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Research Article

The Boundary Value Problem of the Equations with Nonnegative Characteristic Form

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We study the generalized Keldys-Fichera boundary value problem for a class of higher order equations with nonnegative characteristic. By using the acute angle principle and the Hölder inequalities and Young inequalities we discuss the existence of the weak solution. Then by using the inverse Hölder inequalities, we obtain the regularity of the weak solution in the anisotropic Sobolev space.

1. Introduction

Keldys [1] studies the boundary problem for linear elliptic equations with degenerationg on the boundary. For the linear elliptic equations with nonnegative characteristic forms, Oleinik and Radkevich [2] had discussed the Keldys-Fichera boundary value problem. In 1989, Ma and Yu [3] studied the existence of weak solution for the Keldys-Fichera boundary value of the nonlinear degenerate elliptic equations of second-order. Chen [4] and Chen and Xuan [5], Li [6], and Wang [7] had investigated the existence and the regularity of degenerate elliptic equations by using different methods. In this paper, we study the generalized Keldys-Fichera boundary value problem which is a kind of new boundary conditions for a class of higher-order equations with nonnegative characteristic form. We discuss the existence and uniqueness of weak solution by using the acute angle principle, then study the regularity of solution by using inverse Hölder inequalities in the anisotropic Sobolev Space.

We firstly study the following linear partial differential operator

$$\begin{aligned} Lu = & \sum_{|\alpha|=|\beta|=m,|\gamma|=m-1} (-1)^m D^\alpha \Big(a_{\alpha\beta}(x) D^\beta u + b_{\alpha\gamma}(x) D^\gamma u \Big) \\ & + \sum_{|\theta|,|\lambda|\le m-1} (-1)^{|\theta|} D^\theta \Big(d_{\theta\lambda}(x) D^\lambda u \Big), \end{aligned} \tag{1.1}$$

where $x \in \Omega$, $\Omega \subset R^n$ is an open set, the coefficients of L are bounded measurable, and the leading term coefficients satisfy

$$a_{\alpha\beta}(x)\xi_\alpha\xi_\beta \geq 0. \tag{1.2}$$

We investigate the generalized Keldys-Fichera boundary value conditions as follows:

$$D^\alpha u|_{\partial\Omega} = 0, \quad |\alpha| \leq m-2, \tag{1.3}$$

$$\sum_{j=1}^{N_{m-1}} C_{ij}^B(x) D^{\lambda^j} u|_{\sum_i^B} = 0, \quad \left|\lambda^j\right| = m-1,\ 1 \leq i \leq N_{m-1}, \tag{1.4}$$

$$\sum_{j=1}^{N_m} C_{ij}^M(x) D^{\alpha^j - \delta_{k_j}} u \cdot n_{k_j}|_{\sum_i^M} = 0, \quad \forall \delta_{k_j} \leq \alpha^j, \tag{1.5}$$

with $|\alpha^j| = m$ and $1 \leq i \leq N_m$, where $\delta_{k_j} = \{\underbrace{0,\ldots,1}_{k_j},\ldots,0\}$.

The leading term coefficients are symmetric, that is, $a_{\alpha\beta}(x) = a_{\beta\alpha}(x)$ which can be made into a symmetric matrix $M(x) = (a_{\alpha^i\alpha^j})$. The odd order term coefficients $b_{\theta\lambda}(x)$ can be made into a matrix $B(x) = (\sum_{k=1}^n b_{\lambda^i\lambda^j(x)} \cdot n_k)$, $\overrightarrow{n} = (n_1,\ldots,n_n)$ is the outward normal at $\partial\Omega$. $\{e_i(x)\}_{i=1}^{N_m}$ and $\{h_i(x)\}_{i=1}^{N_{m-1}}$ are the eigenvalues of matrices $M(x)$ and $B(x)$, respectively. $C_{ij}^B(x)$ and $C_{ij}^M(x)$ are orthogonal matrix satisfying

$$\begin{aligned} C_{ij}^M(x)M(x)C_{ij}^M(x)' &= \left(e_i(x)\delta_{ij}\right)_{i,j=1,\ldots,N_m}, \\ C_{ij}^B(x)B(x)C_{ij}^B(x)' &= \left(h_i(x)\delta_{ij}\right)_{i,j=1,\ldots,N_{m-1}}. \end{aligned} \tag{1.6}$$

The boundary sets are

$$\begin{aligned} \sum_i^M &= \{x \in \partial\Omega \mid e_i(x) > 0\}, \quad 1 \leq i \leq N_m, \\ \sum_i^B &= \{x \in \partial\Omega \mid h_i(x) > 0\}, \quad 1 \leq i \leq N_{m-1}. \end{aligned} \tag{1.7}$$

At last, we study the existence and regularity of the following quasilinear differential operator with boundary conditions (1.3)–(1.5):

$$\begin{aligned} Au = &\sum_{|\alpha|=|\beta|=m,|\gamma|=m-1} (-1)^m D^\alpha \left(a_{\alpha\beta}\left(x, \bigwedge u\right) D^\beta u + b_{\alpha\gamma}(x) D^\gamma u\right) \\ &+ \sum_{|\gamma|=|\theta|=m-1} (-1)^{m-1} D^\gamma \left(d_{\gamma\theta}\left(x, \bigwedge u\right) D^\theta u\right) + \sum_{|\lambda|\leq m-1} (-1)^{|\lambda|} D^\lambda g_\lambda\left(x, \bigwedge u\right), \end{aligned} \tag{1.8}$$

where $m \geq 2$ and $\bigwedge u = \{D^\alpha u\}_{|\alpha|\leq m-2}$.

This paper is a generalization of [3, 8–10].

2. Formulation of the Boundary Value Problem

For second-order equations with nonnegative characteristic form, Keldys [1] and Fichera presented a kind of boundary that is the Keldys-Fichera boundary value problem, with that the associated problem is of well-posedness. However, for higher-order ones, the discussion of well-posed boundary value problem has not been seen. Here we will give a kind of boundary value condition, which is consistent with Dirichlet problem if the equations are elliptic, and coincident with Keldys-Fichera boundary value problem when the equations are of second-order.

We consider the linear partial differential operator

$$Lu = \sum_{|\alpha|=|\beta|=m,|\gamma|=m-1} (-1)^m D^\alpha \left(a_{\alpha\beta}(x) D^\beta u + b_{\alpha\gamma}(x) D^\gamma u \right) + \sum_{|\theta|,|\lambda|\le m-1} (-1)^{|\theta|} D^\theta \left(d_{\theta\lambda}(x) D^\lambda u \right), \tag{2.1}$$

where $x \in \Omega$, $\Omega \subset R^n$ is an open set, the coefficients of L are bounded measurable functions, and $a_{\alpha\beta}(x) = a_{\beta\alpha}(x)$.

Let $\{g_{\alpha\beta}(x)\}$ be a series of functions with $g_{\alpha\beta} = g_{\beta\alpha}, |\alpha| = |\beta| = k$. If in certain order we put all multiple indexes α with $|\alpha| = k$ into a row $\{\alpha^1, \dots, \alpha^{N_k}\}$, then $\{g_{\alpha\beta}(x)\}$ can be made into a symmetric matrix $(g_{\alpha^i\alpha^j})$. By this rule, we get a symmetric leading term matrix of (2.1), as follows:

$$M(x) = \left(a_{\alpha^i\alpha^j}(x) \right)_{i,j=1,\dots,N_m}. \tag{2.2}$$

Suppose that the matrix $M(x)$ is semipositive, that is,

$$0 \le a_{\alpha^i\alpha^j}(x) \xi_i \xi_j, \quad \forall x \in \overline{\Omega},\ \xi \in R^{N_m}, \tag{2.3}$$

and the odd order part of (2.1) can be written as

$$\sum_{|\alpha|=m,|\gamma|=m-1} (-1)^m D^\alpha \left(b_{\alpha\gamma}(x) D^\gamma u \right) = \sum_{i=1}^{n} \sum_{|\lambda|=|\theta|=m-1} (-1)^m D^{\lambda+\delta_i} \left(b^i_{\lambda\theta}(x) D^\theta u \right), \tag{2.4}$$

where $\delta_i = \{\delta_{i1}, \dots, \delta_{in}\}, \delta_{ij}$ is the Kronecker symbol. Assume that for all $1 \le i \le n$, we have

$$b^i_{\lambda\theta}(x) = b^i_{\theta\lambda}(x), \quad x \in \Omega. \tag{2.5}$$

