}(t)$ *satisfies the following properties:*

(P1) $\lim_{\boldsymbol{\lambda} \to \infty} \|u_{\boldsymbol{\lambda}}\| = \infty$;

(P2) $u_{\boldsymbol{\lambda}}(t)$ *is strictly increasing in* $\boldsymbol{\lambda}$, *that is,*

$$\boldsymbol{\lambda} > \boldsymbol{\mu} > \mathbf{0} \Longrightarrow u_{\boldsymbol{\lambda}}(t) > u_{\boldsymbol{\mu}}(t), \quad t \in [0,1]; \tag{3.1}$$

(P3) $u_{\boldsymbol{\lambda}}(t)$ *is continuous in* $\boldsymbol{\lambda}$, *that is, for any given* $\boldsymbol{\mu} > \mathbf{0}$,

$$\boldsymbol{\lambda} \longrightarrow \boldsymbol{\mu} \Longrightarrow \|u_{\boldsymbol{\lambda}} - u_{\boldsymbol{\mu}}\| \longrightarrow 0. \tag{3.2}$$

Proof. Let $P = \{u \in X \mid u(t) \geq 0, \ t \in [0,1]\}$. Then P is a normal solid cone in X with $\overset{\circ}{P} = \{u \in X \mid u(t) > 0, \ t \in [0,1]\}$. For any $\boldsymbol{\lambda} > \mathbf{0}$, if we define an operator $T_{\boldsymbol{\lambda}} : \overset{\circ}{P} \to X$ as follows:

$$T_{\boldsymbol{\lambda}} u(t) = \int_0^1 G_1(t,s) \int_{\xi_1}^{\xi_2} G_2(s,\tau) f(\tau, u(\tau)) d\tau\, ds + \sum_{i=1}^{4} \lambda_i \phi_i(t), \quad t \in [0,1], \tag{3.3}$$

then it is not difficult to verify that u is a positive solution of the BVP (1.5)–(1.7) if and only if u is a fixed point of T_λ.

Now, we will prove that T_λ has a unique fixed point by using Lemma 2.3.

First, in view of Lemma 2.5, we know that $T_\lambda : \overset{\circ}{P} \to \overset{\circ}{P}$.

Next, we claim that $T_\lambda : \overset{\circ}{P} \to \overset{\circ}{P}$ is a θ-concave operator.

In fact, for any $k \in (0,1)$ and $u \in \overset{\circ}{P}$, it follows from (3.3) and (A3) that

$$\begin{aligned} T_\lambda(ku)(t) &= \int_0^1 G_1(t,s)\int_{\xi_1}^{\xi_2} G_2(s,\tau)f(\tau,ku(\tau))d\tau\, ds + \sum_{i=1}^4 \lambda_i\phi_i(t) \\ &\geq k^\theta \int_0^1 G_1(t,s)\int_{\xi_1}^{\xi_2} G_2(s,\tau)f(\tau,u(\tau))d\tau\, ds + \sum_{i=1}^4 \lambda_i\phi_i(t) \\ &\geq k^\theta \left(\int_0^1 G_1(t,s)\int_{\xi_1}^{\xi_2} G_2(s,\tau)f(\tau,u(\tau))d\tau\, ds + \sum_{i=1}^4 \lambda_i\phi_i(t)\right) \\ &= k^\theta T_\lambda u(t), \quad t \in [0,1], \end{aligned} \tag{3.4}$$

which shows that T_λ is θ-concave.

Finally, we assert that $T_\lambda : \overset{\circ}{P} \to \overset{\circ}{P}$ is an increasing operator.

Suppose that $u, v \in \overset{\circ}{P}$ and $u \leq v$. By (3.3) and (A2), we have

$$\begin{aligned} T_\lambda u(t) &= \int_0^1 G_1(t,s)\int_{\xi_1}^{\xi_2} G_2(s,\tau)f(\tau,u(\tau))d\tau\, ds + \sum_{i=1}^4 \lambda_i\phi_i(t) \\ &\leq \int_0^1 G_1(t,s)\int_{\xi_1}^{\xi_2} G_2(s,\tau)f(\tau,v(\tau))d\tau\, ds + \sum_{i=1}^4 \lambda_i\phi_i(t) \\ &= T_\lambda v(t), \quad t \in [0,1], \end{aligned} \tag{3.5}$$

which indicates that T_λ is increasing.

