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Research Article

Multiple Positive Solutions of the Singular Boundary Value Problem for Second-Order Impulsive Differential Equations on the Half-Line

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This paper uses a fixed point theorem in cones to investigate the multiple positive solutions of a boundary value problem for second-order impulsive singular differential equations on the half-line. The conditions for the existence of multiple positive solutions are established.

1. Introduction

Consider the following nonlinear singular Sturm-Liouville boundary value problems for second-order impulsive differential equation on the half-line:

$$\begin{gathered}
\left(p(t)u'(t)\right)' + f(t,u) = 0, \quad \forall t \in J_+', \\
\Delta u'(t_k) = I_k(u(t_k)), \quad k = 1,2,\ldots,n, \\
\alpha u(0) - \beta \lim_{t\to 0^+} p(t)u'(t) = 0, \\
\gamma u(\infty) + \delta \lim_{t\to +\infty} p(t)u'(t) = 0,
\end{gathered} \tag{1.1}$$

where $J = [0,+\infty)$, $0 < t_1 < \cdots < t_n$, $J_+ = (0,+\infty)$, $J_+' = J_+ \setminus \{t_1,\ldots,t_n\}$, $f \in C[J_+ \times J_+, J_+]$, $p \in C[J, J_+] \cap C^1[J_+, J_+]$ with $p > 0$ on J_+, and $\int_0^{+\infty}(1/p(s))ds < +\infty$; $\alpha,\beta,\gamma,\delta \geq 0$ with $\rho = \beta\gamma + \alpha\delta + \alpha\gamma B(0,+\infty) > 0$, in which $B(t,s) = \int_t^s (1/p(\sigma))d\sigma$. $\Delta u'(t_k) = u'(t_k^+) - u'(t_k^-)$, where $u'(t_k^-)$ and $u'(t_k^+)$ are, respectively, the left and right limits of $u'(t)$ at t_k, $k = 1, \dots, n$, $1 \le n < +\infty$.

The theory of singular impulsive differential equations has been emerging as an important area of investigation in recent years. For the theory and classical results, we refer the monographs to [1, 2] and the papers [3–19] to readers. We point out that in a second-order differential equation $u'' = f(t, u, u')$, one usually considers impulses in the position u and the velocity u'. However, in the motion of spacecraft one has to consider instantaneous impulses depending on the position that result in jump discontinuities in velocity, but with no change in position [20]. The impulses only on the velocity occur also in impulsive mechanics [21].

In recent paper [3], by using the Krasnoselskii's fixed point theorem, Kaufmann has discussed the existence of solutions for some second-order boundary value problem with impulsive effects on an unbounded domain. In [22] Sun et al. and [23] Liu et al., respectively, discussed the existence and multiple positive solutions for singular Sturm-Liouville boundary value problems for second-order differential equation on the half-line. But the Multiple positive solutions of this case with both singularity and impulses are not to be studied. The aim of this paper is to fill up this gap.

The rest of the paper is organized as follows. In Section 2, we give several important lemmas. The main theorems are formulated and proved in Section 3. And in Section 4, we give an example to demonstrate the application of our results.

2. Several Lemmas

Lemma 2.1 (see [23]). *If conditions $\int_0^{+\infty}(1/p(s))ds < +\infty$ and $\rho > 0$ are satisfied, then the boundary value problem*

$$\begin{gathered}
(p(t)u'(t))' + v(t) = 0, \quad \forall t \in J_+, \\
\alpha u(0) - \beta \lim_{t\to 0^+} p(t)u'(t) = 0, \\
\gamma u(\infty) + \delta \lim_{t\to +\infty} p(t)u'(t) = 0
\end{gathered} \tag{2.1}$$

has a unique solution for any $v \in L[J_+, R]$. Moreover, this unique solution can be expressed in the form

$$u(t) = \int_0^{\infty} G(t,s)v(s)ds, \tag{2.2}$$

where $G(t,s)$ is defined by

$$G(t,s) = \frac{1}{\rho}\begin{cases} (\beta + \alpha B(0,s))(\delta + \gamma B(t,\infty)), & 0 \le s \le t < +\infty, \\ (\beta + \alpha B(0,t))(\delta + \gamma B(s,\infty)), & 0 \le t \le s < +\infty. \end{cases} \tag{2.3}$$

