Hindawi Publishing Corporation
Boundary Value Problems
Volume 2010, Article ID 410986, 26 pages
doi:10.1155/2010/410986

Research Article

Optimal Conditions for Maximum and Antimaximum Principles of the Periodic Solution Problem

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Received 18 September 2009; Accepted 11 April 2010

Academic Editor: Pavel Drábek

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Given a periodic, integrable potential $q(t)$, we will study conditions on $q(t)$ so that the operator $L_q x = x'' + qx$ admits the maximum principle or the antimaximum principle with respect to the periodic boundary condition. By exploiting Green functions, eigenvalues, rotation numbers, and their estimates, we will give several optimal conditions.

1. Introduction and Main Results

Maximum Principle (MP) and AntiMaximum Principle (AMP) are fundamental tools in many problems. Generally speaking, criteria for MP and AMP are related to the location of relevant eigenvalues. See, for example, [1–5]. We also refer the reader to Campos et al. [6] for a recent abstract setting of MP and AMP.

In this paper we are studying criteria of MP and AMP for the periodic solution problem of ODEs. For such a problem, MP and AMP are not only related to periodic eigenvalues, but also to antiperiodic eigenvalues. Though there exist several sufficient conditions of MP and AMP for the periodic solution problem in literature like [7–9] (for a brief explanation to these conditions, see Section 4.3), an optimal characterization on MP and AMP is not available. The main aim of this paper is to give several optimal criteria of MP and AMP of the periodic solution problem of ODEs which are expressed using eigenvalues, Green functions, or rotation numbers.

Mathematically, let $\mathbb{S} := \mathbb{R}/\mathbb{Z}$ be the circle of length 1. Given a 1-periodic potential $q \in \mathcal{L}^1 := L^1(\mathbb{S}, \mathbb{R})$, which defines a linear differential operator $L_q : \mathcal{W}^{2,1} := W^{2,1}(\mathbb{S}, \mathbb{R}) \to \mathcal{L}^1$

by

$$\left(L_q x\right)(t) = x''(t) + q(t)x(t), \tag{1.1}$$

we say that $L_q : \mathcal{W}^{2,1} \to \mathcal{L}^1$ admits the antimaximum principle if

(i) $L_q : \mathcal{W}^{2,1} \to \mathcal{L}^1$ is invertible, and, moreover,

(ii) for any $h \in \mathcal{L}^1$ with $h \succ 0$, one has $\min_t(L_q^{-1}h)(t) > 0$. Here $h \succ 0$ means that $h(t) \geq 0$ a.e. t and $h(t) > 0$ on a subset of positive measure.

In an abstract setting, these mean that $L_q^{-1} : \mathcal{L}^1 \to \mathcal{W}^{2,1}$ is a strictly positive operator with respect to the ordering $h_1 \geq h_2$ defined by $h_1(t) \geq h_2(t)$ a.e. t.

In terminology of differential equations, L_q admits AMP if and only if

(i) for any $h \in \mathcal{L}^1$, the following equation:

$$x'' + q(t)x = h(t) \tag{1.2}$$

has a unique 1-periodic solution $x = x_h \in \mathcal{W}^{2,1}$, and, moreover,

(ii) if $h \succ 0$, one has $x_h(t) > 0$ for all t.

We say that L_q admits the maximum principle if $\max_t(L_q^{-1}h)(t) < 0$ for all $h \in \mathcal{L}^1$ such that $h \succ 0$.

Using periodic and antiperiodic eigenvalues of Hill's equations [10, 11], we will obtain the following complete characterizations on MP and AMP.

Theorem 1.1. *Let $q \in \mathcal{L}^1$. Then L_q admits MP iff $\overline{\lambda}_0(q) > 0$, and L_q admits AMP iff $\overline{\lambda}_0(q) < 0 \leq \underline{\lambda}_1(q)$.*

Here $\overline{\lambda}_0(q)$ and $\underline{\lambda}_1(q)$ are the smallest 1-periodic and the smallest 1-antiperiodic eigenvalues of

$$x'' + \left(\lambda + q(t)\right)x = 0, \tag{1.3}$$

respectively. For the precise meaning of these eigenvalues, see Section 2.2.

Given an arbitrary potential $q \in \mathcal{L}^1$, by introducing the parameterized potentials $\lambda + q$, $\lambda \in \mathbb{R}$, Theorem 1.1 can be stated as follows.

Theorem 1.2. *Let $q \in \mathcal{L}^1$. Then $L_{\lambda+q}$ admits MP iff $\lambda \in (-\infty, \overline{\lambda}_0(q))$, and $L_{\lambda+q}$ admits AMP iff $\lambda \in (\overline{\lambda}_0(q), \underline{\lambda}_1(q)]$.*

We will also use Green functions to give complete characterizations on MP and AMP of L_q. See Theorem 4.1 and Corollary 4.4.

The paper is organized as follows. In Section 2, we will briefly introduce some concepts on Hill's equations [10, 12, 13], including the Poincaré matrixes P_q, eigenvalues $\{\underline{\lambda}_m(q), \overline{\lambda}_m(q)\}$ and rotation numbers $\varrho(q)$ and oscillation of solutions. In Section 3, we will use the Poincaré matrixes and fundamental matrix solutions to give the formula of the Green functions $G_q(t,s)$ of the periodic solution problem (1.2). We will introduce for each potential $q \in \mathcal{L}^1$ two matrixes, N_q and M_q, and two functions, $\widetilde{G}_q(t,s)$ and $\widehat{G}_q(t)$. They are related with the Poincaré matrix P_q and the Green function $G_q(t,s)$, respectively. Some remarkable properties on these new objects will be established.

Section 4 is composed of three subsections. At first, in Section 4.1, we will use the sign of Green functions $G_q(t,s)$ to establish in Theorem 4.1 and Corollary 4.4 optimal conditions for MP and AMP. Then, in Section 4.2, we will use eigenvalues to give a complete description for the sign of Green functions. The proofs of Theorems 1.1 and 1.2 will be given. One may notice that in the deduction of the sign of Green functions, besides eigenvalues, rotation numbers, and oscillation of solutions, some important estimates on Poincaré matrixes in [10, 12] will be used. Moreover, in the deduction of AMP, a very remarkable reduction for elliptic Hill's equations by Ortega [14, 15] is effectively used to simplify the argument. Note that such a reduction is originally used to deduce the formula for the first Birkhoff twist coefficient of periodic solutions of nonlinear, scalar Newtonian equations. Finally, in Section 4.3, we will outline how the known sufficient conditions on AMP can be easily deduced from Theorem 1.1.

2. Basic Facts on Hill's Equations

2.1. Fundamental Solutions and Poincaré Matrixes

Given $q \in \mathcal{L}^1$, let us introduce some basic concepts on the Hill's equation

$$x'' + q(t)x = 0. \tag{2.1}$$

Let $\varphi_i(t) = \varphi_{i,q}(t)$, $i = 1,2$, be the fundamental solutions of (2.1), that is, $\varphi_i(t)$ are solutions satisfying the initial values

$$\begin{pmatrix} \varphi_1(0) \\ \varphi_1'(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \begin{pmatrix} \varphi_2(0) \\ \varphi_2'(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{2.2}$$

The *fundamental matrix solution* of (2.1) is

$$\Phi(t) = \Phi_q(t) := \begin{pmatrix} \varphi_1(t) & \varphi_2(t) \\ \varphi_1'(t) & \varphi_2'(t) \end{pmatrix}, \quad t \in \mathbb{R}. \tag{2.3}$$

The Liouville theorem asserts that $\det \Phi(t) \equiv +1$. That is,

$$\Phi(t) \in \mathrm{SL}(2,\mathbb{R}) := \left\{ A \in \mathbb{R}^{2\times 2} : \det A = +1 \right\}, \tag{2.4}$$

the symplectic group of $\mathbb{R}^2$.

The *Poincaré matrix* of (2.1) is

$$P = P_q := \begin{pmatrix} \varphi_1(1) & \varphi_2(1) \\ \varphi_1'(\mathrm{t}) & \varphi_2'(1) \end{pmatrix} =: \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{R}). \tag{2.5}$$

In particular,

$$\det P = ad - bc = +1. \tag{2.6}$$

The *Floquet multipliers* of (2.1) are eigenvalues $\mu_{1,2} = \mu_{1,2}(q)$ of P. Then $\mu_1 \cdot \mu_2 = +1$, following from (2.6).

We say that (2.1) is *elliptic*, *hyperbolic* or *parabolic*, respectively, if $|\mu_{1,2}(q)| = 1$ and $\mu_{1,2}(q) \neq \pm 1$, $|\mu_{1,2}(q)| \neq 1$, or $\mu_{1,2}(q) = \pm 1$, respectively. We write the sets of those potentials as $\mathcal{E}^1$, $\mathcal{H}^1$ and $\mathcal{P}^1$, respectively.