We introduce another symmetric matrix

$$B(x) = \left(\sum_{k=1}^{n} b^k_{\lambda^i\lambda^j}(x) \cdot n_k \right)_{i,j=1,\dots,N_{m-1}}, \quad x \in \partial\Omega, \tag{2.6}$$

where $\overrightarrow{n} = \{n_1, n_2, \ldots, n_n\}$ is the outward normal at $x \in \partial\Omega$. Let the following matrices be orthogonal:

$$C^M(x) = \left(C_{ij}^M(x)\right)_{i,j=1,\ldots,N_m}, \quad x \in \Omega,$$
$$C^B(x) = \left(C_{ij}^B(x)\right)_{i,j=1,\ldots,N_{m-1}}, \quad x \in \partial\Omega, \tag{2.7}$$

satisfying

$$C^M(x)M(x)C^M(x)' = \left(e_i(x)\delta_{ij}\right)_{i,j=1,\ldots,N_m},$$
$$C^B(x)B(x)C^B(x)' = \left(h_i(x)\delta_{ij}\right)_{i,j=1,\ldots,N_{m-1}}, \tag{2.8}$$

where $C(x)'$ is the transposed matrix of $C(x), \{e_i(x)\}_{i=1}^{N_m}$ are the eigenvalues of $M(x)$ and $\{h_i(x)\}_{i=1}^{N_{m-1}}$ are the eigenvalues of $B(x)$. Denote by

$$\sum_i^M = \{x \in \partial\Omega \mid e_i(x) > 0\}, \quad 1 \le i \le N_m,$$
$$\sum_i^B = \{x \in \partial\Omega \mid h_i(x) > 0\}, \quad 1 \le i \le N_{m-1}, \tag{2.9}$$
$$\sum_i^C = \partial\Omega \setminus \sum_i^B, \quad 1 \le i \le N_{m-1}.$$

For multiple indices $\alpha, \beta, \alpha \le \beta$ means that $\alpha_i \le \beta_i$, for all $1 \le i \le n$. Now let us consider the following boundary value problem,

$$Lu = f(x), \quad x \in \Omega, \tag{2.10}$$

$$D^\alpha u|_{\partial\Omega} = 0, \quad |\alpha| \le m-2, \tag{2.11}$$

$$\sum_{j=1}^{N_{m-1}} C_{ij}^B(x) D^{\lambda^j} u|_{\sum_i^B} = 0, \quad \left|\lambda^j\right| = m-1, \ 1 \le i \le N_{m-1}, \tag{2.12}$$

$$\sum_{j=1}^{N_m} C_{ij}^M(x) D^{\alpha^j - \delta_{k_j}} u \cdot n_{k_j}|_{\sum_i^M} = 0, \tag{2.13}$$

for all $\delta_{k_j} \le \alpha^j$, $|\alpha^j| = m$ and $1 \le i \le N_m$, where $\delta_{k_j} = \{\underbrace{0, \ldots, 1}_{k_j}, \ldots, 0\}$.

We can see that the item (2.13) of boundary value condition is determined by the leading term matrix (2.2), and (2.12) is defined by the odd term matrix (2.6). Moreover, if the operator L is a not elliptic, then the operator

$$L'u = \sum_{|\theta|,|\lambda|\leq m-1} (-1)^{|\theta|} D^{\theta}\left(d_{\theta\lambda}(x) D^{\lambda} u\right) \tag{2.14}$$

has to be elliptic.

In order to illustrate the boundary value conditions (2.11)–(2.13), in the following we give an example.

Example 2.1. Given the differential equation

$$\frac{\partial^4 u}{\partial x_1^4} + \frac{\partial^4 u}{\partial x_1^2 \partial x_2^2} + \frac{\partial^3 u}{\partial x_2^3} - \Delta u = f, \quad x \in \Omega \subset R^2. \tag{2.15}$$

Here $\Omega = \{(x_1, x_2) \in R^2 \mid 0 < x_1 < 1,\ 0 < x_2 < 1\}$. Let $\alpha^1 = \{2,0\}$, $\alpha^2 = \{1,1\}$. $\alpha^3 = \{0,2\}$ and $\lambda^1 = \{1,0\}$, $\lambda^2 = \{0,1\}$, then the leading and odd term matrices of (2.15) respectively are

$$M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ 0 & n_2 \end{pmatrix}, \tag{2.16}$$

and the orthogonal matrices are

$$C^M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad C^B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{2.17}$$

We can see that $\sum_1^M = \partial\Omega$, $\sum_2^M = \partial\Omega$, $\sum_3^M = \phi$, and $\sum_1^B = \phi$, $\sum_2^B$ as shown in Figure 1.

The item (2.12) is

$$\sum_{j=1}^{2} C_{2j}^B D^{\lambda^j} u|_{\sum_2^B} = D^{\lambda^2} u|_{\sum_2^B} = \left.\frac{\partial u}{\partial x_2}\right|_{\sum_2^B} = 0, \tag{2.18}$$

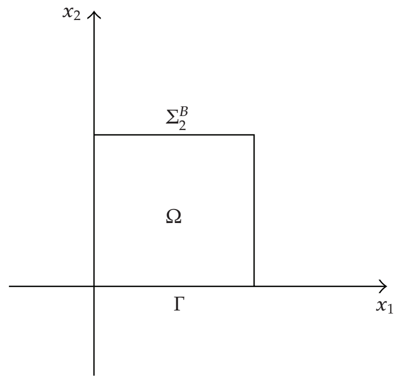

Figure 1

and the item (2.13) is

$$\begin{aligned}&\sum_{j=1}^{3} C_{1j}^{M} D^{\alpha^j-\delta_{k_j}} u \cdot n_{k_j}|_{\Sigma_1^M} = D^{\alpha^1-\delta_{k_1}} u \cdot n_{k_1}|_{\Sigma_1^M} = 0,\\ &\sum_{j=1}^{3} C_{2j}^{M} D^{\alpha^j-\delta_{k_j}} u \cdot n_{k_j}|_{\Sigma_2^M} = D^{\alpha^2-\delta_{k_2}} u \cdot n_{k_2}|_{\Sigma_2^M} = 0,\end{aligned} \tag{2.19}$$

for all $\delta_{k_1} \le \alpha^1$ and $\delta_{k_2} \le \alpha^2$. Since only $\delta_{k_1} = \{1,0\} \le \alpha^1 = \{2,0\}$, hence we have

$$D^{\alpha^1-\delta_{k_1}} u \cdot n_{k_1}|_{\Sigma_1^M} = \frac{\partial u}{\partial x_1} \cdot n_1|_{\partial\Omega} = 0, \tag{2.20}$$

however, $\delta_{k_2} = \{1,0\} < \alpha^2 = \{1,1\}$ and $\delta_{k_2} = \{0,1\} < \alpha^2$, therefore,

$$D^{\alpha^2-\delta_{k_2}} u \cdot n_{k_2}|_{\Sigma_2^M} = \begin{cases} \dfrac{\partial u}{\partial x_2} \cdot n_1|_{\partial\Omega} = 0, \\ \dfrac{\partial u}{\partial x_1} \cdot n_2|_{\partial\Omega} = 0. \end{cases} \tag{2.21}$$

Thus the associated boundary value condition of (2.15) is as follows:

$$u|_{\partial\Omega} = 0, \qquad \left.\frac{\partial u}{\partial x_2}\right|_{\partial\Omega/\Gamma} = 0, \qquad \left.\frac{\partial u}{\partial x_1}\right|_{\partial\Omega} = 0, \tag{2.22}$$

which implies that $\partial u/\partial x_2$ is free on $\Gamma = \{(x_1, x_2) \in \partial\Omega \mid 0 < x_1 < 1,\ x_2 = 0\}$.

Remark 2.2. In general the matrices $M(x)$ and $B(x)$ arranged are not unique, hence the boundary value conditions relating to the operator L may not be unique.

Remark 2.3. When all leading terms of L are zero, (2.10) is an odd order one. In this case, only (2.11) and (2.12) remain.

Now we return to discuss the relations between the conditions (2.11)–(2.13) with Dirichlet and Keldys-Fichera boundary value conditions.