Remark 2.2. It is easy to prove that $G(t,s)$ has the following properties:

(1) $G(t,s)$ is continuous on $J_+ \times J_+$,

(2) $G(t,s)$ is continuous differentiable on $J_+ \times J_+$, except $t = s$,

(3) $\partial_t G(t,s)|_{t=s^+} - \partial_t G(t,s)|_{t=s^-} = (p(s))^{-1}$,

(4) $G(t,s) \le G(s,s) \le \rho^{-1}(\beta + \alpha B(0,s))(\delta + \gamma B(s,\infty)) < +\infty$,

(5) $\overline{G}(s) = \lim_{t\to+\infty} G(t,s) < +\infty$,

(6) for all $t \in [a,b] \subset (0,+\infty)$, $s \in [0,+\infty)$, $G(t,s) \ge \omega G(s,s)$, where

$$w = \min\left\{ \frac{\beta + \alpha B(b,\infty)}{\beta + \alpha B(0,\infty)}, \frac{\delta + \gamma B(b,\infty)}{\delta + \gamma B(0,\infty)} \right\}. \tag{2.4}$$

Obviously, $0 < \omega < 1$.

For the interval $[a,b]$, $0 < a < t_1$, $t_n < b < \infty$, and the corresponding ω in Remark 2.2, we define $PC^1[J,R] = \{u \in C[J,R] : u' \in C[J'_+,R],\ u'(t_k^-)$ and $u'(t_k^+)$ exist, and $u'(t_k) = u'(t_k^-)\}$. $BPC^1[J,R] = \{u \in PC^1[J,R] : \lim_{t\to\infty} u(t)$ exists$\}$. $K = \{u \in BPC^1[J,R] : u(t) > 0, t \in J_+$ and $\min_{t\in[a,b]} u(t) \ge \omega\|u\|\}$. It is easy to see that $BPC^1[J,R]$ is a Banach space with the norm $\|u\| = \sup_{t\in J}|u(t)|$, and K is a positive cone in $BPC^1[J,R]$. For details of the cone theory, see [1]. $u \in PC^1[J,R] \cap C^2[J'_+R]$ is called a positive solution of BVP (1.1) if $u(t) > 0$ for all $t \in J$ and $u(t)$ satisfies (1.1).

As we know that the Ascoli-Arzela Theorem does not hold in infinite interval J, we need the following compactness criterion:

Lemma 2.3 (see [22]). *Let $M \subset BPC^1[J,R]$. Then M is relatively compact in $BPC^1[J,R]$ if the following conditions hold.*

(i) *M is uniformly bounded in $BPC^1[J,R]$.*

(ii) *The functions from M are equicontinuous on any compact interval of $[0,+\infty)$.*

(iii) *The functions from M are equiconvergent, that is, for any given $\varepsilon > 0$, there exists a $T = T(\varepsilon) > 0$ such that $|f(t) - f(+\infty)| < \varepsilon$, for any $t > T$, $f \in M$.*

The main tool of this work is a fixed point theorem in cones.

Lemma 2.4 (see [4]). *Let X be a Banach space and K is a positive cone in X. Assume that Ω_1, Ω_2 are open subsets of X with $0 \in \Omega_1$, $\overline{\Omega}_1 \subset \Omega_2$. Let $T : K \cap (\overline{\Omega}_2 \setminus \Omega_1) \to K$ be a completely continuous operator such that*

(i) *$\|Tu\| \le \|u\|$ for all $u \in K \cap \partial\Omega_1$.*

(ii) *there exists a $\Phi \in K$ such that $u \ne Tu + \lambda\Phi$, for all $u \in K \cap \partial\Omega_2$ and $\lambda > 0$.*

Then T has a fixed point in $K \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

Remark 2.5. If (i) is satisfied for $u \in K \cap \partial\Omega_2$ and (ii) is satisfied for $u \in K \cap \partial\Omega_1$, then Lemma 2.4 is still true.