By introducing the trace

$$\mathrm{tr}(q) := \mathrm{trace}(P_q) = a + d \in \mathbb{R}, \tag{2.7}$$

we have the following classification.

Lemma 2.1 (see [10]). *Equaqtion* (2.1) *is elliptic, hyperbolic, or parabolic, iff* $|\mathrm{tr}(q)| < 2$, $|\mathrm{tr}(q)| > 2$, *or* $|\mathrm{tr}(q)| = 2$, *respectively. In particular,* $q \in \mathcal{E}^1$ *implies that* $bc \neq 0$.

Proof. We need to prove the last conclusion. Suppose that $q \in \mathcal{E}^1$. If $bc = 0$, we have $ad = \det P = 1$ and $|a + d| = |\mathrm{tr}(q)| < 2$. These are impossible. □

2.2. Eigenvalues, Rotation Numbers, and Oscillation of Solutions

Given $q \in \mathcal{L}^1$, consider eigenvalue problems of (1.3) with respect to the 1-periodic boundary condition

$$x(1) - x(0) = x'(1) - x'(0) = 0, \tag{2.8}$$

or with respect to the 1-antiperiodic boundary condition

$$x(1) + x(0) = x'(1) + x'(0) = 0. \tag{2.9}$$

It is well known that one has (real) sequences

$$\overline{\lambda}_0(q) < \underline{\lambda}_1(q) \leq \overline{\lambda}_1(q) < \cdots < \underline{\lambda}_m(q) \leq \overline{\lambda}_m(q) < \cdots \tag{2.10}$$

such that

(i) $\underline{\lambda}_m(q) \to +\infty$ and $\overline{\lambda}_m(q) \to +\infty$ as $m \to \infty$;

(ii) λ is an eigenvalue of problem (1.3)–(2.8) (of problem (1.3)–(2.9), resp.) iff $\lambda = \underline{\lambda}_m(q)$ or $\lambda = \overline{\lambda}_m(q)$ where $m \in \mathbb{Z}^+$ is even ($m \in \mathbb{N}$ is odd, resp.). Here $\underline{\lambda}_0(q)$ is void;

(iii) λ is a periodic (an antiperiodic, resp.) eigenvalue of (1.3) iff

$$\mathrm{tr}(\lambda + q) = +2 \quad (\mathrm{tr}(\lambda + q) = -2, \ \text{resp.}). \tag{2.11}$$

For these general results, one can refer to [10, 11]. Note that in [10] only piecewise continuous potentials are considered. However, these are also true for $\mathcal{L}^1$ potentials. See [12, 16].

Denote

$$\mathcal{O}^1 := \left\{ q \in \mathcal{L}^1 : \text{the associated operator } L_q : \mathcal{W}^{2,1} \longrightarrow \mathcal{L}^1 \text{ is invertible} \right\}. \tag{2.12}$$

Using periodic eigenvalues or traces of Poincaré matrixes, the set $\mathcal{O}^1$ can be characterized as

$$\begin{aligned} \mathcal{O}^1 &= \left\{ q \in \mathcal{L}^1 : \underline{\lambda}_{2m}(q) \neq 0, \ \overline{\lambda}_{2m}(q) \neq 0 \ \forall m \in \mathbb{Z}^+ \right\} \\ &= \left\{ q \in \mathcal{L}^1 : \mathrm{tr}(q) = a + d \neq +2 \right\}. \end{aligned} \tag{2.13}$$

Here the equivalence of (2.13) follows from (2.11).

Let us introduce the rotation number for (2.1). Under the transformation $(x, x') = (r\sin\theta, r\cos\theta)$, we know from (2.1) that the argument θ satisfies

$$\theta' = \cos^2\theta + q(t)\sin^2\theta. \tag{2.14}$$

Definition 2.2 (see [17–19]). Given $q \in \mathcal{L}^1$. Define

$$\varrho(q) := \lim_{t \to +\infty} \frac{\theta(t)}{2\pi t}, \tag{2.15}$$

where $\theta(t)$ is any solution of (2.14). The limit (2.15) does exist and is independent of the choice of $\theta(t)$. Such a number $\varrho(q)$ is called the *rotation number* of (2.1). An alternative definition for (2.15) is

$$\varrho(q) := \lim_{b-a \to +\infty} \frac{\#\{s \in [a, b) : x(s) = 0\}}{2(b-a)} \in [0, \infty), \tag{2.16}$$

where $x(t)$ is any nonzero solution of (2.1).

The connection between eigenvalues and oscillation of solutions is as follows.

Lemma 2.3. *Given $q \in \mathcal{L}^1$, consider the parameterized Hill's equations* (1.3) *where $\lambda \in \mathbb{R}$. Then*

(i) *in case $\lambda \le \overline{\lambda}_0(q)$, any nonzero solution $x(t)$ of* (1.3) *is nonoscillatory. More precisely, $x(t)$ has at most one zero in the whole line $\mathbb{R}$;*

(ii) *in case $\lambda > \overline{\lambda}_0(q)$, any nonzero solution $x(t)$ of* (1.3) *is oscillatory. More precisely, $x(t)$ has infinitely many zeros.*

2.3. Continuous Dependence on Potentials

Associated with the Hill's equation (2.1), we have the objects $\Phi_q(t)$, P_q, $\{\underline{\lambda}_m(q), \overline{\lambda}_m(q)\}$, and $\varrho(q)$. All are determined by the potential $q \in \mathcal{L}^1$. It is a classical result that all of these objects are continuously dependent on $q \in \mathcal{L}^1$ when the L^1 topology $\|\cdot\|_1 := \|\cdot\|_{L^1(0,1)}$ is considered. For the fundamental matrix solutions, this can be stated as follows.

Lemma 2.4 (see [12, 13]). *Given $t \in \mathbb{R}$, the following mapping:*

$$\left(\mathcal{L}^1, \|\cdot\|_1\right) \ni q \longrightarrow \Phi_q(t) \in \mathbb{R}^{2\times 2} \tag{2.17}$$

is continuously Frechét differentiable. Moreover, the Frechét derivatives can be expressed using φ_i.

In the space $\mathcal{L}^1$, one has also the weak topology w_1 which is defined by

$$q_n \longrightarrow q_0 \quad \text{in } \left(\mathcal{L}^1, w_1\right) \Longleftrightarrow \int_{[0,1]} g(t) q_n(t) \mathrm{d}t \longrightarrow \int_{[0,1]} g(t) q_0(t) \mathrm{d}t \quad \forall g \in L^\infty([0,1], \mathbb{R}). \tag{2.18}$$

In a recent paper [20], Zhang has proved that these objects have stronger dependence on potentials q. Some statements of these facts are as follows.

Lemma 2.5 (Zhang [20]). *The following mapping is continuous:*

$$\left(\mathcal{L}^1, w_1\right) \ni q \longmapsto \Phi_q(\cdot) \in \left(C\left([0,1], \mathbb{R}^{2\times 2}\right), \|\cdot\|_{C([0,1])}\right). \tag{2.19}$$

Moreover, the following (nonlinear) functionals:

$$q \longmapsto \underline{\lambda}_m(q), \qquad q \longmapsto \overline{\lambda}_m(q), \qquad q \longmapsto \varrho(q) \tag{2.20}$$

are also continuous in $q \in (\mathcal{L}^1, w_1)$.

From this lemma, the set $\mathcal{O}^1$ is open in $(\mathcal{L}^1, \|\cdot\|_1)$ and in $(\mathcal{L}^1, w_1)$.

3. Green Functions and Their Variants

3.1. Green Functions

Let $q \in \mathcal{O}^1$. Then, for each $h \in \mathcal{L}^1$, (1.2) has a unique solution $x = x_h$ satisfying the 1-periodic boundary condition (2.8). From the Fredholm principle, $x = x_h$ can be represented as

$$x(t) = L_q^{-1}h(t) = \int_{[0,1]} G_q(t,s)h(s)\mathrm{d}s, \quad t \in [0,1], \tag{3.1}$$

where

$$G = G_q : D \overset{\text{def}}{=} [0,1]^2 \longrightarrow \mathbb{R} \tag{3.2}$$

is the so-called Green function of the periodic solution problem (1.2)–(2.8). Another definition of the Green function is

$$G_q(t,s) = \left(L_q^{-1}\delta_s\right)(t). \tag{3.3}$$

Here δ_s is the 1-periodic unit Dirac measure located at s. The Green function G_q can be expressed using $\varphi_i(t)$ and P as follows.

