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Research Article

Approximate Controllability of a Reaction-Diffusion System with a Cross-Diffusion Matrix and Fractional Derivatives on Bounded Domains

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We study the following reaction-diffusion system with a cross-diffusion matrix and fractional derivatives $u_t = a_1\Delta u + a_2\Delta v - c_1(-\Delta)^{\alpha_1}u - c_2(-\Delta)^{\alpha_2}v + 1_\omega f_1(x,t)$ in $\Omega\times]0,t^*[$, $v_t = b_1\Delta u + b_2\Delta v - d_1(-\Delta)^{\beta_1}u - d_2(-\Delta)^{\beta_2}v + 1_\omega f_2(x,t)$ in $\Omega\times]0,t^*[$, $u = v = 0$ on $\partial\Omega\times]0,t^*[$, $u(x,0) = u_0(x)$, $v(x,0) = v_0(x)$ in $x \in \Omega$, where $\Omega \subset \mathbb{R}^N (N \geq 1)$ is a smooth bounded domain, $u_0, v_0 \in L^2(\Omega)$, the diffusion matrix $M = \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix}$ has semisimple and positive eigenvalues $0 < \rho_1 \leq \rho_2$, $0 < \alpha_1, \alpha_2, \beta_1, \beta_2 < 1$, $\omega \subset \Omega$ is an open nonempty set, and 1_ω is the characteristic function of ω. Specifically, we prove that under some conditions over the coefficients $a_i, b_i, c_i, d_i (i = 1, 2)$, the semigroup generated by the linear operator of the system is exponentially stable, and under other conditions we prove that for all $t^* > 0$ the system is approximately controllable on $[0, t^*]$.

1. Introduction

In this paper we prove controllability for the following reaction-diffusion system with cross diffusion matrix:

$$\begin{gathered}
u_t = a_1\Delta u + a_2\Delta v - c_1(-\Delta)^{\alpha_1}u - c_2(-\Delta)^{\alpha_2}v + 1_\omega f_1(x,t) \quad \text{in } \Omega\times]0,t^*[, \\
v_t = b_1\Delta u + b_2\Delta v - d_1(-\Delta)^{\beta_1}u - d_2(-\Delta)^{\beta_2}v + 1_\omega f_2(x,t) \quad \text{in } \Omega\times]0,t^*[, \\
u = v = 0 \quad \text{on } \partial\Omega\times]0,t^*[, \\
u(x,0) = u_0(x), \quad v(x,0) = v_0(x) \quad \text{in } x \in \Omega,
\end{gathered} \tag{1.1}$$

where ω is an open nonempty set of Ω and 1_ω is the characteristic function of ω.

We assume the following assumptions.

(H1) Ω is a smooth bounded domain in $\mathbb{R}^N$ $(N \geq 1)$.

(H2) The diffusion matrix $M = \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix}$ has semisimple and positive eigenvalues $0 < \rho_1 \leq \rho_2$.

(H3) c_j, d_j $(j = 1, 2)$ are real constants, α_j, β_j $(j = 1, 2)$ are real constants belonging to the interval $]0, 1[$.

(H4) $u_0, v_0 \in L^2(\Omega)$.

(H5) The distributed controls $f_1, f_2 \in L^2([0, t^*]; L^2(\Omega))$.

Specifically, we prove the following statements.

(i) If $c_2 = d_1 = 0$ and $\min\{c_1 + \lambda_1^{1-\alpha_1}\rho_1, d_2 + \lambda_1^{1-\beta_2}\rho_1\} > 0$, where λ_1 is the first eigenvalue of $-\Delta$ with Dirichlet condition, or if $c_2 \neq 0$, $d_1 \neq 0$, $c_1 \geq 0$, and $d_2 \geq 0$; then, under the hypotheses (H1)–(H3), the semigroup generated by the linear operator of the system is exponentially stable.

(ii) If $c_2 = d_1 = 0$ and under the hypotheses (H1)–(H5), then, for all $t^* > 0$ and all open nonempty subset ω of Ω the system is approximately controllable on $[0, t^*]$.

This paper has been motivated by the work done in [1] and the work done by H. Larez and H. Leiva in [2]. In the work [1], the auther studies the asymptotic behavior of the solution of the system

$$
\begin{aligned}
u_t &= a\frac{\partial^2 u}{\partial x^2} + \beta\frac{\partial u}{\partial x} + b\frac{\partial^2 v}{\partial x^2} + f(t, u, v), \quad x \in \mathbb{R},\ t > 0, \\
v_t &= c\frac{\partial^2 u}{\partial x^2} + d\frac{\partial^2 v}{\partial x^2} + \beta\frac{\partial v}{\partial x} + g(t, u, v), \quad x \in \mathbb{R},\ t > 0
\end{aligned}
\tag{1.2}
$$

supplemeted with the initial conditions

$$
u(x, 0) = u_0(x), \quad v(x, 0) = v_0(x), \quad x \in \mathbb{R}. \tag{1.3}
$$

The author proved that in the Banach space $X \times X$ where $X = C_{\text{ub}}(\mathbb{R})$ is the space of bounded uniformly continuous real valued functions on $\mathbb{R}$, if f and g are locally Lipshitz and under some conditions over the coefficients a, b, c, d, β, and if $u_0, v_0 \in C_+ = \{u \in C_{\text{ub}}(\mathbb{R}) : \lim_{x\to+\infty} u(x) \text{ exist}\}$, then $u(t)$, $v(t) \in C_+$ for all $t < t_{\max}$. Moreover, $U(t) = \lim_{x\to+\infty} u(x)$ and $V(t) = \lim_{x\to+\infty} v(x)$ satisfy the system of ordinary differential equations

$$
\begin{aligned}
U'(t) &= f(t, U(t), V(t), \\
V'(t) &= g(t, U(t), V(t)
\end{aligned}
\tag{1.4}
$$

with the initial data

$$U(0) = \lim_{x\to+\infty} u_0(x), \qquad V(0) = \lim_{x\to+\infty} v_0(x). \tag{1.5}$$

The same result holds for $C_- = \{u \in C_{\text{ub}}(\mathbb{R}) : \lim_{x\to-\infty} u(x) \text{ exist}\}$.