It is easy to verify that the problem (2.10)–(2.13) is the Dirichlet problem provided the operator L being elliptic (see [11]). In this case, $\sum_i^M = \partial\Omega$ for all $1 \le i \le N_m$. Besides, (2.13) run over all $1 \le i \le N_m$ and $\delta_{k_j} \le \alpha^i$, moreover $C^B(x)$ is nondegenerate for any $x \in \partial\Omega$. Solving the system of equations, we get $D^\alpha u|_{\partial\Omega} = 0$, for all $|\alpha| = m - 1$.

When $m = 1$, namely, L is of second-order, the condition (2.12) is the form

$$u|_{\sum^B} = 0, \quad \sum^B = \left\{ x \in \partial\Omega \mid \sum_{i=1}^n b_i(x) n_i > 0 \right\}, \tag{2.23}$$

and (2.13) is

$$\sum_{j=1}^n C_{ij}^M(x) n_j u|_{\sum_i^M} = 0, \quad 1 \le i \le n. \tag{2.24}$$

Noticing

$$\sum_{i,j=1}^n a_{ij}(x) n_i n_j = \sum_{i=1}^n e_i(x) \left(\sum_{j=1}^n C_{ij}^M(x) n_j \right)^2, \tag{2.25}$$

thus the condition (2.13) is the form

$$u|_{\sum^M} = 0, \quad \sum^M = \left\{ x \in \partial\Omega \mid \sum_{i,j=1}^n a_{ij}(x) n_i n_j > 0 \right\}. \tag{2.26}$$

It shows that when $m = 1$, (2.12) and (2.13) are coincide with Keldys-Fichera boundary value condition.

Next, we will give the definition of weak solutions of (2.10)–(2.13) (see [12]). Let

$$X = \left\{ v \in C^\infty\left(\overline{\Omega}\right) \mid D^\alpha v|_{\partial\Omega} = 0,\ |\alpha| \le m - 2,\ \text{and } v \ \text{ satisfy (2.13)},\ \|v\|_2 < \infty \right\}, \tag{2.27}$$

where $\|\cdot\|_2$ is defined by

$$\|v\|_2 = \left[\int_\Omega \sum_{|\alpha| \le m} |D^\alpha v|^2 dx + \int_{\partial\Omega} \sum_{|\gamma| = m-1} |D^\gamma v|^2 ds \right]^{1/2}. \tag{2.28}$$

We denote by X_2 the completion of X under the norm $\|\cdot\|_2$ and by X_1 the completion of X with the following norm

$$\|v\|_1 = \left[\int_\Omega \left(\sum_{|\alpha|=|\beta|=m} a_{\alpha\beta}(x) D^\alpha v D^\beta v + \sum_{|\gamma|\le m-1} |D^\gamma v|^2\right) dx \right. \\ \left. + \int_{\partial\Omega} \sum_{i=1}^{N_{m-1}} |h_i(x)| \left(\sum_{j=1}^{N_{m-1}} C_{ij}^B D^{\gamma^j} v\right)^2 ds \right]^{1/2}. \tag{2.29}$$

Definition 2.4. $u \in X_1$ is a weak solution of (2.10)–(2.13) if for any $v \in X_2$, the following equality holds:

$$\int_\Omega \left[\sum_{|\alpha|=|\beta|=m, |\gamma|=m-1} \left(a_{\alpha\beta}(x) D^\beta u + b_{\alpha\gamma}(x) D^\gamma u\right) D^\alpha v + \sum_{|\theta|,|\lambda|\le m-1} d_{\theta\lambda}(x) D^\lambda u D^\theta v\right] dx \\ - \sum_{i=1}^{N_{m-1}} \int_{\Sigma_i^C} h_i(x) \left(\sum_{j=1}^{N_{m-1}} C_{ij}^B D^{\gamma^j} u\right) \left(\sum_{j=1}^{N_{m-1}} C_{ij}^B D^{\gamma^j} v\right) ds = \int_\Omega f(x) v \, dx. \tag{2.30}$$

We need to check the reasonableness of the boundary value problem (2.10)–(2.13) under the definition of weak solutions, that is, the solution in the classical sense are necessarily the solutions in weak sense, and conversely when a weak solution satisfies certain regularity conditions, it will surely satisfy the given boundary value conditions. Here, we assume that all coefficients of L are sufficiently smooth.

Let u be a classical solution of (2.10)–(2.13). Denote by

$$\langle Lu, v\rangle = \int_\Omega Lu \cdot v \, dx, \quad \forall v \in X. \tag{2.31}$$

Thanks to integration by part, we have

$$\int_\Omega Lu \cdot v \, dx \\ = \int_\Omega \left[\sum_{|\alpha|=|\beta|=m, |\gamma|=m-1} \left(a_{\alpha\beta}(x) D^\beta u + b_{\alpha\gamma}(x) D^\gamma u\right) D^\alpha v + \sum_{|\theta|,|\lambda|\le m-1} d_{\theta\lambda}(x) D^\lambda u D^\theta v\right] dx \\ - \int_{\partial\Omega} \left[\sum_{|\alpha|=|\beta|=m} a_{\alpha\beta}(x) D^\beta u D^{\alpha-\delta_k} v \cdot n_k + \sum_{|\lambda|=|\theta|=m-1} \sum_{i=1}^{n} b_{\lambda\theta}^i(x) \cdot n_i D^\theta u D^\lambda v\right] ds. \tag{2.32}$$

Since $v \in X$, we have

$$\int_{\partial\Omega} \sum_{|\alpha|=|\beta|=m} a_{\alpha\beta}(x) D^{\beta} u D^{\alpha-\delta_k} v \cdot n_k ds = \int_{\partial\Omega} \sum_{i=1}^{N_m} e_i(x) \left(\sum_{j=1}^{N_m} C_{ij}^{M} D^{\alpha^j} u \right) \left(\sum_{j=1}^{N_m} C_{ij}^{M} D^{\alpha^j - \delta_{k_j}} v \cdot n_{k_j} \right) ds = 0. \tag{2.33}$$

Because u satisfies (2.12),

$$\begin{aligned} \int_{\partial\Omega} \sum_{|\lambda|=|\theta|=m-1} \sum_{i=1}^{n} b_{\lambda\theta}^{i}(x) \cdot n_i D^{\theta} u D^{\lambda} v \, ds &= \int_{\partial\Omega} \sum_{i=1}^{N_{m-1}} h_i(x) \left(\sum_{j=1}^{N_{m-1}} C_{ij}^{B} D^{\gamma^j} u \right) \left(\sum_{j=1}^{N_{m-1}} C_{ij}^{B} D^{\gamma^j} v \right) ds \\ &= \sum_{i=1}^{N_{m-1}} \int_{\Sigma_i^C} h_i(x) \left(\sum_{j=1}^{N_{m-1}} C_{ij}^{B} D^{\gamma^j} u \right) \left(\sum_{j=1}^{N_{m-1}} C_{ij}^{B} D^{\gamma^j} v \right) ds. \end{aligned} \tag{2.34}$$

From the three equalities above we obtain (2.30).

Let $u \in X_1$ be a weak solution of (2.10)–(2.13). Then the boundary value conditions (2.11) and (2.13) can be reflected by the space X_1. In fact, we can show that if $u \in X_1$, then u satisfies

$$\sum_{i=1}^{N_m} \int_{\Sigma_i^M} e_i(x) \left(\sum_{j=1}^{N_m} C_{ij}^{M} D^{\alpha^j - \delta_{k_j}} u \cdot n_{k_j} \right) \left(\sum_{j=1}^{N_m} C_{ij}^{M} D^{\alpha^j} v \right) ds = 0, \quad \forall v \in X_1 \cap W^{m+1,2}(\Omega). \tag{2.35}$$

Evidently, when $u \in X, v \in X_1 \cap W^{m+1,2}(\Omega)$, we have

$$\int_{\Omega} \sum_{|\alpha|=|\beta|=m} a_{\alpha\beta}(x) D^{\beta} u D^{\alpha} v \, dx = -\int_{\Omega} \sum_{|\alpha|=|\beta|=m} D_i \left(a_{\alpha\beta}(x) D^{\alpha} v \right) D^{\beta-\delta_i} u \, dx. \tag{2.36}$$

If we can verify that for any $u \in X_1$, (2.36) holds true, then we get

$$\int_{\partial\Omega} \sum_{|\alpha|=|\beta|=m} a_{\alpha\beta}(x) D^{\alpha} v D^{\beta-\delta_i} u \cdot n_i ds = 0, \tag{2.37}$$

which means that (2.35) holds true. Since X is dense in X_1, for $u \in X_1$ given, let $u_k \in X$ and $u_k \to u$ in X_1. Then

$$\lim_{k\to\infty}\int_\Omega \sum_{|\alpha|=|\beta|=m} a_{\alpha\beta}D^\beta u_k D^\alpha v\,dx = \int_\Omega \sum_{|\alpha|=|\beta|=m} a_{\alpha\beta}D^\beta u D^\alpha v\,dx,$$
$$\lim_{k\to\infty}\int_\Omega \sum_{|\alpha|=|\beta|=m} D_i\left(a_{\alpha\beta}D^\alpha v\right)D^{\beta-\delta_i}u_k dx = \int_\Omega \sum_{|\alpha|=|\beta|=m} D_i\left(a_{\alpha\beta}D^\alpha v\right)D^{\beta-\delta_i}u\,dx. \tag{2.38}$$

Due to u_k satisfying (2.36), hence $u \in X_1$ satisfies (2.36). Thus (2.31) is verified.