Lemma 2.6 (see [3]). *The function* $u \in K \cap C^2[J'_+, R]$ *is a solution of the BVP* (1.1) *if and only if* $u \in K$ *satisfies the equation*

$$u(t) = \int_0^{+\infty} G(t,s) f(s,u(s))ds + \sum_{k=1}^{n} G(t,t_k)p(t_k)I_k(u(t_k)), \quad t \in J. \tag{2.5}$$

The proof of this result is based on the properties of the Green function, so we omit it as elementary.

Define

$$(Tu)(t) = \int_0^{+\infty} G(t,s) f(s,u(s))ds + \sum_{k=1}^{n} G(t,t_k)p(t_k)I_k(u(t_k)), \quad t \in J. \tag{2.6}$$

Obviously, the BVP (1.1) has a solution u if and only if $u \in K$ is a fixed point of the operator T defined by (2.6).

Let us list some conditions as follows.

(A_1) There exist two nonnegative functions: $a \in C[J_+, J]$, $g \in C[J, J]$ such that $f(t,u) \leq a(t)g(u)$. $f(t,u)$, $a(t)$ may be singular at $t = 0$. $I_k : J \to J$, $k = 1, \dots, n$, are continuous.

(A_2) $0 < \int_0^{+\infty} G(s,s)a(s)ds < +\infty$, $0 < G(t_k,t_k)p(t_k) < +\infty$, $k = 1, \dots, n$.

Lemma 2.7. *If* (A_1) *and* (A_2) *are satisfied, then for any bounded open set* $\Omega \subset BPC^1[J,R]$, $T : \overline{\Omega} \cap K \to K$ *is a completely continuous operator.*

Proof. For any bounded open set $\Omega \subset BPC^1[J,R]$, there exists a constant $M > 0$ such that $\|u\| \leq M$ for any $u \in \overline{\Omega}$.

First, we show that $T : \overline{\Omega} \cap K \to K$ is well defined. Let $u \in \overline{\Omega} \cap K$. From (A_1), we have $S_M = \max\{S_1, S_2\}$, where $S_1 = \sup\{g(u) : 0 \leq u \leq M\}$, $S_2 = \sup\{I_k(u) : 0 \leq u \leq M, k = 1, \dots, n\}$, and

$$\begin{aligned} &\int_0^{+\infty} G(t,s) f(s,u(s))ds + \sum_{k=1}^{n} G(t,t_k)p(t_k)I_k(u(t_k)) \\ &\quad \leq S_M \left(\int_0^{+\infty} G(s,s)a(s)ds + \sum_{k=1}^{n} G(t_k,t_k)p(t_k) \right) < +\infty. \end{aligned} \tag{2.7}$$

Hence, T is well defined. For any $t_1, t_2 \in J$, we have

$$\int_0^{+\infty} |G(t_1,s) - G(t_2,s)|a(s)ds \leq 2\int_0^{+\infty} G(s,s)a(s)ds < +\infty. \tag{2.8}$$

Thus, by the Lebesgue dominated convergence theorem and the fact that $G(s,t)$ is continuous on t, we have, for any $t_1, t_2 \in J$, $u \in \overline{\Omega} \cap K$,

$$\begin{aligned}
&|(Tu)(t_1) - (Tu)(t_2)| \\
&\quad \le \int_0^{+\infty} |G(t_1,s) - G(t_2,s)| f(s,u(s))ds \\
&\qquad + \sum_{k=1}^{n} |G(t_1,t_k) - G(t_2,t_k)| p(t_k) I_k(u(t_k)) \\
&\quad \le S_M \left(\int_0^{+\infty} |G(t_1,s) - G(t_2,s)| a(s)ds + \sum_{k=1}^{n} |G(t_1,t_k) - G(t_2,t_k)| p(t_k) \right) \\
&\quad \longrightarrow 0, \quad (t_1 \longrightarrow t_2).
\end{aligned} \tag{2.9}$$

Therefore, $Tu \in C[J,R]$. By the property (3) of $G(s,t)$, it is easy to get $Tu \in PC^1[J,R]$.