In the work done in [2], the authers studied the system (1.1) with $c_2 = d_1 = 0$, $c_1 = d_2$, and $\alpha_1 = \beta_2 = 1/2$. They proved that if the diffusion matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ has semi-simple and positive eigenvalues $0 < \rho_1 \le \rho_2$, $f_1, f_2 \in L^2([0,\tau[; L^2(\Omega))$, then if $\lambda_1^{1/2}\rho_1 + \beta > 0$ (λ_1 is the first eigenvalue of $-\Delta$), the system is approximately controllable on $[0,\tau]$ for all open nonempty subset ω of Ω.

2. Notations and Preliminaries

In the following we denote by

$\mathcal{M}_2(\mathbb{R})$ the set of 2×2 matrices with entries from $\mathbb{R}$,

$L^2(\Omega)$ the set of all measurable functions $u : \Omega \to \mathbb{R}$ such that $\int_\Omega |u|^2 dx < \infty$,

$H^1(\Omega)$ the set of all the functions $u \in L^2(\Omega)$ that have generalized derivatives $\partial u/\partial x_j \in L^2(\Omega)$ for all $j = 1, \dots, N$,

$H_0^1(\Omega)$ the closure of the set $C_0^\infty(\Omega)$ in the Hilbert space $H^1(\Omega)$,

$H^2(\Omega)$ the set of all the functions $u \in L^2(\Omega)$ that have generalized derivatives $\partial u/\partial x_j, \partial^2 u/\partial x_j \partial x_k \in L^2(\Omega)$ for all $j, k = 1, \dots, N$.

We will use the following results.

Theorem 2.1 (cf. [3]). *Let us consider the following classical boundary-eigenvalue problem for the laplacien:*

$$\begin{aligned} -\Delta u &= \lambda u, \quad \text{on } \Omega, \\ u &= 0, \quad \text{on } \partial\Omega, \end{aligned} \tag{2.1}$$

where Ω is a nonempty bounded open set in $\mathbb{R}^N$ and $D(-\Delta) = H^2(\Omega) \cap H_0^1(\Omega)$.

This problem has a countable system of eigenvalues $0 < c \le \lambda_1 < \lambda_2 < \cdots < \lambda_j < \cdots$ and $\lambda_j \to +\infty$ as $j \to \infty$.

(i) All the eigenvalues λ_j have finite multiplicity m_j equal to the dimension of the corresponding eigenspace S_j.

(ii) Let $\{\varphi_{jk}\}_{k=1}^{m_j}$ be a basis of the S_j for every j, then the eigenvectors $\{\varphi_{jk}\}_{k=1,j=1}^{m_j,\infty}$ form a complete orthonormal system in the space $L^2(\Omega)$. Hence for all $u \in L^2(\Omega)$ we have $u = \sum_{j=1}^{\infty} \sum_{k=1}^{m_j} \langle u, \varphi_{jk}\rangle \varphi_{jk}$. If we put $E_j u = \sum_{k=1}^{m_j} \langle u, \varphi_{jk}\rangle \varphi_{jk}$ then we get $u = \sum_{j=1}^{\infty} E_j u$.

(iii) Also, the eigenfunctions $\{\varphi_{jk}\}_{k=1,j=1}^{m_j,\infty} \subset C_0^\infty(\Omega)$, where $C_0^\infty(\Omega)$ is the space of infinitely continuously differentiable functions on Ω and compactly supported in Ω.

(iv) For all $u \in D(-\Delta)$ we have $-\Delta u = \sum_{j=1}^{\infty} \lambda_j E_j u$.

(v) The operator Δ generates an analytic semigroup $\{T_\Delta(t)\}$ on $L^2(\Omega)$ defined by

$$T_\Delta(t)u = \sum_{j=1}^{\infty} e^{-\lambda_j t} E_j u. \tag{2.2}$$

Definition 2.2. Let $0 < \alpha < 1$ a real number, the operator $(-\Delta)^\alpha$ is defined by

$$\begin{gathered}(-\Delta)^\alpha : D\big((-\Delta)^\alpha\big) \subset L^2(\Omega) \longrightarrow L^2(\Omega), \\ D\big((-\Delta)^\alpha\big) = \left\{ u \in L^2(\Omega) \mid \sum_{j=1}^{\infty} \sum_{k=1}^{m_j} \left| \lambda_j^\alpha \langle u, \varphi_{jk} \rangle \right|^2 < \infty \right\}, \\ (-\Delta)^\alpha u = \sum_{j=1}^{\infty} \sum_{k=1}^{m_j} \lambda_j^\alpha \langle \varphi_{jk}, u \rangle \varphi_{jk}.\end{gathered} \tag{2.3}$$

In particular, we obtain $\varphi_{jk} \in D((-\Delta)^\alpha)$ and $(-\Delta)^\alpha \varphi_{jk} = \lambda_j^\alpha \varphi_{jk}$. Since $\{\varphi_{jk}\}_{k=1,j=1}^{m_j,\infty}$ form a complete orthonormal system in the space $L^2(\Omega)$, then it is dense in $L^2(\Omega)$, and hence $D(-\Delta)^\alpha$ is dense in $L^2(\Omega)$.