Remark 2.5. When (2.2) is a diagonal matrix, then (2.13) is the form

$$D^\gamma u|_{\sum_\gamma^M} = 0, \quad \text{for } |\gamma| = m-1, \tag{2.39}$$

where $\sum_\gamma^M = \{x \in \partial\Omega \mid \sum_{i=1}^n a_{\gamma+\delta_{i\gamma}+\delta_i}(x)\cdot n_i^2 > 0\}$. In this case, the corresponding trace embedding theorem can be set, and the boundary value condition (2.13) is naturally satisfied. On the other hand, if the weak solution u of (2.10)–(2.13) belongs to $X_1 \cap W^{m,p}(\Omega)$ for some $p > 1$, then by the trace embedding theorems, the condition (2.13) also holds true.

It remains to verify the condition (2.12). Let $u_0 \in X_1 \cap W^{m+1,2}(\Omega)$ satisfy (2.30). Since $W^{m+1,2}(\Omega) \hookrightarrow X_2$, hence we have

$$\int_\Omega \left[\sum_{|\alpha|=|\beta|=m,|\gamma|=m-1}\left(a_{\alpha\beta}(x)D^\beta u_0 + b_{\alpha\gamma}(x)D^\gamma u_0\right)D^\alpha u_0 + \sum_{|\theta|,|\lambda|\le m-1} d_{\theta\lambda}(x)D^\lambda u_0 D^\theta u_0 - f u_0\right]ds$$
$$-\sum_{i=1}^{N_{m-1}}\int_{\sum_i^C} h_i(x)\left(\sum_{j=1}^{N_{m-1}} C_{ij}^B D^{\gamma_j}u_0\right)^2 ds = 0. \tag{2.40}$$

On the other hand, by (2.30), for any $v \in C_0^\infty(\Omega)$, we get

$$\int_\Omega \left[-\sum_{|\alpha|=|\beta|=m} D_i\left(a_{\alpha\beta}(x)D^\alpha u_0\right)D^{\beta-\delta_i}v + \sum_{|\theta|,|\lambda|\le m-1} d_{\theta\lambda}(x)D^\lambda u_0 D^\theta v\right.$$
$$\left. -fv - D_i\left(\sum_{|\theta|=|\gamma|=m-1} b_{\theta\gamma}^i(x)D^\gamma u_0\right)D^\theta v\right]dx = 0. \tag{2.41}$$

Because the coefficients of L are sufficiently smooth, and C_0^∞ is dense in $W_0^{m-1,2}(\Omega)$, equality (2.41) also holds for any $v \in W_0^{m-1,2}(\Omega)$. Therefore, due to $u_0 \in W_0^{m-1,2}(\Omega)$, we have

$$\begin{aligned}\int_\Omega \Bigg[&- \sum_{|\alpha|=|\beta|=m} D_i\left(a_{\alpha\beta}(x) D^\alpha u_0\right) D^{\beta-\delta_i} u_0 + \sum_{|\theta|,|\lambda|\le m-1} d_{\theta\lambda}(x) D^\lambda u_0 D^\theta u_0 \\ &- f u_0 - D_i\left(\sum_{|\theta|=|\gamma|=m-1} b^i_{\theta\gamma}(x) D^\gamma u_0 \right) D^\theta u_0 \Bigg] dx = 0.\end{aligned} \tag{2.42}$$

From (2.36), one drives

$$-\int_\Omega \sum_{|\alpha|=|\beta|=m} D_i\left(a_{\alpha\beta}(x) D^\alpha u_0\right) D^{\beta-\delta_i} u_0 dx = \int_\Omega \sum_{|\alpha|=|\beta|=m} a_{\alpha\beta}(x) D^\alpha u_0 D^\beta u_0 \, dx, \tag{2.43}$$

Furthermore,

$$\begin{aligned}&-\int_\Omega D_i\left(\sum_{|\theta|=|\gamma|=m-1} b^i_{\theta\gamma}(x) D^\gamma u_0 \right) D^\theta u_0 dx \\ &= \int_\Omega \sum_{|\alpha|=m,|\gamma|=m-1} b_{\alpha\gamma}(x) D^\gamma u_0 D^\alpha u_0 dx - \sum_{i=1}^{N_{m-1}} \int_{\Sigma_i^C \cup \Sigma_i^B} h_i(x) \left(\sum_{j=1}^{N_{m-1}} C^B_{ij} D^{\gamma^j} u_0 \right)^2 ds.\end{aligned} \tag{2.44}$$

From (2.30) and (2.42), one can see that

$$\sum_{i=1}^{N_{m-1}} \int_{\Sigma_i^B} h_i(x) \left(\sum_{j=1}^{N_{m-1}} C^B_{ij} D^{\gamma^j} u_0 \right)^2 ds = 0. \tag{2.45}$$

Noticing $h_i(x) > 0$ in $\sum_i^B$, one deduces that u_0 satisfies (2.12) provided $u_0 \in X_1 \cap W^{m+1,2}(\Omega)$. Finally, we discuss the well-posedness of the boundary value problem (2.10)–(2.13).

Let X be a linear space, and X_1, X_2 be the completion of X, respectively, with the norm $\|\cdot\|_1, \|\cdot\|_2$. Suppose that X_1 is a reflexive Banach space and X_2 is a separable Banach space.

Definition 2.6. A mapping $G : X_1 \to X_2{}^*$ is called to be weakly continuous, if for any $x_n, x_0 \in X_1, x_n \rightharpoonup x_0$ in X_1, one has

$$\lim_{n\to\infty} \langle G x_n, y\rangle = \langle G x_0, y\rangle, \quad \forall y \in X_2. \tag{2.46}$$

Lemma 2.7 (see [3]). *Suppose that* $G : X_1 \to X_2{}^*$ *is a weakly continuous, if there exists a bounded open set* $\Omega \subset X_1$, *such that*

$$\langle Gu, u\rangle \ge 0, \quad \forall u \in \partial\Omega \cap X, \tag{2.47}$$

then the equation $Gu = 0$ *has a solution in* X_1.