On the other hand, by (2.6) we have, for any $u \in \overline{\Omega} \cap K$ and $t \in J_+$,

$$\begin{aligned}
&\left| (Tu)(t) - \int_0^{+\infty} \overline{G}(s) f(s,u(s))ds \right| \\
&\quad \le \int_0^{+\infty} \left| G(t,s) - \overline{G}(s) \right| f(s,u(s))ds + \sum_{k=1}^{n} \left| G(t,t_k) - \overline{G}(t_k) \right| p(t_k) I_k(u(t_k)) \\
&\quad \le S_M \left(\int_0^{+\infty} \left| G(t,s) - \overline{G}(s) \right| a(s)ds + \sum_{k=1}^{n} \left| G(t,t_k) - \overline{G}(t_k) \right| p(t_k) \right).
\end{aligned} \tag{2.10}$$

Then by (A_2), the property (5) of Remark 2.2 and the Lebesgue dominated convergence theorem, we have

$$\lim_{t \to +\infty} (Tu)(t) = \int_0^{+\infty} \overline{G}(s) f(s,u(s))ds + \sum_{k=1}^{n} \overline{G}(t_k) p(t_k) I_k(u(t_k)) < +\infty. \tag{2.11}$$

Thus $Tu \in BPC^1[J,R]$.

For any $u \in \overline{\Omega} \cap K$, we get

$$
\begin{aligned}
(Tu)(t) &= \int_0^{+\infty} G(t,s) f(s,u(s))ds + \sum_{k=1}^{n} G(t,t_k)p(t_k)I_k(u(t_k)) \\
&\le \int_0^{+\infty} G(s,s) f(s,u(s))ds + \sum_{k=1}^{n} G(t_k,t_k)p(t_k)I_k(u(t_k)).
\end{aligned}
\tag{2.12}
$$

So

$$
\|Tu\| \le \int_0^{+\infty} G(s,s) f(s,u(s))ds + \sum_{k=1}^{n} G(t_k,t_k)p(t_k)I_k(u(t_k)). \tag{2.13}
$$

On the other hand, for $t \in [a,b]$ we obtain

$$
(Tu)(t) \ge \omega \left(\int_0^{+\infty} G(s,s) f(s,u(s))ds + \sum_{k=1}^{n} G(t_k,t_k)p(t_k)I_k(u(t_k)) \right) \ge \omega \|Tu\|. \tag{2.14}
$$

Thus $T : \overline{\Omega} \cap K \to K$.

Next, we prove that T is continuous. Let $u_n \to u_0$ in $\overline{\Omega} \cap K$, then $\|u_n\| \le M$ $(n = 1,2,\ldots)$. We prove that $Tu_n \to Tu_0$. For any $\varepsilon > 0$, by (A_2), there exists a constant $A_0 > 0$ such that

$$
S_M \int_{A_0}^{+\infty} G(s,s)a(s)ds \le \frac{\varepsilon}{6}. \tag{2.15}
$$

On the other hand, by the continuities of $f(t,u)$ on $(0, A_0] \times (0, M]$ and the continuities of I_k on J, for the above $\varepsilon > 0$, there exists a $\delta > 0$ such that, for any $u, v \in (0, M]$, $|u - v| < \delta$,

$$
\begin{gathered}
\left| f(t,u) - f(t,v) \right| < \frac{\varepsilon}{3} \left(\int_0^{A_0} G(s,s)ds \right)^{-1}, \quad t \in (0, A_0], \\
G(t_k,t_k)p(t_k)|I_k(u(t_k)) - I_k(v(t_k))| < \frac{\varepsilon}{3n}.
\end{gathered}
\tag{2.16}
$$

From $\|u_n - u_0\| \to 0$, for the above δ, there exists a sufficiently large number N such that, when $n > N$, we have

$$
\begin{gathered}
|u_n(t) - u_0(t)| \le \|u_n - u_0\| < \delta, \quad t \in (0, A_0], \\
|u_n(t_k) - u_0(t_k)| \le \|u_n - u_0\| < \delta.
\end{gathered}
\tag{2.17}
$$

Therefore, by (2.15)–(2.17), we have, for $n > N$,

$$\begin{aligned}\|Tu_n - Tu_0\| &\leq \int_0^{+\infty} G(s,s)\big|f(s,u_n(s)) - f(s,u_0(s))\big|ds \\ &\quad + \sum_{k=1}^{n} G(t_k,t_k)p(t_k)|I_k(u_n(t_k)) - I_k(u_0(t_k))| \\ &\leq 2S_M\int_{A_0}^{+\infty} G(s,s)a(s)ds + \int_0^{A_0} G(s,s)\big|f(s,u_n(s)) - f(s,u_0(s))\big|ds \\ &\quad + \sum_{k=1}^{n} G(t_k,t_k)p(t_k)|I_k(u(t_k)) - I_k(u_0(t_k))| \\ &\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.\end{aligned} \tag{2.18}$$

This implies that the operator T is continuous.