Proposition 2.3 (cf. [4]). *Let X be a Hilbert separable space and $\{A_j\}_{j\geq 1}$ and $\{P_j\}_{j\geq 1}$ two families of bounded linear operators in X, with $\{P_j\}_{j\geq 1}$ a family of complete orthogonal projections such that $A_j P_j = P_j A_j$, $j \geq 1$.*

Define the following family of linear operators $S(t)w = \sum_{j=1}^{\infty} e^{A_j t} P_j w$, $w \in X$, $t \geq 0$. Then

(a) *$S(t)$ is a linear and bounded operator if $\|e^{A_j t}\| \leq g(t)$, $j \geq 1$ with $g(t) \geq 0$, continiuous for $t \geq 0$,*

(b) *under the above condition (a), $\{S(t)\}_{t\geq 0}$ is a strongly continiuous semigroup in the Hilbert space X, whose infinitesimal generator A is given by*

$$Aw = \sum_{j=1}^{\infty} A_j P_j w, \quad w \in D(A), \quad D(A) = \left\{ w \in X \mid \sum_{j=1}^{\infty} \left\| A_j P_j w \right\|^2 < \infty \right\}. \tag{2.4}$$

Theorem 2.4 (cf. [5]). *Suppose Ω is connected, f is a real function in Ω, and $f = 0$ on a nonempty open subset of Ω. Then $f \equiv 0$ in Ω.*

3. Abstract Formulation of the Problem

In this section we consider the following notations.

(i) $X = L^2(\Omega) \times L^2(\Omega)$. X is a Hilbert space with the inner product

$$\langle (u_1, v_1), (u_2, v_2) \rangle = \langle u_1, u_2 \rangle + \langle v_1, v_2 \rangle. \tag{3.1}$$

(ii) We define

$$A_{11}(u,v) = a_1\Delta u + a_2\Delta v - c_1(-\Delta)^{\alpha_1}u - c_2(-\Delta)^{\alpha_2}v,$$
$$A_{12}(u,v) = b_1\Delta u + b_2\Delta v - d_1(-\Delta)^{\beta_1}u - d_2(-\Delta)^{\beta_2}v. \tag{3.2}$$

(iii) Let $w = (u,v)$, then we can define the linear operator

$$A : D(A) \subset X \longrightarrow X,$$
$$D(A) = \left(H^2(\Omega;\mathbb{R}) \cap H_0^1(\Omega;\mathbb{R})\right)^2, \tag{3.3}$$
$$Aw = -\left(M\Delta - c_1B_1(-\Delta)^{\alpha_1} - c_2B_2(-\Delta)^{\alpha_2} - d_1B_3(-\Delta)^{\beta_1} - d_2B_4(-\Delta)^{\beta_2}\right)w,$$

where

$$M = \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix}, \qquad B_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad B_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$
$$B_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad B_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \tag{3.4}$$

Therefore, for all $w \in D(A)$

$$A_{11}(u,v) = a_1\sum_{j=1}^{\infty}\lambda_jE_ju + a_2\sum_{j=1}^{\infty}\lambda_jE_jv + c_1\sum_{j=1}^{\infty}\lambda_j^{\alpha_1}E_ju + c_2\sum_{j=1}^{\infty}\lambda_j^{\alpha_2}E_jv,$$
$$A_{12}(u,v) = b_1\sum_{j=1}^{\infty}\lambda_jE_ju + b_2\sum_{j=1}^{\infty}\lambda_jE_jv + d_1\sum_{j=1}^{\infty}\lambda_j^{\beta_1}E_ju + d_2\sum_{j=1}^{\infty}\lambda_j^{\beta_2}E_jv. \tag{3.5}$$

If we put

$$P_j = \begin{pmatrix} E_j & 0 \\ 0 & E_j \end{pmatrix}, \quad j = 1,2, \tag{3.6}$$

then (3.3) can be written as

$$Aw \equiv \begin{pmatrix} A_{11}(u,v) \\ A_{12}(u,v) \end{pmatrix} = \sum_{j=1}^{\infty}\left(\lambda_jM + \lambda_j^{\alpha_1}c_1B_1 + \lambda_j^{\alpha_2}c_2B_2 + \lambda_j^{\beta_1}d_1B_3 + \lambda_j^{\beta_2}d_2B_4\right)P_jw, \tag{3.7}$$

and we have for all $w \in X$

$$w = \sum_{j=1}^{\infty}P_jw, \qquad \|w\|^2 = \sum_{j=1}^{\infty}\left\|P_jw\right\|^2. \tag{3.8}$$

Consequently, system (1.1) can be written as an abstract differential equation in the Hilbert space X in the following form:

$$\begin{aligned} \dot{w} &= -Aw + B_\omega f(t), \quad \text{in } \Omega \times]0, t^*[, \\ w &= 0, \quad \text{on }]0, t^*[\times \partial\Omega, \\ w(0) &= w_0, \quad \text{in } x \in \Omega, \end{aligned} \tag{3.9}$$

where $f \equiv \operatorname{col}(f_1, f_2) \in L^2([0,T]; X)$) and $B_\omega = \begin{pmatrix} 1_\omega & 0 \\ 0 & 1_\omega \end{pmatrix}$ is a bounded linear operator from U into X.