Therefore, it follows from Lemma 2.3 that T_λ has a unique fixed point $u_\lambda \in \overset{\circ}{P}$, which is the unique positive solution of the BVP (1.5)–(1.7). The first part of the theorem is proved.

In the rest of the proof, we will prove that such a positive solution $u_\lambda(t)$ satisfies properties (P1), (P2), and (P3).

First,

$$\begin{aligned} u_\lambda(t) &= T_\lambda u_\lambda(t) \\ &= \int_0^1 G_1(t,s)\int_{\xi_1}^{\xi_2} G_2(s,\tau)f(\tau,u_\lambda(\tau))d\tau\, ds + \sum_{i=1}^4 \lambda_i\phi_i(t), \quad t \in [0,1], \end{aligned} \tag{3.6}$$

which together with $\phi_i(t) > 0 \quad (i = 1,2,3,4)$ for $t \in [0,1]$ implies (P1).

Next, we show (P2). Assume that $\boldsymbol{\lambda} > \boldsymbol{\mu} > \mathbf{0}$. Let

$$\overline{\chi} = \sup\{\chi > 0 : u_{\boldsymbol{\lambda}}(t) \geq \chi u_{\boldsymbol{\mu}}(t),\ t \in [0,1]\}. \tag{3.7}$$

Then $u_{\boldsymbol{\lambda}}(t) \geq \overline{\chi} u_{\boldsymbol{\mu}}(t)$ for $t \in [0,1]$. We assert that $\overline{\chi} \geq 1$. Suppose on the contrary that $0 < \overline{\chi} < 1$. Since $T_{\boldsymbol{\lambda}}$ is a θ-concave increasing operator and for given $u \in \mathring{P}$, $T_{\boldsymbol{\lambda}} u$ is strictly increasing in $\boldsymbol{\lambda}$, we have

$$\begin{aligned} u_{\boldsymbol{\lambda}}(t) &= T_{\boldsymbol{\lambda}} u_{\boldsymbol{\lambda}}(t) \geq T_{\boldsymbol{\lambda}}(\overline{\chi} u_{\boldsymbol{\mu}})(t) > T_{\boldsymbol{\mu}}(\overline{\chi} u_{\boldsymbol{\mu}})(t) \\ &\geq (\overline{\chi})^{\theta} T_{\boldsymbol{\mu}} u_{\boldsymbol{\mu}}(t) = (\overline{\chi})^{\theta} u_{\boldsymbol{\mu}}(t) > \overline{\chi} u_{\boldsymbol{\mu}}(t), \quad t \in [0,1], \end{aligned} \tag{3.8}$$

which contradicts the definition of $\overline{\chi}$. Thus, we get $u_{\boldsymbol{\lambda}}(t) \geq u_{\boldsymbol{\mu}}(t)$ for $t \in [0,1]$. And so,

$$u_{\boldsymbol{\lambda}}(t) = T_{\boldsymbol{\lambda}} u_{\boldsymbol{\lambda}}(t) \geq T_{\boldsymbol{\lambda}} u_{\boldsymbol{\mu}}(t) > T_{\boldsymbol{\mu}} u_{\boldsymbol{\mu}}(t) = u_{\boldsymbol{\mu}}(t), \quad t \in [0,1], \tag{3.9}$$

which indicates that $u_{\boldsymbol{\lambda}}(t)$ is strictly increasing in $\boldsymbol{\lambda}$.