Finally we show that $T : \overline{\Omega} \cap K \to K$ is a compact operator. In fact for any bounded set $D \subset \overline{\Omega}$, there exists a constant $M_1 > 0$ such that $\|u\| \leq M_1$ for any $u \in D \cap K$. Hence, we obtain

$$\|Tu\| \leq S_{M_1}\left(\int_0^{+\infty} G(s,s)a(s)ds + \sum_{k=1}^{n} G(t_k,t_k)p(t_k)\right) < +\infty. \tag{2.19}$$

Therefore, $T(D \cap K)$ is uniformly bounded in $BPC^1[J,R]$.

Given $r > 0$, for any $u \in D \cap K$, as the proof of (2.9), we can get that $\{Tu : u \in D\cap K\}$ are equicontinuous on $[0,r]$. Since $r > 0$ is arbitrary, $\{Tu : u \in D \cap K\}$ are locally equicontinuous on J_+. By (2.6), (A_1), (A_2), and the Lebesgue dominated convergence theorem, we have

$$\begin{aligned}|Tu(t) - Tu(+\infty)| &\leq S_{M_1}\left(\int_0^{+\infty} \Big|G(t,s) - \overline{G}(s)\Big|a(s)ds + \sum_{k=1}^{n}\Big|G(t,t_k) - \overline{G}(t_k)\Big|p(t_k)\right) \\ &\longrightarrow 0, \quad (t \longrightarrow +\infty).\end{aligned} \tag{2.20}$$

Hence, the functions from $\{Tu : u \in D \cap K\}$ are equiconvergent. By Lemma 2.3, we have that $\{Tu : u \in D\cap K\}$ is relatively compact in $BPC^1[J,R]$. Therefore, $T : \overline{\Omega} \cap K \to K$ is completely continuous. This completed the proof of Lemma 2.7. □

3. Main Results

For convenience and simplicity in the following discussion, we use the following notations:

$$\begin{gathered}
f_0=\liminf_{u\to 0}\min_{t\in[a,b]}\frac{f(t,u)}{u},\qquad g_0=\liminf_{u\to 0}\frac{g(u)}{u},\qquad I_0(k)=\liminf_{u\to 0}\frac{p(t_k)I_k(u)}{u},\\
f_\infty=\liminf_{u\to \infty}\min_{t\in[a,b]}\frac{f(t,u)}{u},\qquad g_\infty=\liminf_{u\to \infty}\frac{g(u)}{u},\qquad I_\infty(k)=\liminf_{u\to \infty}\frac{p(t_k)I_k(u)}{u},\\
I^q(k)=\limsup_{u\to q}\frac{p(t_k)I_k(u)}{u},\qquad g^\infty=\limsup_{u\to \infty}\frac{g(u)}{u},\qquad I^\infty(k)=\limsup_{u\to \infty}\frac{p(t_k)I_k(u)}{u},\\
g^q=\limsup_{u\to q}\frac{g(u)}{u},\qquad g^0=\limsup_{u\to 0}\frac{g(u)}{u},\qquad I^0(k)=\limsup_{u\to 0}\frac{p(t_k)I_k(u)}{u},
\end{gathered}\tag{3.1}$$

Theorem 3.1. *Let* (A_1) *and* (A_2) *hold. Then the BVP* (1.1) *has at least two positive solutions satisfying* $0<\|u_1\|<q<\|u_2\|$ *if the following conditions hold:*

(H_1) $\omega(f_0\int_a^b G(s,s)ds+\sum_{k=1}^n G(t_k,t_k)I_0(k))>1$, $\omega(f_\infty\int_a^b G(s,s)ds+\sum_{k=1}^n G(t_k,t_k)\cdot I_\infty(k))>1$,

(H_2) *there exists a* $q>0$ *such that* $g^q\int_0^{+\infty}G(s,s)a(s)ds+\sum_{k=1}^n G(t_k,t_k)I^q(k)<1$, *for all* $\omega q\le u\le q$, *a.e.* $t\in[0,+\infty)$.