Lemma 3.1. *Given $q \in \mathcal{O}^1$, we have the following results.*

(i) *$G_q(t,s)$ is given by*

$$G_q(t,s) = \begin{cases} \dfrac{(b\varphi_1(s) + (d-1)\varphi_2(s))\varphi_1(t) + ((1-a)\varphi_1(s) - c\varphi_2(s))\varphi_2(t)}{2-a-d} & \text{for } 0 \le s \le t \le 1, \\ \dfrac{(b\varphi_1(s) + (1-a)\varphi_2(s))\varphi_1(t) + ((d-1)\varphi_1(s) - c\varphi_2(s))\varphi_2(t)}{2-a-d} & \text{for } 0 \le t \le s \le 1. \end{cases} \tag{3.4}$$

(ii) *$G_q(t,s)$ is continuous in D and is symmetric*

$$G_q(t,s) \equiv G_q(s,t) \quad \text{on } D. \tag{3.5}$$

Moreover, $G_q(t,s)$ can be extended to a continuous 1-periodic function in both arguments, that is, $G_q \in C(\mathbb{T}^2)$, $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$.

Proof. (i) Formula (3.4) can be found from related references. For completeness, let us give the proof.

Given $h \in \mathcal{L}^1$. By the constant-of-variant formula, solutions of (1.2) are given by

$$x(t) = c_1\varphi_1(t) + c_2\varphi_2(t) + \int_{[0,t]} (\varphi_2(t)\varphi_1(s) - \varphi_1(t)\varphi_2(s))h(s)\mathrm{d}s, \quad t \in [0,1], \tag{3.6}$$

where $c_i \in \mathbb{R}$ are constants. In order that $x(t)$ is 1-periodic, it is necessary and sufficient that $x(t)$ satisfies (2.8), that is, c_i satisfy

$$c_1 = ac_1 + bc_2 + \int_{[0,1]} \big(b\varphi_1(s) - a\varphi_2(s)\big)h(s)\mathrm{d}s,$$
$$c_2 = cc_1 + dc_2 + \int_{[0,1]} \big(d\varphi_1(s) - c\varphi_2(s)\big)h(s)\mathrm{d}s. \tag{3.7}$$

Since $\operatorname{tr}(q) = a + d \neq 2$, we know that

$$c_1 = \frac{1}{2-a-d}\int_{[0,1]} \big(b\varphi_1(s) + (1-a)\varphi_2(s)\big)h(s)\mathrm{d}s,$$
$$c_2 = \frac{1}{2-a-d}\int_{[0,1]} \big((d-1)\varphi_1(s) - c\varphi_2(s)\big)h(s)\mathrm{d}s. \tag{3.8}$$

Hence

$$\begin{aligned} L_q^{-1}h(t) &= \int_{[0,1]} \frac{\big(b\varphi_1(s) + (1-a)\varphi_2(s)\big)\varphi_1(t) + \big((d-1)\varphi_1(s) - c\varphi_2(s)\big)\varphi_2(t)}{2-a-d} h(s)\mathrm{d}s \\ &\quad + \int_{[0,t]} \big(-\varphi_2(s)\varphi_1(t) + \varphi_1(s)\varphi_2(t)\big)h(s)\mathrm{d}s \\ &= \int_{[0,t]} G_q(t,s)h(s)\mathrm{d}s, \end{aligned} \tag{3.9}$$

where $G_q(t,s)$ has the form of (3.4).

(ii) From formula (3.4), the symmetry (3.5) is obvious. Moreover, $G_q \in C(D)$. Finally, let us show that G_q can be extended to a continuous function on the torus $\mathbb{T}^2$. By using (2.2), (2.5), and (2.6), one has from (3.4)

$$G_q(0,s) = G_q(1,s) = \frac{b\varphi_1(s) + (1-a)\varphi_2(s)}{2-a-d}, \quad s \in [0,1]. \tag{3.10}$$

By the symmetry (3.5), one has

$$G_q(t,0) = G_q(0,t) = G_q(1,t) = G_q(t,1), \quad t \in [0,1]. \tag{3.11}$$

Thus $G_q(t,s)$ can be understood as a function on $\mathbb{T}^2$. □

In general, $G_q(t,s)$ is not differentiable at the diagonal $t = s$.

3.2. Two Matrixes and Two Functions

Let us introduce, for any $q \in \mathcal{L}^1$, the following two matrixes:

$$N = N_q := \begin{pmatrix} b & d-1 \\ 1-a & -c \end{pmatrix}, \tag{3.12}$$

$$M = M_q := \frac{1}{2}\left(N_q + N_q^{\tau}\right) = \begin{pmatrix} b & \dfrac{(d-a)}{2} \\ \dfrac{(d-a)}{2} & -c \end{pmatrix}. \tag{3.13}$$

Note that M is a symmetric matrix. Using the Poincaré matrix P, N and M can be rewritten as

$$N = J - JP^{\tau} = J - P^{-1}J, \qquad M = \frac{1}{2}\left(PJ + (PJ)^{\tau}\right). \tag{3.14}$$

Here τ denotes the transpose of matrixes, I is the identity matrix, and

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{3.15}$$

Some results on N_q and M_q and their connections with the Poincaré matrix P_q are as follows. All of them can be verified directly.

Lemma 3.2. *Given $q \in \mathcal{L}^1$, let $N = N_q$, $M = M_q$, and $P = P_q$. Then*

$$\det N = 2 - \operatorname{tr}(q), \tag{3.16}$$

$$\det M = \frac{4 - (\operatorname{tr}(q))^2}{4}, \tag{3.17}$$

$$PN = N^{\tau}, \tag{3.18}$$

$$PNP^{\tau} = N. \tag{3.19}$$

From equalities in Lemma 3.2, we have the following statements.

Lemma 3.3. *Given $q \in \mathcal{L}^1$, then*

(i) *N_q is nonsingular iff $q \in \mathcal{O}^1$, and M_q is nonsingular iff $q \in \mathcal{H}^1 \cup \mathcal{E}^1$;*

(ii) *Equation* (2.1) *is elliptic, hyperbolic, or parabolic, iff* $\det M_q > 0$, $\det M_q < 0$, *or* $\det M_q = 0$, *respectively.*

Since $q \in \mathcal{L}^1$ is 1-periodic, one has the following equality for the fundamental matrix solution

$$\Phi(t+1) = \Phi(t)P, \quad t \in \mathbb{R}. \tag{3.20}$$

Let us introduce the vector-valued function

$$\varphi(t) = \varphi_q(t) \overset{\text{def}}{=} \begin{pmatrix} \varphi_1(t) \\ \varphi_2(t) \end{pmatrix}, \quad t \in \mathbb{R}, \tag{3.21}$$

which is composed by the fundamental solutions $\varphi_i(t)$ of (2.1). Then

$$\begin{gathered} (\varphi(t), \varphi'(t)) = \Phi^\tau(t), \\ (\varphi(t+1), \varphi'(t+1)) = \Phi^\tau(t+1) = (\Phi(t)P)^\tau = P^\tau(\varphi(t), \varphi'(t)). \end{gathered} \tag{3.22}$$

Hence

$$\varphi(t+1) \equiv P^\tau \varphi(t), \quad t \in \mathbb{R}. \tag{3.23}$$

In the following, we use $\langle x, y\rangle = x^\tau y$ to denote the Euclidean inner product on $\mathbb{R}^2$. In case $q \in \mathcal{O}^1$, the Green function $G_q(t,s)$ in (3.4) can be rewritten as

$$G_q(t,s) = \begin{cases} \dfrac{1}{2-\operatorname{tr}(q)} \langle \varphi(t), N\varphi(s)\rangle & \text{for } (t,s) \in D_1 \overset{\text{def}}{=} \{(t,s) : 0 \le s \le t \le 1\}, \\ \dfrac{1}{2-\operatorname{tr}(q)} \langle \varphi(t), N^\tau\varphi(s)\rangle & \text{for } (t,s) \in D_2 \overset{\text{def}}{=} \{(t,s) : 0 \le t \le s \le 1\}. \end{cases} \tag{3.24}$$

Here $N = N_q$ is as in (3.12). Note that $D_1 \cup D_2 = D$.

Suggested by (3.24), let us introduce for any $q \in \mathcal{L}^1$ two functions

$$\widetilde{G}_q(t,s) := \langle \varphi(t), N\varphi(s)\rangle, \quad (t,s) \in \mathbb{R}^2, \tag{3.25}$$

$$\widehat{G}_q(t) := \widetilde{G}_q(t,t) \equiv \langle \varphi(t), M\varphi(t)\rangle, \quad t \in \mathbb{R}, \tag{3.26}$$

where $N = N_q$ and $M = M_q$ are as in (3.12) and (3.13). Note that these functions are well defined on the whole plane and the whole line, respectively. Some properties are as follows.