Theorem 2.8 (existence theorem). *Let $\Omega \subset R^n$ be an arbitrary open set, $f \in L^2(\Omega)$ and $b_{\alpha\gamma} \in C^1(\overline{\Omega})$. If there exist a constant $C > 0$ and $g \in L^1(\Omega)$ such that*

$$C \sum_{|\gamma|=m-1} \left|\xi_\gamma\right|^2 + C|\xi_i|^2 - g \leq \sum_{|\lambda|,|\theta|\leq m-1} d_{\theta\lambda}(x)\xi_\theta\xi_\lambda - \frac{1}{2}\sum_{i=1}^{n} \sum_{|\gamma|=|\beta|=m-1} D_i b^i_{\gamma\beta}(x)\xi_\gamma\xi_\beta, \tag{2.48}$$

where ξ_α is the component of $\xi \in R^{N_{m-1}}$ corresponding to $D^\alpha u$, then the problem (2.10)–(2.13) *has a weak solution in X_1.*

Proof. Let $\langle Lu, v\rangle$ be the inner product as in (2.31). It is easy to verify that $\langle Lu, v\rangle$ defines a bounded linear operator $L : X_1 \to {X_2}^*$. Hence L is weakly continuous (see [3]). From (2.42), for $u \in X$ we drive that

$$\begin{aligned}
\langle Lu, u\rangle = &\int_\Omega \left[\sum_{|\alpha|=|\beta|=m} a_{\alpha\beta}(x)D^\alpha u D^\beta u + \sum_{i=1}^{n}\sum_{|\lambda|=|\theta|=m-1} b^i_{\lambda\theta}(x)D^\theta u D^{\lambda+\delta_i}u \right.\\
&\left. + \sum_{|\gamma|,|\alpha|\leq m-1} d_{\gamma\alpha}(x)D^\gamma u D^\alpha u\right] dx\\
&- \sum_{i=1}^{N_{m-1}} \int_{\sum_i^C} h_i(x)\left(\sum_{j=1}^{N_{m-1}} C^B_{ij} D^{\gamma^j} u\right)^2 ds\\
= &\int_\Omega \left[\sum_{|\alpha|=|\beta|=m} a_{\alpha\beta}(x)D^\alpha u D^\beta u + \sum_{|\gamma|,|\alpha|\leq m-1} d_{\gamma\alpha}(x)D^\gamma u D^\alpha u \right.\\
&\left. -\frac{1}{2}\sum_{i=1}^{n}\sum_{|\gamma|=|\beta|=m-1} D_i b^i_{\gamma\beta}(x)D^\gamma u D^\beta u\right] dx\\
&+\frac{1}{2}\sum_{i=1}^{N_{m-1}}\left[\int_{\sum_i^B} - \int_{\sum_i^C} h_i(x)\left(\sum_{j=1}^{N_{m-1}} C^B_{ij}D^{\gamma^j}u\right)^2\right] ds\\
\geq &\int_\Omega \left[\sum_{|\alpha|=|\beta|=m} a_{\alpha\beta}(x)D^\alpha u D^\beta u + C\sum_{|\gamma|=m-1}|D^\gamma u|^2 + Cu^2 - g(x)\right]\\
&+\frac{1}{2}\sum_{i=1}^{N_{m-1}}\left[\int_{\sum_i^B\cup\sum_i^C} |h_i(x)|\left(\sum_{j=1}^{N_{m-1}} C^B_{ij}(x)D^{\gamma^j}u\right)^2\right] ds.
\end{aligned}\tag{2.49}$$

Hence we obtain

$$\langle Lu, u\rangle \geq C\|u\|_1^2 - C, \quad \forall u \in X. \tag{2.50}$$

Thus by Hölder inequality (see [13]), we have

$$\langle Lu - f, u\rangle \geq 0, \quad \forall u \in X,\ \|u\|_1 = R \text{ great enough.} \tag{2.51}$$

By Lemma 2.7, the theorem is proven. □

Theorem 2.9 (uniqueness theorem). *Under the assumptions of Theorem 2.8 with* $g(x) = 0$ *in* (2.48)*. If the problem* (2.10)–(2.13) *has a weak solution in* $X_1 \cap W^{m,p}(\Omega) \cap W^{m-1,q}(\Omega)((1/p) + (1/q) = 1)$*, then such a solution is unique. Moreover, if* $b_{\alpha\gamma}(x) = 0$ *in* L*, for all* $|\alpha| = m,\ |\gamma| = m-1$*, then the weak solution* $u \in X_1$ *of* (2.10)–(2.13) *is unique.*

Proof. Let $u_0 \in X_1 \cap W^{m,p}(\Omega) \cap W^{m-1,q}$ be a weak solution of (2.10)–(2.13). We can see that (2.30) holds for all $v \in X_1 \cap W^{m,p} \cap W^{m-1,q}(\Omega)$. Hence Lu_0, u_0 is well defined. Let $u_1 \in X_1 \cap W^{m,p} \cap W^{m-1,q}(\Omega)$. Then from (2.49) it follows that $< Lu_1 - Lu_0, u_1 - u_0 >= 0$, we obtain $u_1 = u_0$, which means that the solution of (2.10)–(2.13) in $X_1 \cap W^{m,p} \cap W^{m-1,q}(\Omega)$ is unique. If all the odd terms $b_{\alpha\gamma}(x)$ of L, then (2.30) holds for all $v \in X_1$, in the same fashion we known that the weak solution of (2.10)–(2.13) in X_1 is unique. The proof is complete. □

Remark 2.10. In next subsection, we can see that under certain assumptions, the weak solutions of degenerate elliptic equations are in $X_1 \cap W^{m,p}(\Omega) \cap W^{m-1,q}(\Omega)((1/p) + (1/q) = 1)$.

3. Existence of Higher-Order Quasilinear Equations

Given the quasilinear differential operator

$$\begin{aligned} Au = &\sum_{|\alpha|=|\beta|=m,|\gamma|=m-1} (-1)^m D^\alpha \left(a_{\alpha\beta}\left(x, \bigwedge u\right) D^\beta u + b_{\alpha\gamma}(x) D^\gamma u\right) \\ &+ \sum_{|\gamma|=|\theta|=m-1} (-1)^{m-1} D^\gamma \left(d_{\gamma\theta}\left(x, \bigwedge u\right) D^\theta u\right) \\ &+ \sum_{|\lambda|\leq m-1} (-1)^{|\lambda|} D^\lambda g_\lambda\left(x, \bigwedge u\right), \end{aligned} \tag{3.1}$$

where $m \geq 2$ and $\bigwedge u = \{D^\alpha u\}_{|\alpha|\leq m-2}$.

Let $a_{\alpha\beta}(x,\xi) = a_{\beta\alpha}(x,\xi)$, the odd order part of (3.1) be as that in (2.4), $b_{\alpha\gamma} \in C^1(\overline{\Omega})$, and $\sum_i^B \sum_i^C$, be the same as those in Section 2. The leading matrix is

$$M(x,\xi) = (a_{\alpha^i\alpha^j}(x,\xi))_{i,j=1,\ldots,N_m}, \tag{3.2}$$

and the eigenvalues are $\{e_i(x,\xi)\}_{i=1}^{N_m}$. We denote $\sum_i^M = \{x \in \partial\Omega \mid e_i(x,0) > 0\},\ 1 \leq i \leq N_m$.

We consider the following problem:

$$
\begin{gathered}
Au = f(x), \quad x \in \Omega, \\
\bigwedge u|_{\partial\Omega} = 0, \\
\sum_{j=1}^{N_{m-1}} C_{ij}^{B}(x) D^{\lambda^j} u|_{\sum_i^B} = 0, \quad \left|\lambda^j\right| = m-1, \ 1 \le i \le N_{m-1}, \\
\sum_{j=1}^{N_m} C_{ij}^{M}(x,0) D^{\alpha^j - \delta_{k_j}} u \cdot n_{k_j}|_{\sum_i^M} = 0, \quad \forall \delta_{k_j} \le \alpha^j, \\
\text{with } \left|\alpha^j\right| = m, \quad 1 \le i \le N_m, \quad \delta_{k_j} = \Bigg\{ \underbrace{0,\ldots,1}_{k_j},\ldots,0 \Bigg\}.
\end{gathered}
\tag{3.3}
$$

Denote the anisotropic Sobolev space by

$$
W_{|\alpha| \le k}^{p_\alpha}(\Omega) = \left\{ u \in L^{p_0}(\Omega) \mid p_0 \ge 1, \ D^\alpha u \in L^{p_\alpha}(\Omega), \ \ \forall 1 \le |\alpha| \le k, \ \text{and} \ \ p_\alpha \ge 1, \ \text{or} \ p_\alpha = 0 \right\},
\tag{3.4}
$$

whose norm is

$$
\|u\| = \sum_{|\alpha| \le k} \operatorname{sign} p_\alpha \|D^\alpha u\|_{L^{p_\alpha}},
\tag{3.5}
$$

when all $p_\alpha = p$ for $|\alpha| = k$, then the space is denoted by $W_{k,|\alpha| \le k-1}^{p,p_\alpha}(\Omega)$. $q_\theta(|\theta| \le k)$ is termed the critical embedding exponent from $W_{k,|\alpha|\le k}^{p_\alpha}(\Omega)$ to $L^p(\Omega)$, if q_θ is the largest number of the exponent p in where $D^\theta u \in L^p(\Omega)$, for all $u \in W_{|\alpha|\le k}^{p_\alpha}(\Omega)$, and the embedding is continuous.