Proof. By the definition of f_0 and I_0, for any $\varepsilon>0$, there exist $r\in(0,q)$ such that

$$\begin{gathered}
f(t,u)\ge(1-\varepsilon)f_0u,\quad \forall\|u\|\le r,\ t\in[a,b],\\
p(t_k)I_k(u)\ge(1-\varepsilon)I_0(k)u,\\
(1-\varepsilon)\omega\left(f_0\int_a^b G(s,s)ds+\sum_{k=1}^n G(t_k,t_k)I_0(k)\right)\ge 1,\quad \forall\|u\|\le r.
\end{gathered}\tag{3.2}$$

Define the open sets

$$\Omega_r=\left\{u\in BPC^1[J,R]:\|u\|<r\right\}.\tag{3.3}$$

Let $\Phi\equiv 1$, then $\Phi\in K$. Now we prove that

$$u\ne Tu+\lambda\Phi,\quad \forall u\in K\cap\partial\Omega_r,\ \lambda>0.\tag{3.4}$$

If not, then there exist $u_0 \in K \cap \partial\Omega_r$ and $\lambda_0 > 0$ such that $u_0 = Tu_0 + \lambda_0 \Phi$. Let $\mu = \min_{t\in[a,b]} u_0(t)$, then for any $t \in [a,b]$, we have

$$\begin{aligned} u_0(t) &= (Tu_0)(t) + \lambda_0 \\ &= \int_0^{+\infty} G(t,s) f(s,u_0(s))ds + \sum_{k=1}^{n} G(t,t_k) p(t_k) I_k(u_0(t_k)) + \lambda_0 \\ &\geq \omega \int_0^{+\infty} G(s,s) f(s,u_0(s))ds + \omega \sum_{k=1}^{n} G(t_k,t_k) p(t_k) I_k(u_0(t_k)) + \lambda_0 \\ &> (1-\varepsilon)\mu\omega \left(f_0 \int_a^b G(s,s)ds + \sum_{k=1}^{n} G(t_k,t_k) I_0(k) \right) + \lambda_0 \\ &\geq \mu + \lambda_0. \end{aligned} \tag{3.5}$$

This implies $\mu > \mu + \lambda_0$, a contradiction. Therefore, (3.4) holds.

That by the definition of f_∞ and I_∞, for any $\varepsilon > 0$ there exist $R > q$ such that

$$\begin{gathered} f(t,u) \geq (1-\varepsilon) f_\infty u, \quad \forall \|u\| \geq R, \ t \in [a,b], \\ p(t_k) I_k(u) \geq (1-\varepsilon) I_\infty(k) u, \\ (1-\varepsilon)\omega \left(f_\infty \int_a^b G(s,s)ds + \sum_{k=1}^{n} G(t_k,t_k) I_\infty(k) \right) \geq 1, \quad \forall \|u\| \geq R. \end{gathered} \tag{3.6}$$

Define the open sets:

$$\Omega_R = \left\{ u \in BPC^1[J,R] : \|u\| < R \right\}. \tag{3.7}$$

As the proof of (3.4), we can get that

$$u \neq Tu + \lambda\Phi, \quad \forall x \in K \cap \partial\Omega_R, \ \lambda > 0. \tag{3.8}$$

On the other hand, for any $\varepsilon > 0$, choose q in (H_2) such that

$$(1+\varepsilon)\left(g^q \int_0^{+\infty} G(s,s) a(s) ds + \sum_{k=1}^{n} G(t_k,t_k) I^q(k) \right) \leq 1, \qquad \omega q \leq u \leq q. \tag{3.9}$$

By the definition of g^q, I^q, for the above $\varepsilon > 0$, there exists $\delta > 0$, when $u \in (q-\delta, q+\delta)$; thus, we have

$$\begin{gathered} g(u) \leq (1+\varepsilon) g^q u, \\ p(t_k) I_k(u) \leq (1+\varepsilon) I^q(k) u. \end{gathered} \tag{3.10}$$