Lemma 3.4. *For any $q \in \mathcal{L}^1$, one has*

$$\widetilde{G}_q(t+k, s+k) \equiv \widetilde{G}_q(t,s), \quad \forall (t,s) \in \mathbb{R}^2,\ k \in \mathbb{Z}, \tag{3.27}$$

$$\widetilde{G}_q(s+1, s) \equiv \widetilde{G}_q(s,s), \quad \forall s \in \mathbb{R}, \tag{3.28}$$

$$\widehat{G}_q(t+k) \equiv \widehat{G}_q(t), \quad \forall t \in \mathbb{R},\ k \in \mathbb{Z}. \tag{3.29}$$

Proof. We need only to verify (3.27) for the case $k = 1$. To this end, one has

$$\begin{aligned}\widetilde{G}_q(t+1,s+1) &= \langle \varphi(t+1), N\varphi(s+1)\rangle \quad (\text{by } (3.25)) \\ &= \langle P^\tau\varphi(t), NP^\tau\varphi(s)\rangle \quad (\text{by } (3.23)) \\ &= \langle \varphi(t), PNP^\tau\varphi(s)\rangle \\ &= \langle \varphi(t), N\varphi(s)\rangle \quad (\text{by } (3.19)) \\ &= \widetilde{G}_q(t,s) \quad (\text{by } (3.25)).\end{aligned} \tag{3.30}$$

For (3.28), we have

$$\begin{aligned}\widetilde{G}_q(s+1,s) &= \langle \varphi(s+1), N\varphi(s)\rangle \quad (\text{by } (3.25)) \\ &= \langle P^\tau\varphi(s), N\varphi(s)\rangle \quad (\text{by } (3.23)) \\ &= \langle \varphi(s), PN\varphi(s)\rangle \\ &= \langle \varphi(s), N^\tau\varphi(s)\rangle \quad (\text{by } (3.18)) \\ &= \langle N\varphi(s), \varphi(s)\rangle \\ &= \widetilde{G}_q(s,s) \quad (\text{by } (3.25)).\end{aligned} \tag{3.31}$$

Finally, equality (3.29) follows simply from (3.26) and (3.27). □

We remark that, in general, $\widetilde{G}_q(s+k,s) = \widetilde{G}_q(s,s)$ is not true for $k \in \mathbb{Z} \setminus \{0,1\}$. Note that (3.29) asserts that $\widehat{G}_q(t)$ is 1-periodic. Some further properties on $\widehat{G}_q(t)$ are as follows.

Lemma 3.5. (i) *Let $q \in \mathcal{E}^1$. Then $\widehat{G}_q(t)$ does not have any zero and therefore does not change sign.*

(ii) *Let $q \in \mathcal{H}^1$. Then $\widehat{G}_q(t)$ has only nondegenerate zeros, if they exist.*

(iii) *Let $q \in \mathcal{P}^1$. Then $\widehat{G}_q(t)$ has a constant sign. Moreover,*

$$\widehat{G}_q(t) \equiv 0 \Longleftrightarrow P_q = \pm I. \tag{3.32}$$

Proof. (i) Suppose that $q \in \mathcal{E}^1$ is elliptic. We have $b \neq 0$ from Lemma 2.1. By (3.17), $\det M > 0$. Hence the symmetric matrix M is either positive definite or negative definite, according to $b > 0$ or $b < 0$. Since $\varphi(t) \neq 0$ for all t, we know that $\widehat{G}_q(t) = \langle \varphi(t), M\varphi(t)\rangle \neq 0$ on $\mathbb{R}$.

(ii) Suppose that $q \in \mathcal{H}^1$. We have $\det M < 0$. Thus there exists an orthogonal transformation $V = V_q$ such that

$$M = V^{-1} \cdot \operatorname{diag}(\nu_1, \nu_2) \cdot V. \tag{3.33}$$

Here $\nu_k = \nu_k(q)$ are eigenvalues of M and satisfy $\nu_1 \cdot \nu_2 < 0$. Then

$$\widehat{G}_q(t) = \nu_1 \varphi_1^2(t) + \nu_2 \varphi_2^2(t), \quad \begin{pmatrix} \varphi_1(t) \\ \varphi_2(t) \end{pmatrix} := V \begin{pmatrix} \varphi_1(t) \\ \varphi_2(t) \end{pmatrix}. \tag{3.34}$$

Note that $\{\varphi_1, \varphi_2\}$ is also a system of fundamental solutions of (2.1). As $\nu_1 \cdot \nu_2 < 0$, we have

$$\widehat{G}_q(t) = \nu_1 \varphi_+(t) \varphi_-(t), \tag{3.35}$$

where

$$\varphi_\pm(t) := \varphi_1(t) \pm c\varphi_2(t), \qquad c := \sqrt{\left|\frac{\nu_2}{\nu_1}\right|} > 0. \tag{3.36}$$

Note that $\{\varphi_+, \varphi_-\}$ is a linearly independent system of solutions of (2.1). From (3.35), $\widehat{G}_q(t)$ has only nondegenerate zeros, if they exist. In fact, suppose that $\widehat{G}_q(t_0) = 0$, say $\varphi_+(t_0) = 0$. We have ${\varphi'}_+(t_0) \neq 0$ and $\varphi_-(t_0) \neq 0$. Thus

$$\widehat{G}_q'(t_0) = \nu_1 \varphi_+'(t_0) \varphi_-(t_0) \neq 0. \tag{3.37}$$

(iii) Suppose that $q \in \mathcal{P}^1$. We have $\det M_q = 0$. Then one eigenvalue of M_q is 0 and another is $\operatorname{trace}(M_q) = b - c$. In this case,

$$\widehat{G}_q(t) \equiv (b - c)\varphi^2(t), \tag{3.38}$$

where $\varphi(t)$ is a nonzero solution of (2.1). This shows that $\widehat{G}_q(t)$ does not change sign.

We distinguish two cases.

(i) $q \in \mathcal{P}^1$ is stable-parabolic, that is, $P_q = \pm I$. In this case, one has $M_q = 0$ and $\widehat{G}_q(t) \equiv 0$.

(ii) $q \in \mathcal{P}^1$ is unstable-parabolic, that is, $P_q \neq \pm I$. In this case, we assert that $b - c \neq 0$.

Otherwise, assume $b - c = 0$. Then

$$P_q = \begin{pmatrix} a & b \\ b & d \end{pmatrix}. \tag{3.39}$$

Since $ad - b^2 = \det P_q = 1$ and $a + d = \operatorname{trace}(P_q) = \pm 2$, we obtain $(a \mp 1)^2 + b^2 = 0$. Hence $a = \pm 1$ and $b = 0$. Moreover, $d = a = \pm 1$. Thus $P_q = \pm I$ and q is stable-parabolic. In conclusion, for unstable-parabolic case, we have $b - c \neq 0$. Now it follows from (3.38) that $\widehat{G}_q(t) \not\equiv 0$. As proved before, $\widehat{G}_q(t)$ does not change sign. Moreover, it is easy to see from (3.38) that all zeros of $\widehat{G}_q(t)$ must be degenerate, if they exist.

From these, (3.32) is clear. □

4. Optimal Conditions for MP and AMP

4.1. Complete Characterizations of MP and AMP Using Green Functions

Using Green functions $G_q(t,s)$, we have the following characterizations on MP and AMP.

Theorem 4.1. *Let $q \in \mathcal{O}^1$ with the Green function $G_q(t,s)$. Then L_q admits MP iff $\max_{(t,s)\in D} G_q(t,s) \le 0$, and L_q admits AMP iff $\min_{(t,s)\in D} G_q(t,s) \ge 0$.*

Proof. We give only the proof for AMP.

The sufficiency is as follows. Suppose that $q \in \mathcal{O}^1$ satisfies $\min_{(t,s)\in D} G_q(t,s) \ge 0$. Then, for any $h \succ 0$, it is easy to see from (3.1) that $x_h(t) \ge 0$ for all $t \in [0,1]$. We will show that $x_h(t) > 0$ for all t and consequently (1.2) admits AMP.

Otherwise, suppose that $x_h(t_0) = 0$ for some $t_0 \in [0,1]$, that is,

$$\int_{[0,1]} G_q(t_0,s)h(s)\mathrm{d}s = 0. \tag{4.1}$$

Since $G_q(t_0,\cdot)h(\cdot) \ge 0$, we have necessarily

$$G_q(t_0,s)h(s) = 0 \quad \text{a.e. } s \in [0,1]. \tag{4.2}$$

From (3.24), we know that

(i) on the interval $[0,t_0]$,

$$G_q(t_0,s) = \frac{1}{2-\mathrm{tr}(q)}\left\langle N^{\tau}\varphi(t_0), \varphi(s)\right\rangle \tag{4.3}$$

is a solution of (2.1);

(ii) on the interval $[t_0,1]$,

$$G_q(t_0,s) = \frac{1}{2-\mathrm{tr}(q)}\left\langle N\varphi(t_0), \varphi(s)\right\rangle \tag{4.4}$$

is also a solution of (2.1).