Finally, we prove (P3). For any given $\boldsymbol{\mu} > \mathbf{0}$, we first suppose that $\boldsymbol{\lambda} \to \boldsymbol{\mu}$ with $\boldsymbol{\mu}/2 < \boldsymbol{\lambda} < \boldsymbol{\mu}$. From (P2), we know that

$$u_{\boldsymbol{\lambda}}(t) < u_{\boldsymbol{\mu}}(t), \quad t \in [0,1]. \tag{3.10}$$

Let

$$\overline{\sigma} = \sup\{\sigma > 0 : u_{\boldsymbol{\lambda}}(t) \geq \sigma u_{\boldsymbol{\mu}}(t),\ t \in [0,1]\}. \tag{3.11}$$

Then $0 < \overline{\sigma} < 1$ and $u_{\boldsymbol{\lambda}}(t) \geq \overline{\sigma} u_{\boldsymbol{\mu}}(t)$ for $t \in [0,1]$. If we define

$$\omega(\boldsymbol{\lambda}) = \min\left\{\frac{\lambda_i}{\mu_i} : \mu_i > 0\right\}, \tag{3.12}$$

then $0 < \omega(\boldsymbol{\lambda}) < 1$ and

$$\begin{aligned}
u_{\lambda}(t) &= T_{\lambda} u_{\lambda}(t) \\
&\geq T_{\lambda}(\overline{\sigma} u_{\mu})(t) \\
&= \int_0^1 G_1(t,s) \int_{\xi_1}^{\xi_2} G_2(s,\tau) f(\tau, \overline{\sigma} u_{\mu}(\tau)) d\tau\, ds + \sum_{i=1}^{4} \lambda_i \phi_i(t) \\
&\geq \int_0^1 G_1(t,s) \int_{\xi_1}^{\xi_2} G_2(s,\tau) f(\tau, \overline{\sigma} u_{\mu}(\tau)) d\tau\, ds + \omega(\boldsymbol{\lambda}) \sum_{i=1}^{4} \mu_i \phi_i(t) \\
&\geq \omega(\boldsymbol{\lambda}) \left(\int_0^1 G_1(t,s) \int_{\xi_1}^{\xi_2} G_2(s,\tau) f(\tau, \overline{\sigma} u_{\mu}(\tau)) d\tau\, ds + \sum_{i=1}^{4} \mu_i \phi_i(t) \right) \\
&= \omega(\boldsymbol{\lambda}) T_{\mu}(\overline{\sigma} u_{\mu})(t) \\
&\geq \omega(\boldsymbol{\lambda}) (\overline{\sigma})^{\theta} T_{\mu} u_{\mu}(t) \\
&= \omega(\boldsymbol{\lambda}) (\overline{\sigma})^{\theta} u_{\mu}(t), \quad t \in [0,1],
\end{aligned} \tag{3.13}$$

which together with the definition of $\overline{\sigma}$ implies that

$$\omega(\boldsymbol{\lambda}) (\overline{\sigma})^{\theta} \leq \overline{\sigma}. \tag{3.14}$$

So,

$$\overline{\sigma} \geq (\omega(\boldsymbol{\lambda}))^{1/(1-\theta)}. \tag{3.15}$$

Therefore,

$$u_{\lambda}(t) \geq \overline{\sigma} u_{\mu}(t) \geq (\omega(\boldsymbol{\lambda}))^{1/(1-\theta)} u_{\mu}(t), \quad t \in [0,1]. \tag{3.16}$$

In view of (3.10) and (3.16), we obtain that

$$\|u_{\lambda} - u_{\mu}\| \leq \left(1 - (\omega(\boldsymbol{\lambda}))^{1/(1-\theta)}\right) \|u_{\mu}\|, \tag{3.17}$$

which together with the fact that $\omega(\boldsymbol{\lambda}) \to 1$ as $\boldsymbol{\lambda} \to \boldsymbol{\mu}$ shows that

$$\|u_{\lambda} - u_{\mu}\| \longrightarrow 0 \quad \text{as } \boldsymbol{\lambda} \longrightarrow \boldsymbol{\mu} \text{ with } \boldsymbol{\lambda} < \boldsymbol{\mu}. \tag{3.18}$$

Similarly, we can also prove that

$$\|u_{\lambda} - u_{\mu}\| \longrightarrow 0 \quad \text{as } \boldsymbol{\lambda} \longrightarrow \boldsymbol{\mu} \text{ with } \boldsymbol{\lambda} > \boldsymbol{\mu}. \tag{3.19}$$

Hence, (P3) holds. □

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