4. Main Results

4.1. Generation of a C_0-Semigroup

Theorem 4.1. *If $c_2 = d_1 = 0$, then, under hypotheses (H1)–(H3), the linear operator $-A$ defined by* (3.3) *is the infinitesimal generator of strongly continuous semigroup $\{S(t)\}_{t\ge 0}$ given by*

$$S(t)w = \sum_{j=1}^{\infty} e^{A_j t} P_j w, \quad w \in X, \tag{4.1}$$

where

$$M_j = -\lambda_j M - \lambda_j^{\alpha_1} c_1 B_1 - \lambda_j^{\alpha_2} c_2 B_2 - \lambda_j^{\beta_1} d_1 B_3 - \lambda_j^{\beta_2} d_2 B_4, \tag{4.2}$$

$$A_j = M_j P_j. \tag{4.3}$$

Moreover, if

$$\min\left\{ c_1 + \lambda_1^{1-\alpha_1} \rho_1, d_2 + \lambda_1^{1-\beta_2} \rho_1 \right\} > 0, \tag{4.4}$$

then the C_0-semigoup $\{S(t)\}_{t\ge 0}$ is exponentially stable, that is, there exist two positives constants c, δ such that

$$\|S(t)\| \le c e^{-\delta t}, \quad \textit{for all } t \ge 0. \tag{4.5}$$

Proof. In order to apply the Proposition 2.3, we observe that $-A$ can be written as follows:

$$-Aw = \sum_{j=1}^{\infty} A_j P_j w, \quad w \in D(A), \tag{4.6}$$

where

$$A_j = -\left(\lambda_j M + \lambda_j^{\alpha_1} c_1 B_1 + \lambda_j^{\alpha_2} c_2 B_2 + \lambda_j^{\beta_1} d_1 B_3 + \lambda_j^{\beta_2} d_2 B_4\right) P_j. \tag{4.7}$$

Therefore, $A_j = M_j P_j$ and $A_j P_j = P_j A_j$.

Now, we have to verify condition (a) of the Proposition 2.3. We shall suppose that $0 < \rho_1 < \rho_2$. Then, there exists a set $\{Q_1, Q_2\} \in [\mathcal{M}_2(\mathbb{R})]^2$ of complementary projections on $\mathbb{R}^2$ such that

$$e^{Mt} = e^{\rho_1 t} Q_1 + e^{\rho_2 t} Q_2. \tag{4.8}$$

If $G = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}$ is the matrix passage from the canonical basis of $\mathbb{R}^2$ to the basis composed with the eigenvectors of M, then

$$Q_1 = \frac{1}{\rho_1 \rho_2} \begin{pmatrix} g_{11} g_{22} & -g_{11} g_{12} \\ g_{21} g_{22} & -g_{12} g_{21} \end{pmatrix}, \qquad Q_2 = \frac{1}{\rho_1 \rho_2} \begin{pmatrix} -g_{12} g_{21} & g_{11} g_{12} \\ -g_{21} g_{22} & g_{11} g_{22} \end{pmatrix}. \tag{4.9}$$

Hence,

$$e^{-\lambda_j M t} = e^{-\lambda_j \rho_1 t} Q_1 + e^{-\lambda_j \rho_2 t} Q_2. \tag{4.10}$$

We have also

$$\begin{gathered} e^{-\lambda_j^{\alpha_1} c_1 B_1 t} = \begin{pmatrix} e^{-\lambda_j^{\alpha_1} c_1 t} & 0 \\ 0 & 1 \end{pmatrix}, \qquad e^{-\lambda_j^{\alpha_2} c_2 B_2 t} = \begin{pmatrix} 1 & -\lambda_j^{\alpha_2} c_2 t \\ 0 & 1 \end{pmatrix}, \\ e^{-\lambda_j^{\beta_1} d_1 B_3 t} = \begin{pmatrix} 1 & 0 \\ -\lambda_j^{\beta_1} d_1 t & 1 \end{pmatrix}, \qquad e^{-\lambda_j^{\beta_1} d_2 B_4 t} = \begin{pmatrix} 1 & 0 \\ 0 & e^{-\lambda_j^{\beta_2} d_2 t} \end{pmatrix}. \end{gathered} \tag{4.11}$$

From (4.10)-(4.11) into (4.7) we obtain

$$e^{A_j t} = \left(e^{-\lambda_j \rho_1 t} Q_1 + e^{-\lambda_j \rho_2 t} Q_2\right) K_j(t) P_j, \tag{4.12}$$

where

$$K_j(t) = \begin{pmatrix} e^{-\lambda_j^{\alpha_1} c_1 t} + \lambda_j^{\alpha_2 + \beta_1} c_2 d_1 t^2 e^{-\lambda_j^{\alpha_1} c_1 t} & -\lambda_j^{\alpha_2} c_2 t e^{-(\lambda_j^{\alpha_1} c_1 + \lambda_j^{\beta_2} d_2) t} \\ -\lambda_j^{\beta_1} d_1 t & e^{-\lambda_j^{\beta_2} d_2 t} \end{pmatrix}. \tag{4.13}$$

As $c_2 = d_1 = 0$ we get

$$K_j(t) = \begin{pmatrix} e^{-\lambda_j^{\alpha_1} c_1 t} & 0 \\ 0 & e^{-\lambda_j^{\beta_2} d_2 t} \end{pmatrix}. \tag{4.14}$$

As $\lambda_j \to +\infty$ as $j \to \infty$, then this implies the existence of a positive number c and a real number δ such that $\|e^{A_j t}\| \leq ce^{\delta t}$, for every $j \geq 1$. Therefore $-A$ is a strongly continious semigroup $\{S(t)\}_{t\geq 0}$ given by (4.1). We can even estimate the constants c and δ as follows.