We assert that these solutions are nonzero when the corresponding intervals are nontrivial. As $\varphi(s)$ is composed of two linearly independent solutions $\varphi_i(s)$, the nontriviality of these solutions is the same as

$$N^{\tau}\varphi(t_0) \ne 0, \qquad N\varphi(t_0) \ne 0, \tag{4.5}$$

which are evident because $\varphi(t_0) \ne 0$ and (3.16) shows that $\det N \ne 0$.

From the above assertion, we know that $G_q(t_0,s)$ (≥ 0) has only isolated zeros for $s \in [0,1]$. As $h \succ 0$, we have $G_q(t_0,\cdot)h(\cdot) \succ 0$, a contradiction with (4.2).

For the necessity, let us assume that $\min_{(t,s)\in D} G_q(t,s) < 0$. Then one has some $(t_0, s_0) \in D$ so that $G_q(t_0, s_0) < 0$. Hence one has some $\delta_0 > 0$ such that

$$G_q(t_0, s) \le \frac{G_q(t_0, s_0)}{2} < 0 \quad \forall s \in [0,1] \text{ with } |s - s_0| \le \delta_0. \tag{4.6}$$

Let us choose $h \in C^\infty(\mathbb{S}) \subset \mathcal{L}^1$ such that

$$h(s) = \begin{cases} > 0 & \text{for } |s - s_0| \le \delta_0, \\ 0 & \text{for } |s - s_0| > \delta_0. \end{cases} \tag{4.7}$$

Then $h \succ 0$. However, the corresponding periodic solution $x = x_h$ of (1.2) satisfies

$$x_h(t_0) = \int_{[0,1]\cap[s_0-\delta_0, s_0+\delta_0]} G_q(t_0, s) h(s) \mathrm{d}s < 0. \tag{4.8}$$

Hence L_q does not admit AMP. □

In order to apply Theorem 4.1, it is important to compute the signs of the following nonlinear functionals of potentials:

$$\mathcal{O}^1 \ni q \longmapsto \min_{(t,s)\in D} G_q(t,s) \in \mathbb{R}, \qquad \mathcal{O}^1 \ni q \longmapsto \max_{(t,s)\in D} G_q(t,s) \in \mathbb{R}. \tag{4.9}$$

To this end, let us establish some relation between $G_q(t,s)$ and $\widetilde{G}_q(t,s)$.

For general $q \in \mathcal{L}^1$, denote

$$s_q := \operatorname{sign}(2 - a - d) = \operatorname{sign}(2 - \operatorname{tr}(q)) \in \{0, \pm 1\}. \tag{4.10}$$

Suppose that $q \in \mathcal{O}^1$ so that $G_q(t,s)$ is meaningful. We assert that

$$|2 - \operatorname{tr}(q)| \cdot G_q(t,s) = \begin{cases} s_q \widetilde{G}_q(t,s), & (t,s) \in D_1, \\ s_q \widetilde{G}_q(t+1,s), & (t,s) \in D_2. \end{cases} \tag{4.11}$$

In fact, for $(t,s) \in D_1$, the first case of (4.11) follows immediately from the defining equalities (3.24), (3.25), and (4.10). On the other hand, for $(t,s) \in D_2$, from the second case of (3.24), one has

$$\begin{aligned}
|2-\operatorname{tr}(q)|\cdot G_q(t,s) &= s_q\langle \varphi(t), N^\tau \varphi(s)\rangle \\
&= s_q\left\langle (P^\tau)^{-1}\varphi(t+1), N^\tau\varphi(s)\right\rangle \quad (\text{by } (3.23)) \\
&= s_q\left\langle \varphi(t+1), P^{-1}N^\tau\varphi(s)\right\rangle \\
&= s_q\langle \varphi(t+1), N\varphi(s)\rangle \qquad (\text{by } (3.18)) \\
&= s_q\widetilde{G}_q(t+1,s) \qquad (\text{by } (3.25)).
\end{aligned} \tag{4.12}$$

Hence (4.11) is also true for this case.

By introducing the domain

$$\widetilde{D} \stackrel{\text{def}}{=} \{(t,s) : s \le t \le s+1,\ 0 \le s \le 1\} \tag{4.13}$$

and the following nonlinear functionals $\underline{\mathbf{G}}$, $\overline{\mathbf{G}} : \mathcal{L}^1 \to \mathbb{R}$

$$\underline{\mathbf{G}}(q) \stackrel{\text{def}}{=} \min_{(t,s)\in\widetilde{D}} s_q\widetilde{G}_q(t,s), \qquad \overline{\mathbf{G}}(q) \stackrel{\text{def}}{=} \max_{(t,s)\in\widetilde{D}} s_q\widetilde{G}_q(t,s), \tag{4.14}$$

we have the following statements.

Lemma 4.2. *There hold, for all $q \in \mathcal{O}^1$,*

$$\min_{(t,s)\in D} G_q(t,s) = \frac{\underline{\mathbf{G}}(q)}{|2-\operatorname{tr}(q)|}, \qquad \max_{(t,s)\in D} G_q(t,s) = \frac{\overline{\mathbf{G}}(q)}{|2-\operatorname{tr}(q)|}. \tag{4.15}$$

Proof. We only prove the first equality of (4.15) because the second one is similar. By (4.11), for any $s \in [0,1]$, we have

$$\begin{aligned}
|2-\operatorname{tr}(q)|\cdot \min_{t\in[s,1]} G_q(t,s) &= \min_{t\in[s,1]} s_q\widetilde{G}_q(t,s), \\
|2-\operatorname{tr}(q)|\cdot \min_{t\in[0,s]} G_q(t,s) &= \min_{t\in[0,s]} s_q\widetilde{G}_q(t+1,s) = \min_{t\in[1,s+1]} s_q\widetilde{G}_q(t,s).
\end{aligned} \tag{4.16}$$

Hence

$$\begin{aligned}
|2-\operatorname{tr}(q)|\cdot \min_{t\in[0,1]} G_q(t,s) &= \min\left\{ \min_{t\in[0,s]} s_q\widetilde{G}_q(t,s), \min_{t\in[s,1]} s_q\widetilde{G}_q(t,s) \right\} \\
&= \min\left\{ \min_{t\in[s,1]} s_q\widetilde{G}_q(t,s), \min_{t\in[1,s+1]} s_q\widetilde{G}_q(t,s) \right\} \\
&= \min_{t\in[s,s+1]} s_q\widetilde{G}_q(t,s), \quad \forall s \in [0,1].
\end{aligned} \tag{4.17}$$

Consequently,

$$\left|2-\mathrm{tr}(q)\right|\cdot \min_{(t,s)\in D} G_q(t,s) = \min_{s\in[0,1]}\min_{t\in[s,s+1]} s_q\widetilde{G}_q(t,s) = \min_{(t,s)\in\widetilde{D}} s_q\widetilde{G}_q(t,s) = \underline{\mathbf{G}}(q). \tag{4.18}$$

This is just (4.15) because $\mathrm{tr}(q)\neq 2$. □

Remark 4.3. (i) The functionals $\underline{\mathbf{G}}(q)$ and $\overline{\mathbf{G}}(q)$ are well defined for all potentials $q \in \mathcal{L}^1$. Moreover, by (4.15), $\underline{\mathbf{G}}(q)$ and $\overline{\mathbf{G}}(q)$ have the same signs with the functionals in (4.9).

(ii) Compared with the defining formulas in (4.9), the novelty of formulas in (4.14) is that when s is fixed, $s_q\widetilde{G}_q(\cdot,s)$ is a solution of (2.1), while when t is fixed, $s_q\widetilde{G}_q(t,\cdot)$ is also a solution of (2.1). A similar observation is used in [8] as well.

(iii) Due to the factor s_q which is zero at those $q \in \mathcal{L}^1 \setminus \mathcal{O}^1$, $\underline{\mathbf{G}}(q)$ and $\overline{\mathbf{G}}(q)$ are in general discontinuous at $q \in \mathcal{L}^1 \setminus \mathcal{O}^1$. However, $\underline{\mathbf{G}}(q)$ and $\overline{\mathbf{G}}(q)$ are continuous at $q \in \mathcal{O}^1$ in the L^1 topology $\|\cdot\|_1$ or even in the weak topology w_1. See Lemmas 2.4 and 2.5.

By Lemma 4.2, Theorem 4.1 can be restated as follows.