For example, when Ω is bounded, the space $X = \{u \in L^k(\Omega) \mid k \ge 1, D_i u \in L^2(\Omega), 1 \le i \le n\}$ with norm $\|u\| = \|\nabla u\|_{L^2} + \|u\|_{L^k}$ is an anisotropic Sobolev space, and the critical embedding exponents from X to $L^P(\Omega)$ are $q_i = 2(1 \le i \le n)$, and $q_0 = \max\{k, 2n/(n-2)\}$.

Suppose that the following hold.

(A_1) The coefficients of the leading term of A satisfy one of the following two conditions:

(1) $a_{\alpha\beta}(x,\eta) = a_{\alpha\beta}(x)$;

(2) $a_{\alpha\beta}(x,\eta) = 0$, as $\alpha \ne \beta$.

(A_2) There is a constant $M > 0$ such that

$$
\begin{aligned}
0 \le M \sum_{|\alpha|=|\beta|=m} a_{\alpha\beta}(x,0)\xi_\alpha\xi_\beta &\le \sum_{|\alpha|=|\beta|=m} a_{\alpha\beta}(x,\eta)\xi_\alpha\xi_\beta \\
&\le M^{-1} \sum_{|\alpha|=|\beta|=m} a_{\alpha\beta}(x,0)\xi_\alpha\xi_\beta.
\end{aligned}
\tag{3.6}
$$

(A_3) There are functions $G_i(x,\eta)$ $(i = 0,1,\dots,n)$ with $G_i(x,0) = 0$, for all $1 \le i \le n$, such that

$$\sum_{|\gamma|=m-1} g_\gamma\left(x,\bigwedge u\right) D_\gamma u = \sum_{i=1}^{n} D_i G_i\left(x,\bigwedge u\right) = G_0\left(x,\bigwedge u\right). \tag{3.7}$$

(A_4) There is a constant $C > 0$ such that

$$\begin{gathered} C|\xi|^2 \le \sum_{|\alpha|=|\beta|=m-1} \left[d_{\alpha\beta}(x)\xi_\alpha\xi_\beta - \frac{1}{2}\sum_{i=1}^{n} D_i b^i_{\alpha\beta}(x)\xi_\alpha\xi_\beta \right], \\ C\sum_{|\lambda|\le m-1} \operatorname{sign} p_\lambda |\eta_\lambda|^{p_\lambda} - f_1 \le \sum_{|\theta|\le m-2} g_\theta(x,\eta)\eta_\theta + G_0(x,\eta), \end{gathered} \tag{3.8}$$

where $f_1 \in L^1(\Omega)$, $p_0 > 1$, $p_\lambda > 1$ or $p_\lambda = 0$, for all $1 \le |\lambda| \le m-2$.

(A_5) There is a constant $c > 0$ such that

$$\begin{gathered} \left|a_{\alpha\beta}(x,\eta)\right| \le C, \\ \left|d_{\gamma\theta}(x,\eta)\right| \le C\left[\sum_{|\beta|\le m-2} |\eta_\beta|^{S_\beta} + 1\right], \\ \left|g_\gamma(x,\eta)\right| \le C\left[\sum_{|\beta|\le m-2} |\eta_\beta|^{\overline{S}_\beta} + 1\right], \end{gathered} \tag{3.9}$$

where $1 \le S_\beta < q_{\beta/2}$, $1 \le \overline{S}_\beta < q_\beta$, q_β is a critical embedding exponent from $W^{2,p_\lambda}_{m-1,|\lambda|\le m-1}(\Omega)$ to $L^P(\Omega)$. Let X be defined by (2.27) and X_1 be the completion of X under the norm

$$\begin{aligned} \|v\|_1 = \Bigg[&\int_\Omega \left(\sum_{|\alpha|=|\beta|=m} a_{\alpha\beta}(x,0) D^\alpha v D^\beta v + \sum_{|\gamma|=m-1} |D^\gamma v|^2 \right) dx \\ &+ \int_{\partial\Omega} \sum_{i=1}^{N_{m-1}} |h_i(x)| \left(\sum_{j=1}^{N_{m-1}} C^B_{ij} D^{\gamma^j} v \right)^2 ds \Bigg]^{1/2} + \sum_{|\gamma|\le m-2} \operatorname{sign} p_\gamma \|D^\gamma v\|_{L^{p_\gamma}}, \end{aligned} \tag{3.10}$$

and X_2 be the completion of X with the norm

$$\|v\| = \|v\|_{W^{m,p}} + \|v\|_{W^{m,2}} + \left[\int_{\partial\Omega} \sum_{|\gamma|=m-1} |D^\gamma v|^2 ds\right]^{1/2}, \tag{3.11}$$

where $p \geq \max\{2, q_\beta/(q_\beta - \overline{S}_\beta),\ 2q_\beta/(q_\beta - 2S_\beta)\}$.

$u \in X_1$ is a weak solution of (3.3), if for any $v \in X_2$, we have

$$\begin{aligned}
&\int_\Omega \left[\sum_{|\alpha|=|\beta|=m} a_{\alpha\beta}\left(x, \bigwedge u\right) D^\beta u D^\alpha v + \sum_{|\alpha|=m,|\gamma|=m-1} b_{\alpha\gamma}(x) D^\gamma u D^\alpha v\right. \\
&\quad \left. + \sum_{|\gamma|=|\theta|=m-1} d_{\gamma\theta}\left(x, \bigwedge u\right) D^\theta u D^\gamma v + \sum_{|\lambda|\leq m-1} g_\lambda\left(x, \bigwedge u\right) D^\lambda v - fv\right] dx \\
&\quad - \sum_{i=1}^{N_{m-1}} \int_{\sum_i^C} h_i(x) \left(\sum_{j=1}^{N_{m-1}} C_{ij}^B D^{\gamma^j} u\right)\left(\sum_{j=1}^{N_{m-1}} C_{ij}^B D^{\gamma^j} v\right) ds = 0.
\end{aligned} \tag{3.12}$$

Theorem 3.1. *Under the conditions* (A_1)–(A_5), *if* $f \in L^{p_0'}(\Omega)$, $(1/p_0 + 1/p_0') = 1$, *then the problem* (3.3) *has a weak solution in* X_1.

Proof. Denote by $\langle Au, v\rangle$ the left part of (3.12). It is easy to verify that the inner product $\langle Au, v\rangle$ defines a bounded mapping $A : X_1 \to X_2^*$ by the condition (A_5).

Let $u \in X$, by (A_2)–(A_4), one can deduce that

$$\begin{aligned}
\langle Au, u\rangle \geq &\int_\Omega \left[M \sum_{|\alpha|=|\beta|=m} a_{\alpha\beta}(x, 0) D^\alpha u D^\beta u + C \sum_{|\gamma|=m-1} |D^\gamma u|^2 + C \sum_{|\theta|\leq m-2} \left|D^\theta u\right|^{p_\theta}\right] dx \\
&+ \frac{1}{2}\sum_{i=1}^{N_{m-1}} \left[\left[\int_{\sum_i^B} - \int_{\sum_i^C} h_i(x) \left(\sum_{j=1}^{N_{m-1}} C_{ij}^B(x) D^{\gamma^j} u\right)^2\right] ds - \int_\Omega \left[fu + |f_1|\right] dx.\right.
\end{aligned} \tag{3.13}$$

Noticing that $h_i|_{\sum_i^B} > 0$, $h_i|_{\sum_i^C} \leq 0$, $\sum_i^B \cup \sum_i^C = \partial\Omega$, by Hölder and Young inequalities (see[13]), from (3.13) we can get

$$\langle Au, u\rangle \geq 0, \quad \forall u \in X,\ \|u\|_{X_1} \text{ large enough}. \tag{3.14}$$

Ones can easily show that the mapping $A : X_1 \to X_2^*$ is weakly continuous. Here we omit the details of the proof. By Lemma 2.7, this theorem is proven. □

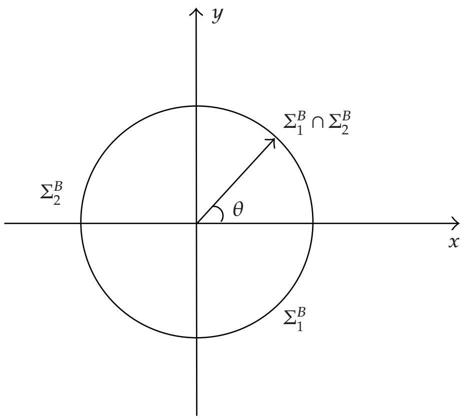

Figure 2

In the following, we take an example to illustrate the application of Theorem 3.1.