(i) If $\min\{c_1 + \lambda_1^{1-\alpha_1}\rho_1, d_2 + \lambda_1^{1-\beta_2}\rho_1\} \leq 0$. As $\lim_{j\to\infty}\{-\lambda_j^{\alpha_1}(c_1 + \lambda_j^{1-\alpha_1}\rho_1)\} = \lim_{j\to\infty}\{-\lambda_j^{\beta_2}(c_1 + \lambda_j^{1-\beta_2}\rho_1)\} = -\infty$, then there exist constants

$$\begin{aligned} \delta_1 &= \max\left\{-\lambda_j^{\alpha_1}\left(c_1 + \lambda_j^{1-\alpha_1}\rho_1\right) \mid \lambda_j^{\alpha_1}\left(c_1 + \lambda_j^{1-\alpha_1}\rho_1\right) \leq 0,\ j \geq 1\right\}, \\ \delta_2 &= \max\left\{-\lambda_j^{\beta_2}\left(d_2 + \lambda_j^{1-\beta_2}\rho_1\right) \mid \lambda_j^{\beta_2}\left(d_2 + \lambda_j^{1-\beta_2}\rho_1\right) \leq 0,\ j \geq 1\right\}, \end{aligned} \tag{4.15}$$

hence, if we put

$$\delta = \max\{\delta_1, \delta_2\} \geq 0, \tag{4.16}$$

$$c_0 = \frac{1}{\rho_1\rho_2}\max\{|g_{11}g_{22}|, |g_{11}g_{12}|, |g_{21}g_{22}|, |g_{12}g_{21}|\}, \tag{4.17}$$

we easily obtain

$$\left\|e^{A_j t}\right\| \leq 4c_0 e^{-\delta t}, \quad j \geq 1. \tag{4.18}$$

(ii) If $\min\{c_1 + \lambda_1^{1-\alpha_1}\rho_1, d_2 + \lambda_1^{1-\beta_2}\rho_1\} > 0$. If we put

$$\delta = \min\left\{\lambda_1^{\alpha 1}\left(c_1 + \lambda_1^{1-\alpha 1}\rho_1\right), \lambda_1^{\beta_2}\left(d_2 + \lambda_1^{1-\beta_2}\rho_1\right)\right\} > 0, \tag{4.19}$$

then we find that

$$\left\|e^{A_j t}\right\| \leq 4c_0 e^{-\delta t}, \quad j \geq 1. \tag{4.20}$$

Therefore, the linear operator $-A$ generates a strongly continuous semigroup $\{S(t)\}_{t\geq 0}$ on X given by expression (4.1).

Finally, if $\min\{c_1 + \lambda_1^{1-\alpha_1}\rho_1, d_2 + \lambda_1^{1-\beta_2}\rho_1\} > 0$, we have already proved (4.20). Using (4.20) into (4.1) we get that the C_0-semigoup $\{S(t)\}_{t\geq 0}$ is exponentially stable. The expression (4.5) is verfied with $c = 4c_0$ and δ is defined by (4.19). □

Theorem 4.2. *If*

$$c_2 \neq 0, \qquad d_1 \neq 0, \qquad c_1 \geq 0, \qquad d_2 \geq 0, \tag{4.21}$$

then, under the hypotheses (H1)–(H3), the linear operator $-A$ defined by (3.3) *is the infinitesimal generator of strongly continuous semigroup exponentially stable $\{S(t)\}_{t\geq 0}$ defined by* (4.1). *Specially, there exist two positives constants c, δ such that*

$$\|S(t)\| \leq ce^{-\delta t}, \quad \forall t \geq 0. \tag{4.22}$$

To prove this result, we need the following lemma.

Lemma 4.3. *For every two real positives constants c and λ, one has for every $0 < \delta < \lambda/c$*

$$cte^{-\lambda t} \leq \frac{1}{e(\lambda/c - \delta)} e^{-\delta ct}, \quad \forall t \geq 0, \tag{4.23}$$

and for every $0 < \delta < \lambda/\sqrt{c}$

$$ct^2 e^{-\lambda t} \leq \frac{4}{e^2(\lambda/\sqrt{c} - \delta)} e^{-\delta\sqrt{c}t}, \quad \forall t \geq 0. \tag{4.24}$$

Proof of Lemma 4.3. It is easy to verify that for every $\varepsilon > 0 : te^{-\varepsilon t} \leq 1/e\varepsilon$, for all $t \geq 0$.

Let $0 < \delta < \lambda/c$ and $\varepsilon = \lambda/c - \delta > 0$, then we get

$$te^{(-\lambda/ct)} \leq \frac{1}{e(\lambda/c - \delta)} e^{-\delta t}, \quad \forall t \geq 0. \tag{4.25}$$

Hence, we get (4.23).

Also, it is easy to verify that for every $\varepsilon > 0 : t^2 e^{-\varepsilon t} \leq 4/e^2\varepsilon^2$, for all $t \geq 0$. Let $0 < \delta < \lambda/\sqrt{c}$ and $\varepsilon = \lambda/\sqrt{c} - \delta > 0$, then we get

$$t^2 e^{-(\lambda\sqrt{c})t} \leq \frac{4}{e^2\left((\lambda/\sqrt{c}) - \delta\right)^2} e^{-\delta t}, \quad \forall t \geq 0. \tag{4.26}$$

Hence, from (4.26) we get $ct^2 e^{-\lambda t} = (\sqrt{c}t)^2 e^{-(\lambda/\sqrt{c})\sqrt{c}t} \leq 4/e^2(\lambda/\sqrt{c} - \delta)^2 e^{-\delta\sqrt{c}t}$ for all $t \geq 0$ and $0 < \delta < \lambda/\sqrt{c}$, which gives (4.24).