Corollary 4.4. *Let $q \in \mathcal{O}^1$. Then L_q admits MP iff $\overline{\mathbf{G}}(q) \le 0$, and L_q admits AMP iff $\underline{\mathbf{G}}(q) \ge 0$.*

4.2. Complete Characterizations of MP and AMP Using Eigenvalues

Lemma 4.5. *Let $q \in \mathcal{L}^1$ be such that $\overline{\lambda}_0(q) > 0$. Then $\overline{\mathbf{G}}(q) < 0$ and L_q admits MP.*

Proof. For simplicity, denote

$$Q_0 := \left\{q \in \mathcal{L}^1 : \overline{\lambda}_0(q) > 0\right\}. \tag{4.19}$$

For any $q \in Q_0$, one has $a+d > 2$ and $s_q = -1$. See [10]. Thus $Q_0 \subset \mathcal{H}^1$. In the following let us fix any $q \in Q_0$.

Step 1. We assert that

$$\widetilde{G}_q(t,t) = \widehat{G}_q(t) \neq 0 \quad \forall t \in \mathbb{R}. \tag{4.20}$$

Since $q \in \mathcal{H}^1$, we can use the representation (3.35) for $\widehat{G}_q(t)$ where $\nu_1 \neq 0$ and $\varphi_\pm(t)$ are nonzero solutions of (2.1). Since $\overline{\lambda}_0(q) > 0$, both $\varphi_\pm(t)$ have at most one zero. See Lemma 2.3. Hence $\widehat{G}_q(t)$ has at most two zeros. However, as $\widehat{G}_q(t)$ is 1-periodic, $\widehat{G}_q(t)$ does not have any zero. This proves (4.20).

Step 2. We assert that

$$\widetilde{G}_q(t,s) \neq 0 \quad \forall (t,s) \in \widetilde{D}. \tag{4.21}$$

If (4.21) is false, there exists $(t_0, s_0) \in \widetilde{D}$ such that $\widetilde{G}_q(t_0, s_0) = 0$. By introducing

$$x_0(t) := \widetilde{G}_q(t, s_0), \quad t \in \mathbb{R}, \tag{4.22}$$

one has

$$x_0(t_0) = 0. \tag{4.23}$$

We know from (3.28) and (4.20) that $x_0(t)$ satisfies

$$x_0(s_0 + 1) = \widetilde{G}_q(s_0 + 1, s_0) = \widetilde{G}_q(s_0, s_0) = x_0(s_0) \neq 0. \tag{4.24}$$

This shows that $t_0 \in (s_0, s_0 + 1)$. Since $x_0(t)$ is a nonzero solution of (2.1), (4.23) implies

$$x_0'(t_0) \neq 0. \tag{4.25}$$

Since $x_0(t)$ has the same nonzero value at the end-points of the interval $[s_0, s_0+1]$, it is easy to see from (4.24) and (4.25) that $x_0(t)$ must have another zero $t_1 \in (s_0, s_0 + 1)$ which is different from t_0. Consequently, the solution $x_0(t)$ of (2.1) has at least zeros t_0 and t_1. This is impossible because $\overline{\lambda}_0(q) > 0$. See Lemma 2.3.

Step 3. Let us notice that

$$b = \langle (1,0)^\tau, M(1,0)^\tau \rangle = \widehat{G}_q(0) = \widetilde{G}_q(0,0) \quad \forall q \in \mathcal{L}^1. \tag{4.26}$$

We assert that

$$b = b(q) > 0 \quad \forall q \in Q_0. \tag{4.27}$$

To prove (4.27), let us fix $q \in Q_0$ and consider $q_\lambda(t) := \lambda + q(t)$, where $\lambda \in (-\infty, 0]$. Then $q_0 = q$. Since $\overline{\lambda}_0(q_\lambda) = -\lambda + \overline{\lambda}_0(q)$, $q_\lambda \in Q_0$ for all $\lambda \in (-\infty, 0]$. When $\lambda \ll -1$, $b(q_\lambda)$ can be estimated. The basic idea is to consider (1.3) as a perturbation of the equation

$$x'' + \lambda x = 0 \tag{4.28}$$

for which

$$b(\lambda) = \frac{\sinh \omega}{\omega}, \quad \lambda = -\omega^2, \ \omega > 0. \tag{4.29}$$

It is well known that the difference $b(q_\lambda) - b(\lambda)$ can be controlled by the norm of the potential q when $\lambda \ll -1$. For piecewise continuous and L^2 potentials, see [10] and [12], respectively. Similar estimates are also true for L^1 potentials. In fact, these can be generalized to Hill's equations with coefficients being measures [16]. We quote from [12, Theorem 3] the following result:

$$\left|b(q_\lambda) - b(\lambda)\right| \leq \frac{\exp(\omega + \|q\|_1)}{\omega^2}. \tag{4.30}$$

Hence

$$\begin{aligned} b(q_\lambda) &\geq \frac{\sinh \omega}{\omega} - \frac{\exp(\omega + \|q\|_1)}{\omega^2} \\ &= \frac{\sinh \omega}{\omega}\left(1 + O\left(\frac{1}{\omega}\right)\right) \\ &\longrightarrow +\infty \end{aligned} \tag{4.31}$$

as $\omega \to +\infty$. We conclude

$$b(q_\lambda) > 0 \quad \forall \lambda \ll -1. \tag{4.32}$$

On the other hand, by (4.21) and (4.26),

$$b(q_\lambda) \neq 0 \quad \forall \lambda \in (-\infty, 0]. \tag{4.33}$$

Moreover, it follows from Lemma 2.4 that $b(q_\lambda)$ is continuous in $\lambda \in (-\infty, 0]$. Thus (4.27) follows simply from (4.32) and (4.33).

Step 4. Since $s_q = -1$, $s_q\widetilde{G}_q(t,s) \equiv -\widetilde{G}_q(t,s)$. It follows from (4.21), (4.26), and (4.27) that, for all $(t,s) \in \widetilde{D}$, $s_q\widetilde{G}_q(t,s) = -\widetilde{G}_q(t,s)$ has the same sign with $-b(q) < 0$. Thus $\overline{\mathbf{G}}(q) < 0$. By Corollary 4.4, L_q admits MP. □

Lemma 4.6. *Suppose that $q \in \mathcal{L}^1$ satisfies $\overline{\lambda}_0(q) < 0$ and $\underline{\lambda}_1(q) > 0$. Then $\underline{\mathbf{G}}(q) > 0$ and L_q admits AMP.*

Proof. For simplicity, denote

$$Q_1 := \left\{q \in \mathcal{L}^1 : \overline{\lambda}_0(q) < 0 < \underline{\lambda}_1(q)\right\}. \tag{4.34}$$

Recall from [11] that eigenvalues $\overline{\lambda}_0(q)$ and $\underline{\lambda}_1(q)$ can be characterized using rotation numbers by

$$\overline{\lambda}_0(q) = \max\{\lambda \in \mathbb{R} : \varrho(\lambda + q) = 0\}, \qquad \underline{\lambda}_1(q) = \min\left\{\lambda \in \mathbb{R} : \varrho(\lambda + q) = \frac{1}{2}\right\}. \tag{4.35}$$

Here $q \in \mathcal{L}^1$ is arbitrary. Hence

$$Q_1 = \left\{ q \in \mathcal{L}^1 : 0 < \varrho(q) < \frac{1}{2} \right\} \subset \mathcal{E}^1. \tag{4.36}$$

In the following, let $q \in \mathcal{E}^1$. We have $\operatorname{tr}(q) \in (-2,2)$, $s_q = +1$ and $s_q \widetilde{G}_q(t,s) \equiv \widetilde{G}_q(t,s)$. Now we argue as in the proof of Lemma 4.5. In this case, result (4.20) can be obtained from Lemma 3.5(i) because $q \in \mathcal{E}^1$. If (4.21) is false at some $(t_0, s_0) \in \widetilde{D}$, we have also $t_0 \in (s_0, s_0+1)$. By letting $x_0(t)$ be as in (4.22), one has also some $t_1 \in (s_0, s_0+1)$ such that $x_0(t_1) = 0$ and $t_1 \neq t_0$. With loss of generality, let us assume that $s_0 < t_0 < t_1 < s_0 + 1$. Notice that the solution $x_0(t) = \widetilde{G}_q(t, s_0)$ of (2.1) has zeros t_0 and t_1. By the Sturm comparison theorem, any nonzero solution $x(t)$ of (2.1) has at least one zero in $[t_0, t_1]$. In particular, for any $n \in \mathbb{N}$, $\widetilde{G}_q(t, s_0 - n)$ is a solution of (2.1). Hence there exists some $\widehat{t}_n \in [t_0, t_1]$ such that

$$\widetilde{G}_q\left(\widehat{t}_n, s_0 - n\right) = 0. \tag{4.37}$$

By equality (3.27),

$$\widetilde{G}_q(t+n, s_0) \equiv \widetilde{G}_q(t, s_0 - n). \tag{4.38}$$

Thus

$$x_0\left(\widetilde{t}_n\right) = \widetilde{G}_q\left(\widetilde{t}_n, s_0\right) = 0, \quad \widetilde{t}_n := \widehat{t}_n + n \in [t_0+n, t_1+n] \subset (s_0+n, s_0+n+1). \tag{4.39}$$

From these, the distribution of zeros of the specific solution $x_0(t) = \widetilde{G}_q(t, s_0)$ satisfies

$$\#\left\{ t \in [0, s_0+n) : \widetilde{G}_q(t, s_0) = 0 \right\} \geq n+1 \quad \forall n \in \mathbb{N}. \tag{4.40}$$

By definition (2.16) for the rotation number, we obtain

$$\varrho(q) = \lim_{n \to +\infty} \frac{\#\left\{ t \in [0, s_0+n) : \widetilde{G}_q(t, s_0) = 0 \right\}}{2(s_0+n)} \geq \frac{1}{2}, \tag{4.41}$$

a contradiction with the characterization of $q \in Q_1$. Thus (4.21) is also true for $q \in Q_1$.