Define

$$\Omega_q = \left\{ u \in BPC^1[J, R] : \|u\| < q \right\}. \tag{3.11}$$

Then, for any $u \in K \cap \partial\Omega_q$ and $t \in [0, +\infty)$, we can obtain

$$\begin{aligned}(Tu)(t) &= \int_0^{+\infty} G(t,s) f(s,u(s))ds + \sum_{k=1}^{n} G(t,t_k)p(t_k)I_k(u(t_k)) \\ &\leq \int_0^{+\infty} G(s,s)a(s)g(u(s))ds + \sum_{k=1}^{n} G(t_k,t_k)p(t_k)I_k(u(t_k)) \\ &\leq (1+\varepsilon)\left(g^q \int_0^{\infty} G(s,s)a(s)ds + \sum_{k=1}^{n} G(t_k,t_k)I^q(k) \right)\|u\| \\ &\leq \|u\|.\end{aligned} \tag{3.12}$$

Therefore, $\|Tu\| \leq \|u\|$.

Thus, we can obtain the existence of two positive solutions u_1 and u_2 satisfying $0 < \|u_1\| < q < \|u_2\|$ by using Lemma 2.4 and Remark 2.5, respectively. □

Using a similar proof of Theorem 3.1, we can get the following conclusions.

Theorem 3.2. *Let* (A_1) *and* (A_2) *hold. Then the BVP* (1.1) *has at least two positive solutions satisfying* $0 < \|u_1\| < q < \|u_2\|$ *if the following conditions hold:*

- (H_3) $g^0 \int_0^{+\infty} G(s,s)a(s)ds + \sum_{k=1}^{n} G(t_k,t_k)I^0(k) < 1$, $g^\infty \int_0^{+\infty} G(s,s)a(s)ds + \sum_{k=1}^{n} G(t_k)I^\infty(k) < 1$,
- (H_4) *there exists* $q > 0$ *such that* $\omega(f_q \int_a^b G(s,s)ds + \sum_{k=1}^{n} G(t_k,t_k)I_q(k)) > 1$, *for all* $\omega q \leq u \leq q$, *a.e.* $t \in [0, +\infty)$.

Corollary 3.3. *In Theorems 3.1 and 3.2, if conditions* (H_1) *and* (H_3) *are replaced by* (H_1^*) *and* (H_3^*), *respectively, then the conclusions also hold.*

- (H_1^*) $f_0 = +\infty$, *or* $\sum_{k=1}^{n} I_0(k) = +\infty$; $f_\infty = +\infty$ *or* $\sum_{k=1}^{n} I_\infty(k) = +\infty$,
- (H_3^*) $g^\infty = 0$, $\sum_{k=1}^{n} I^\infty(k) = 0$, $g^0 = 0$, $\sum_{k=1}^{n} I^0(k) = 0$.

Remark 3.4. Notice that, in the above conclusions, we suppose that the singularity only exist in $f(t,u)$, that is, $\|f(t,u)\| \to +\infty$ as $t \to 0$. If we permit $\|f(t,u)\| \to +\infty$ as $t \to 0^+$ or $u \to 0^+$ and $\|I_k(u_k)\| \to +\infty$ as $u_k \to 0^+$, then the discussion will be much more complex. Now we state the corresponding results.

Let us define the following.

- (A_1^*) There exist four nonnegative functions $a, g \in C[J_+, J]$, $b, h \in C[J, J]$ such that $b(t)h(u) \leq f(t,u) \leq a(t)g(u)$, and $h(u)$ is nondecreasing on J. $I_k : J_+ \to J$, $k = 1, \ldots, n$, are continuous.
- (A_2^*) $0 < \int_0^{+\infty} G(s,s)a(s)ds < +\infty$, $\int_0^{+\infty} G(s,s)b(s)ds \geq (u^*/\omega h^*)$, $0 < G(t_k,t_k)p(t_k) < +\infty$, $k = 1, \ldots, n$, where $u^* \in K$, $h^* = h(0)$.