With the same manner we can prove that for every $0 < \delta < \lambda c^{-1/n}$ and every $n \in \mathbb{N}^*$ we have

$$t^n e^{-\lambda c^{-1/n} t} \leq \frac{n^n}{(e\varepsilon)^n} e^{-\delta t}, \quad \forall t \geq 0, \tag{4.27}$$

and consequently, for every two real positives constants c and λ and every $n \in \mathbb{N}^*$ we have

$$ct^n e^{-\lambda t} \leq \frac{n^n}{(e\varepsilon)^n} e^{-\delta c^{-1/n} t}, \quad \text{for all } t \geq 0 \text{ and every } 0 < \delta < \lambda. \tag{4.28}$$

Now, we are ready to prove Theorem 4.2. □

Proof of Theorem 4.2. By applying Proposition 2.3 we start from formula (4.12) and we put

$$K_j(t) = \begin{pmatrix} K_{11,j}(t) & K_{12,j}(t) \\ K_{21,j}(t) & K_{22,,j}(t) \end{pmatrix}, \tag{4.29}$$

where

$$\begin{aligned} K_{11,j}(t) &= e^{-\lambda_j^{\alpha_1} c_1 t} + \lambda_j^{\alpha_2+\beta_1} c_2 d_1 t^2 e^{-\lambda_j^{\alpha_1} c_1 t}, \qquad K_{12,j}(t) = -\lambda_j^{\alpha_2} c_2 t e^{-(\lambda_j^{\alpha_1} c_1 + \lambda_j^{\beta_2} d_2)t}, \\ K_{21,j}(t) &= -\lambda_j^{\beta_1} d_1 t, \qquad K_{22,,j}(t) = e^{-\lambda_j^{\beta_2} d_2 t}, \quad \forall j \geq 1. \end{aligned} \tag{4.30}$$

To estimate $e^{-\lambda_j \rho_1 t} K_{11,j}(t)$ we have in taking into account $c_1 \geq 0$

$$e^{-(\lambda_j \rho_1 + \lambda_j^{\alpha_1} c_1)t} \leq e^{-\lambda_1 \rho_1 t}, \quad \forall t \geq 0, \tag{4.31}$$

and applying the Lemma 4.3($c = \lambda_j^{\alpha_2+\beta_1} |c_2 d_1|$) we get

$$\begin{aligned} &\lambda_j^{\alpha_2+\beta_1} c_2 d_1 t^2 e^{-(\lambda_j \rho_1 + \lambda_j^{\alpha_1} c_1)t} \\ &\quad \leq \frac{4}{e^2\left(\left(\lambda_j^{1-(\alpha_2+\beta_1)/2} / \sqrt{|c_2 d_1|}\right)\rho_1 + \left(\lambda_j^{\alpha_1-(\alpha_2+\beta_1/2)} / \sqrt{|c_2 d_1|}\right)c_1 - \gamma_1\right)} e^{-\gamma_1 \lambda_j^{\alpha_2+\beta_1/2} \sqrt{|c_2 d_1|} t}, \end{aligned} \tag{4.32}$$

for all $t \geq 0$ and $0 < \gamma_1 < (\lambda_j^{1-(\alpha_2+\beta_1)/2} / \sqrt{|c_2 d_1|})\rho_1 + (\lambda_j^{\alpha_1-(\alpha_2+\beta_1)/2} / \sqrt{|c_2 d_1|})c_1$. But we have $(\lambda_j^{1-(\alpha_2+\beta_1)/2} / \sqrt{|c_2 d_1|})\rho_1 + (\lambda_j^{\alpha_1-(\alpha_2+\beta_1)/2} / \sqrt{|c_2 d_1|})c_1 \geq (\lambda_1^{1-(\alpha_2+\beta_1)/2} / \sqrt{|c_2 d_1|})\rho_1$, for all $j \geq 1$. Then we get for every $0 < \gamma_1 < (\lambda_1^{1-(\alpha_2+\beta_1)/2} / \sqrt{|c_2 d_1|})\rho_1$ that

$$\begin{aligned} &\lambda_j^{\alpha_2+\beta_1} c_2 d_1 t^2 e^{-(\lambda_j \rho_1 + \lambda_j^{\alpha_1} c_1)t} \\ &\quad \leq \frac{4}{e^2\left(\left(\lambda_1^{1-(\alpha_2+\beta_1)/2} / \sqrt{|c_2 d_1|}\right)\rho_1 - \delta_1\right)} e^{-\gamma_1 (\lambda_1^{(\alpha_2+\beta_1)/2} \sqrt{|c_2 d_1|})t}, \quad \forall t \geq 0. \end{aligned} \tag{4.33}$$

From (4.31)-(4.33) we get

$$e^{-\lambda_j \rho_1 t} K_{11,j}(t) \leq \sigma_1 e^{-\delta_1 t}, \quad \forall t \geq 0, \ j \geq 1, \tag{4.34}$$

where

$$\sigma_1 = 1 + 4\left(\frac{\lambda_1^{1-(\alpha_2+\beta_1)/2}}{\sqrt{|c_2 d_1|}}\rho_1 - \delta_1\right)^{-1}, \qquad \delta_1 = \min\left\{\lambda_1\rho_1, \gamma_1\lambda_1^{(\alpha_2+\beta_1)/2}\sqrt{|c_2 d_1|}\right\}, \tag{4.35}$$

and $0 < \gamma_1 < (\lambda_1^{1-(\alpha_2+\beta_1)/2}/\sqrt{|c_2 d_1|})\rho_1$.