Example 3.2. We consider the boundary value problem of odd order equation as follows:

$$\frac{\partial^3 u}{\partial x^3} + \frac{\partial^3 u}{\partial y^3} - \Delta u + u^3 = f(x,y), \quad (x,y) \in \Omega \subset R^2, \tag{3.15}$$

where Ω is an unit ball in R^2, see Figure 2

The odd term matrix is

$$B(x,y) = \begin{pmatrix} n_x & 0 \\ 0 & n_y \end{pmatrix} = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}. \tag{3.16}$$

It is easy to see that

$$\begin{gathered}\sum_1^B = \{x \in \partial\Omega \mid n_x = x > 0\} = \left\{-\frac{\pi}{2} < \theta < \frac{\pi}{2}\right\}, \\ \sum_2^B = \left\{x \in \partial\Omega \mid n_y = y > 0\right\} = \{0 < \theta < \pi\}.\end{gathered} \tag{3.17}$$

The boundary value condition associated with (3.15) is

$$\begin{gathered} u|_{\partial\Omega} = 0, \\ \left.\frac{\partial u}{\partial x}\right|_{\Sigma_1^B} = \frac{\partial u}{\partial x}(\cos\theta, \sin\theta) = 0, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}, \\ \left.\frac{\partial u}{\partial x}\right|_{\Sigma_2^B} = \frac{\partial u}{\partial x}(\cos\theta, \sin\theta) = 0, \quad 0 < \theta < \pi. \end{gathered} \tag{3.18}$$

Applying Theorem 3.1, if $f \in L^{4/3}(\Omega)$, then the problem (3.15)–(3.18) has a weak solution $u \in W^{1,2}(\Omega)$.

4. $W^{m,p}$-Solutions of Degenerate Elliptic Equations

We start with an abstract regularity result which is useful for the existence problem of $W^{m,p}(\Omega)$-solutions of degenerate quasilinear elliptic equations of order $2m$. Let X, X_1, X_2 be the spaces defined in Definition 2.6, and Y be a reflective Banach space, at the same time $Y \hookrightarrow X_1$.

Lemma 4.1. *Under the hypotheses of Lemma 2.7, there exists a sequence of* $\{u_n\} \subset X$, $u_n \rightharpoonup u_0$ *in* X_1 *such that* $\langle Gu_n, u_n \rangle = 0$. *Furthermore, if, we can derive that* $\|u\|_Y < C$, C *is a constant, then the solution* u_0 *of* $Gu = 0$ *belongs to* Y.

In the following, we give some existence theorems of $W^{m,p}$-solutions for the boundary value conditions (4.3)–(4.5) of higher-order degenerate elliptic equations.

First, we consider the quasilinear equations

$$\begin{aligned} \widetilde{A}u = &\sum_{|\alpha|=|\beta|=m,|\gamma|=m-1} (-1)^m D^\alpha \left(a_{\alpha\beta}\left(x, \widetilde{D}u\right) D^\beta u + b_{\alpha\gamma}(x) D^\gamma u \right) \\ &+ \sum_{|\gamma|\le m-1} (-1)^{|\gamma|} D^\gamma g_\gamma\left(x, \widetilde{D}u\right) = f(x), \quad x \in \Omega, \end{aligned} \tag{4.1}$$

where $\widetilde{D}u = \{D^\alpha u\}_{|\alpha|\le m-1}$. Now, we consider the following problem

$$\widetilde{A}u = f(x), \quad x \in \Omega, \tag{4.2}$$

$$\widetilde{D}u|_{\partial\Omega} = 0, \tag{4.3}$$

$$\sum_{j=1}^{N_{m-1}} C_{ij}^B(x) D^{\lambda^j} u|_{\Sigma_i^B} = 0, \quad \left|\lambda^j\right| = m-1, \ 1 \le i \le N_{m-1}, \tag{4.4}$$

$$\begin{aligned} &\sum_{j=1}^{N_m} C_{ij}^M(x,0) D^{\alpha^j - \delta_{k_j}} u \cdot n_{k_j}|_{\Sigma_i^M} = 0, \quad \forall \delta_{k_j} \le \alpha^j, \\ &\left|\alpha^j\right| = m, \ 1 \le i \le N_m, \ \delta_{k_j} = \Bigg\{ \underbrace{0, \dots, 1}_{k_j}, \dots, 0 \Bigg\}. \end{aligned} \tag{4.5}$$

The boundary value condition associated with (4.1) is given by (4.3)–(4.5). Suppose that $\Omega \subset R^n$ is bounded, and the following assumptions hold.

(B_1) The condition (3.6) holds, and there is a continuous function $\lambda(x) \geq 0$ on $\overline{\Omega}$ such that

$$\lambda(x)|\xi|^{2m} \leq \sum_{|\alpha|=|\beta|=m} a_{\alpha\beta}(x,0)\xi^{\alpha}\xi^{\beta}, \quad \forall \xi \in R^n, \tag{4.6}$$

where $\xi^{\alpha} = \xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n}$, $\alpha = (\alpha_1, \dots, \alpha_n)$.

(B_2) $\Omega' = \{x \in \Omega \mid \lambda(x) = 0\}$ is a measure zero set in R^n, and there is a sequence of subdomains Ω_k with cone property such that $\Omega_k \subset\subset \Omega/\Omega'$, $\Omega_k \subset \Omega_{k+1}$ and $\cup_k \Omega_k = \Omega/\Omega'$.

(B_3) The positive definite condition is

$$C \sum_{|\lambda|\leq m-1} |\xi_\lambda|^{p_\lambda} - f_1 \leq \sum_{|\theta|\leq m-1} g_\theta(x,\xi)\xi_\theta - \frac{1}{2}\sum_{i=1}^{n} \sum_{|\gamma|=|\alpha|=m-1} D_i b^i \xi_\alpha \xi_\gamma, \tag{4.7}$$

where C is a constant, $p_0 > 1$, $p_\lambda > 1$ or $p_\lambda = 0$ for $1 \leq |\lambda| \leq m-1$, $f_1 \in L^1(\Omega)$.

(B_4) The structure conditions are

$$\begin{gathered} \left|a_{\alpha\beta}(x,\xi)\right| \leq C, \\ \left|g_\gamma(x,\xi)\right| \leq C\left[\sum_{|\theta|\leq m-1} |\xi_\theta|^{S_\theta} + 1\right], \end{gathered} \tag{4.8}$$

where C is a constant, $0 \leq S_\theta < q_\theta$, q_θ is the critical embedding exponent from $W^{p_\lambda}_{|\lambda|\leq m-1}(\Omega)$ to $L^P(\Omega)$.

Let X be defined by (2.27) and $\widetilde{X}_1$ be the completion of X with the norm

$$\begin{aligned} \|u\| = &\left[\int_\Omega \sum_{|\alpha|=|\beta|=m} a_{\alpha\beta}(x,0) D^\alpha u D^\beta u \, dx\right]^{1/2} + \sum_{|\alpha|\leq m-1} \operatorname{sign} p_\alpha \|D^\alpha u\|_{L^{p^\alpha}} \\ &+ \left[\sum_{i=1}^{N_{m-1}} \int_{\partial\Omega} |h_i(x)| \left(\sum_{j=1}^{N_{m-1}} C_{ij}^B(x) D^{\gamma^j} u\right) ds\right]^{1/2}. \end{aligned} \tag{4.9}$$

Definition 4.2. $u \in \widetilde{X}_1$ is a weak solution of (4.2)–(4.5), if for any $v \in X_2$, the following equality holds:

$$\int_\Omega \left[\sum_{|\alpha|=|\beta|=m} a_{\alpha\beta}\left(x, \widetilde{D}u\right) D^\beta u D^\alpha v + \sum_{|\alpha|=m,|\gamma|=m-1} b_{\alpha\gamma}(x) D^\gamma u D^\alpha v + \sum_{|\gamma|\leq m-1} g_\gamma\left(x, \widetilde{D}u\right) D^\gamma v - fv \right] dx$$
$$- \sum_{i=1}^{N_{m-1}} \int_{\Sigma_i^C} h_i(x) \left(\sum_{j=1}^{N_{m-1}} C_{ij}^B D^{\gamma^j} u \right) \left(\sum_{j=1}^{N_{m-1}} C_{ij}^B D^{\gamma^j} v \right) ds = 0. \tag{4.10}$$