Theorem 3.5. *Suppose* (A_1^*) *and* (A_2^*) *hold, then the BVP* (1.1) *has at least two positive solutions satisfying* $u^* < \|u_1\| < q < \|u_2\|$ *if* (H_1) *and* (H_2) *hold.*

Proof. Define $Q = \{u \in K : u(t) \geq u^*, \text{for all } t \in J\}$. We only need to proove $T : \overline{\Omega} \cap Q \to Q$ is a completely continuous operator. Then the rest of the proof is the same as that Theorem 3.1. Notice that

$$(Tu)(t) \geq \omega \int_0^{+\infty} G(s,s) f(s,u(s)) ds \geq \omega h^* \int_0^{+\infty} G(s,s) b(s) ds \geq u^*, \tag{3.13}$$

and change S_1, S_2 to $S_1 = \sup\{g(u) : u^* \leq u \leq M\}$, $S_2 = \sup\{I_k(u) : u^* \leq u \leq M, k = 1 \ldots, n\}$, then the same as the proof of Lemma 2.7, it is easy to compute that $T : \overline{\Omega} \cap Q \to Q$ is a completely continuous operator. □

Corresponding to Theorem 3.2 and Corollary 3.3, there are Theorem 3.6 and Corollary 3.7. We just list here without proof.

Theorem 3.6. *Suppose* (A_1^*) *and* (A_2^*) *hold, then the BVP* (1.1) *has at least two positive solutions satisfying* $u^* < \|u_1\| < q < \|u_2\|$*, if* (H_3) *and* (H_4) *hold.*

Corollary 3.7. *In Theorems 3.5 and 3.6, if conditions* (H_1) *and* (H_3) *are replaced by* (H_1^*) *and* (H_3^*)*, respectively, then the conclusions also hold.*

4. Example

To illustrate how our main results can be used in practice we present the following example.

Example 4.1. Consider the following boundary value problem:

$$\begin{gathered} \left(e^t u'(t)\right)' + |\ln t| = 0, \quad \forall t \in J_+, \ t \neq 1, \\ \Delta u'\big|_{t_1=1} = u^2(1), \\ u(0) = 0, \qquad u(\infty) = 0. \end{gathered} \tag{4.1}$$

Conclusion 1. BVP (4.1) has at least two positive solutions u_1, u_2 satisfying $0 < \|u_1\| < 1/2 < \|u_2\|$.

Proof. Let $p(t) = e^t$, $g(u) = 1$, $f(t,u) = a(t) = |\ln t|$, $I(u) = u^2$. Then by simple computation we have

$$G(t,s) = \begin{cases} \displaystyle\int_0^s e^{-\sigma} d\sigma \int_t^{+\infty} e^{-\sigma} d\sigma, & 0 \leq s \leq t < +\infty, \\ \displaystyle\int_0^t e^{-\sigma} d\sigma \int_s^{+\infty} e^{-\sigma} d\sigma, & 0 \leq t \leq s < +\infty, \end{cases} \tag{4.2}$$

where $\rho = 1$. Furthermore, $\int_0^{+\infty}(1/p(\sigma))d\sigma = \int_0^{+\infty} e^{-\sigma}d\sigma = 1 < +\infty$ and

$$0 < \int_0^{+\infty} G(s,s)a(s)ds = \int_0^{+\infty} (1 - e^{-s})e^{-s}|\ln s|ds < +\infty,$$
$$0 < G(t_1,t_1)p(t_1) = 1 - e^{-1} < +\infty. \tag{4.3}$$

Let $[a,b] = [1,2] \subset (0,+\infty)$. Then $\omega = e^{-2}$. Thus (A_1) and (A_2) are satisfied. It is easy to get that $f_0 = +\infty, I_\infty(1) = +\infty$. Let $q = 1/2$. Then

$$g^q \int_0^{+\infty} G(s,s)a(s)ds + \sum_{k=1}^{n} G(t_k,t_k)I^q(k) < 1. \tag{4.4}$$

Hence, (H_1^*) and (H_2) are satisfied. Therefore, by Corollary 3.3, problem (4.1) has at least two positive solutions u_1, u_2 satisfying $0 < \|u_1\| < 1/2 < \|u_2\|$. The proof is completed. □

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