Since $s_q = +1$, we have from (4.21) and (4.26) that $\operatorname{sign}(\underline{\mathbf{G}}(q)) = \operatorname{sign}(b) = +1$, because we will prove in Lemma 4.7 that $b = b(q) > 0$ for all $q \in Q_1$. □

Note that Q_1 is the set of potentials which are in the first ellipticity zone. By Lemmas 2.1 or 3.5, $b(q) \neq 0$ for all $q \in Q_1$. It seems that there are several ways to deduce that $b(q) > 0$ for all $q \in Q_1$. However, some remarkable result on elliptic Hill's equations by Ortega [14, 15] can simplify the argument. Let us describe the result. Suppose that $q \in \mathcal{E}^1$. Consider the temporal-spatial transformation

$$\tau = \frac{(t - t_0)}{\alpha}, \qquad y(\tau) = x(t_0 + \alpha\tau), \tag{4.42}$$

where $t_0 \in \mathbb{R}$ and $\alpha > 0$. Then (2.1) is transformed into a new Hill's equation

$$\frac{d^2 y}{d\tau^2} + q^*(\tau) y = 0, \tag{4.43}$$

where q^* is now $T^* := 1/\alpha$ periodic. The result of Ortega shows that it is always possible to choose some $t_0,\ \alpha$ such that the Poincaré matrix P_{q^*} (of the period T^*) of (4.43) is a rigid rotation

$$P_{q^*} = R_\vartheta := \begin{pmatrix} \cos\vartheta & -\sin\vartheta \\ \sin\vartheta & \cos\vartheta \end{pmatrix}. \tag{4.44}$$

See [15, Lemma 4.1] and [21]. We remark that such a result is very important to study the twist character and Lyapunov stability of periodic solutions of nonlinear Newtonian equations and planar Hamiltonian systems. See, for example, [14, 15, 21].

Note that the transformation (4.42) does not change rotation numbers. Recall that the polar coordinates to define rotation numbers are

$$x = r\cos\left(\frac{\pi}{2} - \theta\right), \qquad x' = r\ \sin\left(\frac{\pi}{2} - \theta\right). \tag{4.45}$$

We see from (4.44) that ϑ is related with $\varrho(q)$ via

$$-\vartheta \equiv 2\pi\varrho(q) \mod 2\pi\mathbb{Z}. \tag{4.46}$$

Hence

$$b(q^*) = -\sin\vartheta = \sin 2\pi\varrho(q). \tag{4.47}$$

Lemma 4.7. *We assert that*

$$b(q) > 0 \quad \forall q \in Q_1. \tag{4.48}$$

Proof. We first prove that $\operatorname{sign}(b(q))$, $q \in \mathcal{E}^1$, is invariant under transformations (4.42). In fact, it is well known that P_{q^*} and P_q are conjugate

$$P_{q^*} = V^{-1} P_q V \tag{4.49}$$

for some $V \in \mathrm{SL}(2,\mathbb{R})$. Denote

$$V = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R}). \tag{4.50}$$

From (4.49), one has the explicit relation

$$b(q^*) = b\delta^2 + (a-d)\delta\gamma - c\gamma^2 = \langle v, M_q v\rangle, \quad v := (\delta, -\gamma)^{\tau}. \tag{4.51}$$

Note that the quadratic form $\langle x, M_q x\rangle$ is definite. See the proof of Lemma 3.5(i). Since $v \neq 0$, we have

$$\operatorname{sign}(b(q^*)) = \operatorname{sign}(\langle v, M_q v\rangle) = \operatorname{sign}(\langle (1,0)^{\tau}, M_q(1,0)^{\tau}\rangle) = \operatorname{sign}(b(q)). \tag{4.52}$$

Hence $\operatorname{sign}(b(q))$ is invariant under transformations (4.42).

Now (4.48) can be obtained as follows. Let $q \in Q_1$. Then $\varrho(q) \in (0, 1/2)$. By (4.47), the transformed potential q^* satisfies $b(q^*) > 0$. By the invariance, we have the desired result (4.48). □

Lemma 4.8. *Suppose that $q \in \mathcal{L}^1$ satisfies $\underline{\lambda}_1(q) = 0$. Then $\underline{\mathbf{G}}(q) = 0$ and L_q admits AMP.*

Proof. Since $\underline{\lambda}_1(q) = 0$, we have $\operatorname{tr}(q) = -2$ and $s_q = +1$. See (2.11). Moreover, by (2.10), we have $\overline{\lambda}_0(q) < \underline{\lambda}_1(q) = 0$. Let $q_\varepsilon := q - \varepsilon$. Then $q_\varepsilon \in Q_1$ for all $0 < \varepsilon \ll 1$. We know from Lemma 4.6 that $\underline{\mathbf{G}}(q_\varepsilon) > 0$ for $0 < \varepsilon \ll 1$. Letting $\varepsilon \downarrow 0$ and noticing that $\underline{\mathbf{G}}(q_\varepsilon)$ is continuous at $\varepsilon = 0$, we get

$$\underline{\mathbf{G}}(q) = \lim_{\varepsilon \downarrow 0} \underline{\mathbf{G}}(q_\varepsilon) \geq 0. \tag{4.53}$$

On the other hand, let us take an antiperiodic eigen function $y(t)$ of (2.1) associated with $\underline{\lambda}_1(q) = 0$. Denote by t_0 the smallest nonnegative zero of $y(t)$. Then $t_0 \in [0,1)$. Moreover, both t_0 and $t_0 + 1$ are zeros of $y(t)$ because of the 1-antiperiodicity of $y(t)$. By the Sturm comparison theorem, the solution $\widetilde{G}_q(t, t_0)$ of (2.1) must have some zero in $[t_0, t_0 + 1]$. Hence $\min_{t \in [t_0, t_0+1]} \widetilde{G}_q(t, t_0) \leq 0$. As $s_q = +1$, we obtain

$$\underline{\mathbf{G}}(q) \leq \min_{t \in [t_0, t_0+1]} \widetilde{G}_q(t, t_0) \leq 0. \tag{4.54}$$

In conclusion we have $\underline{\mathbf{G}}(q) = 0$. □

Lemma 4.9. *Suppose that $q \in \mathcal{L}^1$ satisfies $\underline{\lambda}_1(q) < 0$. Then L_q does not admit neither MP nor AMP.*

Proof. We need not to consider the case $q \in \mathcal{L}^1 \setminus \mathcal{O}^1$ because L_q is not invertible.

In the following let us assume that $q \in \mathcal{O}^1$ satisfies $\underline{\lambda}_1(q) < 0$. Then $s_q = \pm 1$. The following is a modification of the last part of the proof of Lemma 4.8.

Let us take an antiperiodic eigenfunction $y(t)$ associated with $\underline{\lambda}_1(q)$. Then the set of all zeros of $y(t)$ is $\{t_0 + k\}_{k\in\mathbb{Z}}$ for some $t_0 \in [0,1)$. Denote

$$x_0(t) := \widetilde{G}_q(t,t_0), \quad t \in \mathbb{R}. \tag{4.55}$$

Then $x_0(t)$ is a nonzero solution of (2.1). Since $\underline{\lambda}_1(q) < 0$, by applying the Sturm comparison theorem to $y(t)$ and $x_0(t)$, we know that $x_0(t)$ must have some zero $\widehat{t}_0$ in (t_0, t_0+1), the interior of the interval $[t_0, t_0+1]$ because t_0 and t_0+1 are consecutive zeros of $y(t)$. As $x_0(t) \not\equiv 0$, one must have

$$\widehat{t}_0 \in (t_0, t_0+1), \quad x_0\left(\widehat{t}_0\right) = 0, \quad x_0'\left(\widehat{t}_0\right) \neq 0. \tag{4.56}$$

Thus $x_0(t)$ changes sign near $\widehat{t}_0$. Consequently,

$$\underline{\mathbf{G}}(q) \leq \min_{t\in[t_0,t_0+1]} s_q \widetilde{G}_q(t,t_0) = \min_{t\in[t_0,t_0+1]} s_q x_0(t) < 0. \tag{4.57}$$