Theorem 4.3. *Under the assumptions* (B_1)–(B_4)*, if* $f \in L^{p_0'}$*, then the problem and* (4.2)–(4.5) *has a weak solution* $u \in \widetilde{X}_1$*. Moreover, if there is a real number* $\delta \geq 1$*, such that*

$$\int_\Omega |\lambda(x)|^{-\delta} dx < \infty, \tag{4.11}$$

then the weak solution $u \in W^{m,p}(\Omega) \cap \widetilde{X}_1$*,* $p = 2\delta/(1+\delta)$*.*

Proof. According to Lemma 4.1, it suffices to prove that there is a constant $C > 0$ such that for any $u \in X$ (X is as that in Section 3) with $\langle \widetilde{A}u, u \rangle = 0$, we have

$$\|u\|_{W^{m,p}} \leq C, \quad p = \frac{2\delta}{1+\delta}. \tag{4.12}$$

From (4.10) we know

$$\left\langle \widetilde{A}u, u \right\rangle = \int_\Omega \left[\sum_{|\alpha|=|\beta|=m} a_{\alpha\beta}\left(x, \widetilde{D}u\right) D^\beta u D^\alpha u + \sum_{|\alpha|=m,|\gamma|=m-1} b_{\alpha\gamma}(x) D^\gamma u D^\alpha u \right.$$
$$\left. + \sum_{|\gamma|\leq m-1} g_\gamma\left(x, \widetilde{D}u\right) D^\gamma u - fu \right] dx \tag{4.13}$$
$$- \sum_{i=1}^{N_{m-1}} \int_{\Sigma_i^C} h_i(x) \left(\sum_{j=1}^{N_{m-1}} C_{ij}^B D^{\gamma^j} u \right)^{1/2} ds, \quad x \in X_1.$$

Due to (B_1) and (B_3) we have

$$\begin{aligned}\left\langle \widetilde{A}u,u\right\rangle &= \int_\Omega \Bigg[\sum_{|\alpha|=|\beta|=m} a_{\alpha\beta}\left(x,\widetilde{D}u\right)D^\beta u D^\alpha u + \sum_{i=1}^{n}\sum_{|\alpha|=|\gamma|=m-1} b^i_{\alpha\gamma(x)} D^\gamma u D^{\alpha+\delta_i}u \\ &\quad + \sum_{|\gamma|\le m-1} g_\gamma\left(x,\widetilde{D}u\right)D^\gamma u - fu\Bigg]dx \\ &\quad - \sum_{i=1}^{N_{m-1}}\int_{\Sigma_i^C} h_i(x)\left(\sum_{j=1}^{N_{m-1}} C^B_{ij}(x)D^{\gamma^j}u\right)^2 ds \\ &= \int_\Omega \Bigg[\sum_{|\alpha|=|\beta|=m} a_{\alpha\beta}\left(x,\widetilde{D}u\right)D^\beta u D^\alpha u - \frac{1}{2}\sum_{i=1}^{n}\sum_{|\alpha|=|\gamma|=m-1} D_i b^i_{\alpha\gamma}(x) D^\gamma u D^{\alpha}u \\ &\quad + \sum_{|\gamma|\le m-1} g_\gamma\left(x,\widetilde{D}u\right)D^\gamma u - fu\Bigg]dx \\ &\quad - \sum_{i=1}^{N_{m-1}}\int_{\Sigma_i^C} h_i(x)\left(\sum_{j=1}^{N_{m-1}} C^B_{ij}(x)D^{\gamma^j}u\right)^2 ds \\ &\ge \int_\Omega\left[\lambda(x)|\nabla u|^{2m} + C\sum_{|\theta|\le m-1}\left|D^\theta u\right|^{p_\theta}\right]dx - \int_\Omega [fu + |f_1|]dx \\ &\quad + \frac{1}{2}\sum_{i=1}^{N_{m-1}}\left[\int_{\Sigma_i^B-\Sigma_i^C} h_i(x)\left(\sum_{j=1}^{N_{m-1}} C^B_{ij}(x)D^{\gamma^j}u\right)^2\right]ds. \end{aligned} \tag{4.14}$$

Noticing that $h_i|_{\Sigma_i^B} > 0$, $h_i|_{\Sigma_i^C} \le 0, \Sigma_i^B \cap \Sigma_i^C = \partial\Omega$, and $f \in L^{p_0'}$ consequently we have

$$\begin{aligned}&\varepsilon\int_\Omega |u|^{p_0'}dx + \int_\Omega\left[C_1|f|^{p_0'} + |f_1|\right]dx \\ &\quad\ge \int_\Omega [fu + |f_1|]dx \ge \int_\Omega\left[\lambda(x)|\nabla u|^{2m} + C\sum_{|\theta|\le m-1}\left|D^\theta u\right|^{P_\theta}\right]dx,\end{aligned} \tag{4.15}$$

where the $p_\theta > 1$ or $p_\theta = 0$, p_θ is the critical embedding exponent from $W^{p_\theta}_{|\theta|\le m-1(\Omega)}$ to $L^p(\Omega)$. By the reversed Hölder inequality (see [14])

$$\int_\Omega \lambda(x)|\nabla u|^{2m} \ge \left[\int_\Omega |\lambda(x)|^{-\delta}dx\right]^{-1/\delta}\left[\int_\Omega |\nabla u|^{2m\delta/(1+\delta)}dx\right]^{(1+\delta)/\delta}. \tag{4.16}$$

Then we obtain

$$C \geq \int_{\Omega} \left[\lambda(x) |\nabla u|^{2m} + C \sum_{|\theta| \leq m-1} \left| D^{\theta} u \right|^{p_{\theta}} \right] dx. \tag{4.17}$$

From (4.15) and (4.17), the estimates (4.12) follows. This completes the proof. □

Next, we consider a quasilinear equation

$$\begin{aligned} &\sum_{|\alpha|=|\beta|=m, |\gamma|=m-1} (-1)^m D^{\alpha} \left(a_{\alpha\beta}(x, \square u) D^{\beta} u + b_{\alpha\beta}(x) D^{\gamma} u \right) \\ &\quad + \sum_{|\gamma| \leq m-1} (-1)^{|\gamma|} D^{\gamma} g_{\gamma}(x, \square u) = f(x), \quad x \in \Omega, \end{aligned} \tag{4.18}$$

where $\square u = \{u, \ldots, D^m u\}$.

Suppose that the following holds.

(B_4') There is a real number $\delta \geq 1$ such that

$$\int_{\Omega} |\lambda(x)|^{-\delta} dx < \infty. \tag{4.19}$$

(B_5') The structural conditions are

$$\begin{aligned} \left| a_{\alpha\beta}(x, \eta) \right| &\leq C, \\ \left| g_{\gamma}(x, \xi) \right| &\leq C \left[\sum_{|\theta| \leq m-1} |\xi_{\theta}|^{S_{\gamma}\theta} + \sum_{|\alpha|=m} |\xi_{\alpha}|^{t_{\gamma}} + 1 \right], \end{aligned} \tag{4.20}$$

where C is a constant, $0 \leq S_{\gamma\theta} < ((q_{\gamma} - 1)/q_{\gamma}) q_{\theta}$, $0 \leq t_{\gamma} < p(q_{\gamma} - 1)/q_{\gamma}$, $p = 2\delta/(1+\delta)$, q_{γ}, q_{θ} are the critical embedding exponents from $W^{p_{\lambda}}_{|\lambda \leq m-1|}(\Omega)$ to $L^q\Omega$.

Theorem 4.4. *Let the conditions* (B_1)–(B_3) *and* $(B_4'), (B_5')$ *be satisfied. If* $f \in L^{p_0'}(\Omega)$, *then the problem* (4.2)–(4.5) *has a weak solution* $u \in W^{m,p}(\Omega) \cap \widetilde{X}_1, p = 2\delta/(1+\delta)$.

The proof of Theorem 4.4 is parallel to that of Theorem 4.3; here we omit the detail.

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