Applying Lemma 4.3 and taking into account (4.21) we get with the same manner that for every $0 < \delta_2 < (\lambda_1^{1-\alpha_2}/|c_2|)\rho_1$

$$e^{-\lambda_j\rho_1 t}K_{12,j}(t) \le \sigma_2 e^{-\delta_2\lambda_1^{\alpha_2}|c_2|t}, \quad \forall t \ge 0,\ j \ge 1, \tag{4.36}$$

where

$$\sigma_2 = \frac{1}{e\left(\left(\lambda_1^{1-\alpha_2}/|c_2|\right)\rho_1 - \delta_2\right)}, \tag{4.37}$$

and or every $0 < \delta_3 < (\lambda_1^{1-\beta_1}/|d_1|)\rho_1$

$$e^{-\lambda_j\rho_1 t}K_{21,j}(t) \le \sigma_3 e^{-\delta_3\lambda_1^{\beta_1}|d_1|t}, \quad \forall t \ge 0,\ j \ge 1, \tag{4.38}$$

where

$$\sigma_3 = \frac{1}{e\left(\left(\lambda_1^{1-\beta_1}/|d_1|\right)\rho_1 - \delta_3\right)}, \tag{4.39}$$

$$e^{-\lambda_j\rho_1 t}K_{22,j}(t) \le e^{-\lambda_1\rho_1 t}, \quad \forall t \ge 0,\ j \ge 1. \tag{4.40}$$

From (4.34)-(4.40) into (4.12) we get

$$\left\|e^{A_j t}\right\| \le 4c_0\sigma e^{-\delta t}, \quad \forall t \ge 0,\ j \ge 1, \tag{4.41}$$

where c_0 is defined by (4.17) and

$$\sigma = 1 + \sigma_1 + \sigma_2 + \sigma_3, \quad 0 < \delta < \min\left\{\delta_1, \delta_2\lambda_1^{\alpha_2}|c_2|, \delta_3\lambda_1^{\beta_1}|d_1|, \lambda_1\rho_1\right\}. \tag{4.42}$$

Using (4.41) into (4.1) we get that the C_0-semigoup $\{S(t)\}_{t\ge 0}$ generated by $-A$ is exponentially stable. Expression (4.22) is verfied with $c = 4c_0\sigma$ and δ is defined by (4.42). □

4.2. Approximate Controllability

Befor giving the definition of the approximate controllabiliy for the sytem (3.9), we have the following known result: for all $w_0 \in X$ and $f \in L^2(]0,T[;U)$, the initial value problem (3.9) admits a unique mild solution given by

$$w(t) = S(t)w_0 + \int_0^t S(t-\tau)B_\omega f(\tau)d\tau, \quad t \in [0,T]. \tag{4.43}$$

This solution is denoted by $w(t;f)$.

Definition 4.4. System (3.9) is said to be *approximately controllable* at time t^* whenever the set $F_{t^*} = \{w(t^*;f) \mid \forall f \in L^2(]0,t^*[;U)\}$ is densely embedded in X; that is,

$$\forall w_0, w_1 \in X, \ \forall \varepsilon > 0; \ \exists f \in L^2(]0,t^*[;U) : \|w(t^*;f) - w_1\| < \varepsilon. \tag{4.44}$$

The following criteria for approximate controllability can be found in [6].

Criteria 1. System (3.9) is approximately controllable on $[0,t^*]$ if and only if

$$B^* S^*(t)w = 0, \quad \forall t \in [0,t^*] \Longrightarrow w = 0. \tag{4.45}$$

Now, we are ready to formulate the third main result of this work.

Theorem 4.5. *If the following condition*

$$c_2 = d_1 = 0 \tag{4.46}$$

is satisfied; then, under hypotheses (H1)–(H5), for all $t^* > 0$ *and all open subset* $\omega \subset \Omega$*, system* (3.9) *is approximately controllable on* $[0,t^*]$.

Proof. The proof of this theorem relies on the Criteria 1 and the following lemma. □

Lemma 4.6. *Let* $\{\alpha_{1j}\}_{j\geq 1}$, $\{\beta_{1j}\}_{j\geq 1}$ *and* $\{\alpha_{2j}\}_{j\geq 1}, \{\beta_{2j}\}_{j\geq 1}$ *be sequences of real numbers such that* $\alpha_{11} > \alpha_{12} > \alpha_{13} > \cdots$, $\alpha_{21} > \alpha_{22} > \alpha_{23} > \cdots$ *and* $\alpha_{1j} > \alpha_{2j}$*, for all* $j \geq 0$*, then for any* $t^* \in \mathbb{R}_+^*$ *one has*

$$\sum_{j=1}^{\infty} \left(e^{\alpha_{1j}t}\beta_{1j} + e^{\alpha_{2j}t}\beta_{2j}\right) = 0, \quad \forall t \in [0,t^*] \Longrightarrow \beta_{1j} = \beta_{2j} = 0, \ \forall j \geq 1. \tag{4.47}$$