Now Corollary 4.4 shows that L_q does not admit AMP. We have also

$$\overline{\mathbf{G}}(q) \geq \max_{t\in[t_0,t_0+1]} s_q x_0(t) > 0. \tag{4.58}$$

Hence L_q does not admit MP. □

Due to the ordering (2.10) of eigenvalues, the statements in Theorems 1.1 and 1.2 are equivalent. Now let us give the proof of Theorem 1.2. Recall that $\overline{\lambda}_0(\lambda+q) = -\lambda + \overline{\lambda}_0(q)$ and $\underline{\lambda}_1(\lambda+q) = -\lambda + \underline{\lambda}_1(q)$ for all $\lambda \in \mathbb{R}$. By Lemma 4.5, if $\lambda \in (-\infty, \overline{\lambda}_0(q))$, $L_{\lambda+q}$ admits MP. By Lemmas 4.6 and 4.8, $L_{\lambda+q}$ admits AMP for $\lambda \in (\overline{\lambda}_0(q), \underline{\lambda}_1(q)]$. By Lemma 4.9, $L_{\lambda+q}$ does not admit MP nor AMP for $\lambda \in (\underline{\lambda}_1(q), \infty)$. Using the ordering (2.10) for eigenvalues, we complete the proof of Theorem 1.2.

From Lemmas 4.5, 4.6, 4.8, and 4.9, the sign of Green functions is clear in all cases.

Definition 4.10. Given $q \in \mathcal{L}^1$, we say that L_q admits *strong antimaximum principle* (SAMP) if L_q admits AMP and, moreover, there exists $c_q > 0$ such that

$$\min_t \left(L_q^{-1} h\right)(t) \geq c_q \|h\|_1 \quad \forall h \in \mathcal{L}^1,\ h \succ 0. \tag{4.59}$$

Then we have the following complete characterizations for SAMP.

Theorem 4.11. *Let $q \in \mathcal{L}^1$. Then L_q admits SAMP iff $\overline{\lambda}_0(q) < 0 < \underline{\lambda}_1(q)$ iff $0 < \varrho(q) < 1/2$.*

4.3. Explicit Conditions for AMP

Let us recall some known sufficient conditions for AMP.

Lemma 4.12 (Torres and Zhang [9]). *Suppose that $q \in \mathcal{L}^1$ satisfies the following two conditions:*

$$q \succ 0, \tag{4.60}$$

$$\underline{\lambda}_1(q) \geq 0. \tag{4.61}$$

Then L_q admits AMP.

In the proof there, the positiveness condition (4.60) is technically used extensively. Some optimal estimates on condition (4.61) can be found in Zhang and Li [22]. For an exponent $\gamma \in [1, \infty]$, let us introduce the following Sobolev constant:

$$\mathbf{K}(\gamma) := \inf_{u \in H_0^1(0,1),\ u \neq 0} \frac{\|u'\|_2^2}{\|u\|_\gamma^2}. \tag{4.62}$$

Here $\|\cdot\|_\gamma = \|\cdot\|_{L^\gamma(0,1)}$. These constants $\mathbf{K}(\gamma)$ can be explicitly expressed using the Gamma function of Euler. The following lower bound for $\underline{\lambda}_1(q)$ is established in [22]:

$$\|q_+\|_p \leq \mathbf{K}(2p^*) \Longrightarrow \underline{\lambda}_1(q) \geq \pi^2 \left(1 - \frac{\|q_+\|_p}{\mathbf{K}(2p^*)}\right) (\geq 0), \tag{4.63}$$

where $p, p^* := p/(p-1) \in [1, \infty]$. Hence one sufficient condition for (4.61) is

$$\|q_+\|_p \leq \mathbf{K}(2p^*). \tag{4.64}$$

Now such an L^p condition (4.64) is quite standard in literature like [8, 23], because in case $p = \infty$, (4.64) reads as the classical condition

$$\|q_+\|_\infty \leq \mathbf{K}(2) = \pi^2. \tag{4.65}$$

In order to overcome the technical assumption (4.60) on positiveness of $q(t)$, one observation is as follows.

Lemma 4.13 (Torres [8, Theorem 2.1]). *Let $q \in \mathcal{L}^1$. Suppose that all gaps of consecutive zeros of all nonzero solutions $x(t)$ of* (2.1) *are strictly greater than the period 1. Then the Green function $G_q(t, s)$ has a constant sign.*

By Theorem 4.1 of this paper, one sees that the hypothesis in Lemma 4.13 on solutions of (2.1) can yield MP or AMP. Combining ideas from [8, 9, 22], Cabada and Cid have overcome the positiveness condition (4.60) to obtain the following criteria.

Lemma 4.14 (Cabada and Cid [7, Theorem 3.1]). *Suppose that $q \in \mathcal{L}^1$ satisfies the following two conditions:*

$$\int_{[0,1]} q(t)\mathrm{d}t > 0, \tag{4.66}$$

$$\|q_+\|_p < \mathbf{K}(2p^*). \tag{4.67}$$

Then L_q admits AMP.

Very recently, Cabada et al. [24, 25] have generalized criteria (4.66)-(4.67) for L_q to AMP of the periodic solutions of the so-called p-Laplacian problem

$$\left(\left|x'\right|^{p-2}x'\right)' + q(t)|x|^{p-2}x = h(t) \quad (1 < p < \infty), \tag{4.68}$$

with the constants $\mathbf{K}(2p^*)$ being replaced by more general Sobolev constants [26].

We end the paper with some remarks.

(i) Recall the following trivial upper bound:

$$\overline{\lambda}_0(q) \le -\int_{[0,1]} q(t)\mathrm{d}t \quad \forall q \in \mathcal{L}^1. \tag{4.69}$$

See, for example, [26]. Criteria (4.66)-(4.67) can be deduced from Theorem 1.1 with the help of estimates (4.63) and (4.69). In fact, by Theorem 4.11, conditions (4.66) and (4.67) guarantee that L_q admits SAMP. For AMP of L_q, condition (4.67) can be improved as

$$\|q_+\|_p \le \mathbf{K}(2p^*). \tag{4.70}$$

Theorem 1.1 shows that condition (4.61) is optimal, while the complete generalization of condition (4.60) is $\overline{\lambda}_0(q) < 0$.

(ii) It is also possible to construct many potentials q for which L_q admits AMP, while (4.70) is violated. For example, let $\tilde{q}_n(t) = \sin 2n\pi t$ and $\hat{q}_n \in \mathcal{L}^1$ be defined by

$$\hat{q}_n(t) = \begin{cases} n & \text{for } t \in \left[0, \dfrac{1}{n\log(n+2)}\right), \\ 0 & \text{for } t \in \left[\dfrac{1}{n\log(n+2)}, 1\right). \end{cases} \tag{4.71}$$

Then $\|\hat{q}_n\|_1 = 1/\log(n+2) \to 0$ and the Riemann-Lebesgue lemma shows that $A\tilde{q}_n + \hat{q}_n \to 0$ in $(\mathcal{L}^1, w_1)$, where $A > 0$ is arbitrarily fixed. In particular, it follows from Lemma 2.5 that

$$\underline{\lambda}_1(A\tilde{q}_n + \hat{q}_n) \longrightarrow \underline{\lambda}_1(0) = \pi^2 > 0. \tag{4.72}$$

Since

$$\int_{[0,1]} \left(A\tilde{q}_n + \hat{q}_n\right) dt = \frac{1}{\log(n+2)} > 0, \tag{4.73}$$

we conclude that for $q = A\tilde{q}_n + \hat{q}_n$ with $n \gg 1$, L_q admits AMP. However, when $A > 0$ is large and $n \gg 1$,

$$\left\| \left(A\tilde{q}_n + \hat{q}_n\right)_+ \right\|_1 = \frac{A}{\pi} + o(1) \tag{4.74}$$

is also large. Hence $A\tilde{q}_n + \hat{q}_n$ does not satisfy (4.70).

(iii) Notice that the lower bound (4.63) has actually shown that, under (4.67) ((4.70), resp.), the gaps of consecutive zeros of all nonzero solutions $x(t)$ of (2.1) are > 1 (≥ 1, resp.). However, for those potentials as in Theorem 1.1, zeros of solutions of (2.1) may not be so evenly distributed. This is the difference between the sufficient conditions in this subsection and our optimal conditions given in Theorem 1.1.

Acknowledgments

The author is supported by the Major State Basic Research Development Program (973 Program) of China (no. 2006CB805903), the Doctoral Fund of Ministry of Education of China (no. 20090002110079), the Program of Introducing Talents of Discipline to Universities (111 Program) of Ministry of Education and State Administration of Foreign Experts Affairs of China (2007), and the National Natural Science Foundation of China (no. 10531010).

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