Proof of Lemma 4.6. By analyticity we get $\sum_{j=1}^{\infty}(e^{\alpha_{1j}t}\beta_{1j} + e^{\alpha_{2j}t}\beta_{2j}) = 0$, $\forall t \geq 0$ and from this we get $\beta_{11} + \sum_{j=2}^{\infty} e^{(\alpha_{1j}-\alpha_{11})t}\beta_{1j} + \sum_{j=1}^{\infty} e^{(\alpha_{2j}-\alpha_{11})t}\beta_{2j} = 0$, $\forall t \geq 0$. Under the assumptions of the lemma we get $\sum_{j=2}^{\infty} e^{(\alpha_{1j}-\alpha_{11})t}\beta_{1j} + \sum_{j=1}^{\infty} e^{(\alpha_{2j}-\alpha_{11})t}\beta_{2j} \to 0$ as $t \to \infty$ and so $\beta_{11} = 0$. If $\alpha_{12} > \alpha_{21}$, we divide $\sum_{j=2}^{\infty} e^{\alpha_{1j}t}\beta_{1j} + \sum_{j=1}^{\infty} e^{\alpha_{2j}t}\beta_{2j} = 0$ by $e^{\alpha_{12}t}$ and we pass $t \to \infty$ we get $\beta_{12} = 0$. If $\alpha_{21} > \alpha_{12}$, we divide $\sum_{j=2}^{\infty} e^{\alpha_{1j}t}\beta_{1j} + \sum_{j=1}^{\infty} e^{\alpha_{2j}t}\beta_{2j} = 0$ by $e^{\alpha_{21}t}$ and we pass $t \to \infty$ and get $\beta_{21} = 0$. If $\alpha_{12} = \alpha_{21}$, we divide $\sum_{j=2}^{\infty} e^{\alpha_{1j}t}\beta_{1j} + \sum_{j=1}^{\infty} e^{\alpha_{2j}t}\beta_{2j} = 0$ by $e^{\alpha_{12}t}$ and we pass $t \to \infty$ and get $\beta_{12} + \beta_{21} = 0$. But in this we case we can integrate under the symbol of sommation over the intervall $[0, t]$ and we get $\beta_{12}e^{\alpha_{21}t} + \beta_{21}e^{\alpha_{12}t} = 0$. Hence $\beta_{12} = \beta_{21} = 0$. Continuing this way we see that $\beta_{1j} = \beta_{2j} = 0$, for all $j \geq 1$.

We are now ready to prove Theorem 4.5. For this purpose, we observe that

$$B_{\omega}^{*} = B_{\omega}, \qquad S^{*}(t)w = \sum_{j=1}^{\infty} e^{M_j^{*}t} P_j^{*} w, \quad w \in X, \ t \geq 0, \tag{4.48}$$

where $\{S(t)\}_{t\geq 0}$ is the C_0-semigroup generated by $-A$.

Without lose of generality, we suppose that $0 < \rho_1 < \rho_2$. Hence

$$B_{\omega}^{*}S^{*}(t)w = \sum_{j=1}^{\infty} B_{\omega}^{*}e^{M_j^{*}t}P_j^{*}w = \sum_{j=1}^{\infty} B_{\omega}^{*}e^{M_j^{*}t}P_j^{*}w = \sum_{j=1}^{\infty}\sum_{s=1}^{2} B_{\omega}^{*}K_j^{*}(t)\left(e^{-\lambda_j\rho_s t}P_{sj}^{*}\right)w, \tag{4.49}$$

where $P_{sj} = Q_sP_j = P_jQ_s$, $s = 1, 2$.

Now, suppose for $w \in X$ that $B_{\omega}^{*}S^{*}(t)w = 0$, for all $t \in [0, t^{*}]$. Then

$$B_{\omega}^{*}S^{*}(t)w = 0 \Longleftrightarrow \sum_{j=1}^{\infty}\sum_{s=1}^{2} B_{\omega}^{*}K_j^{*}(t)\left(e^{-\lambda_j\rho_s t}P_{sj}^{*}\right)w(x) = 0, \quad \forall x \in \Omega. \tag{4.50}$$

If (4.46) is satisfied, then (4.50) take the form

$$\sum_{j=1}^{\infty}\sum_{s=1}^{2}\begin{pmatrix} e^{-(\lambda_j\rho_s+\lambda_j^{\alpha_1}c_1)t} & 0 \\ 0 & e^{-(\lambda_j\rho_s+\lambda_j^{\beta_2}d_2)t}\end{pmatrix}\left(B_{\omega}^{*}P_{sj}^{*}\right)w(x) = 0, \quad \forall x \in \Omega. \tag{4.51}$$

Then, from lemma 4.6 we obtain that for $s = 1, 2$ and all $x \in \omega$

$$\left(B_{\omega}^{*}Q_s^{*}P_j^{*}w\right)(x) = Q_s^{*}\begin{pmatrix} \sum_{k=1}^{m_j}\langle u, \varphi_{jk}\rangle 1_{\omega}\varphi_{jk}(x) \\ \sum_{k=1}^{m_j}\langle v, \varphi_{jk}\rangle 1_{\omega}\varphi_{jk}(x)\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}, \quad j \geq 1. \tag{4.52}$$

Since $Q_1 + Q_2 = I_{\mathbb{R}^2}$, we get that all $x \in \omega$

$$\begin{pmatrix} \sum_{k=1}^{m_j}\langle u, \varphi_{jk}\rangle 1_{\omega}\varphi_{jk}(x) \\ \sum_{k=1}^{m_j}\langle v, \varphi_{jk}\rangle 1_{\omega}\varphi_{jk}(x)\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}, \quad s = 1, 2, \ j \geq 1. \tag{4.53}$$

On the other hand, from Theorem 2.4 we know that φ_{jk} are analytic functions, which implies the analticity of $E_j u = \sum_{k=1}^{m_j} \langle u, \varphi_{jk} \rangle \varphi_{jk}$ and $E_j v = \sum_{k=1}^{m_j} \langle v, \varphi_{jk} \rangle \varphi_{jk}$. Then we can conclude that for $s = 1, 2$ and all $x \in \Omega$

$$\begin{pmatrix} \sum_{k=1}^{m_j} \langle u, \varphi_{jk} \rangle \varphi_{jk}(x) \\ \sum_{k=1}^{m_j} \langle v, \varphi_{jk} \rangle \varphi_{jk}(x) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad j \geq 1. \tag{4.54}$$

Hence $P_j w = 0$, for all $j \geq 1$, which implies that $w = 0$. This completes the proof of Theorem 4.5. □

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