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Research Article

Slowly Oscillating Solutions of a Parabolic Inverse Problem: Boundary Value Problems

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The existence and uniqueness of a slowly oscillating solution to parabolic inverse problems for a type of boundary value problem are established. Stability of the solution is discussed.

1. Introduction

It is well known that the space $\mathcal{AP}(\mathbf{R})$ of almost periodic functions and some of its generalizations have many applications (e.g., [1–13] and references therein). However, little has been done for $\mathcal{AP}(\mathbf{R})$ to inverse problems except for our work in [14–16]. Sarason in [17] studied the space $\mathcal{SO}(\mathbf{R})$ of slowly oscillating functions. This is a C^*-subalgebra of $\mathcal{C}(\mathbf{R})$, the space of bounded, continuous, complex-valued functions f on $\mathbf{R}$ with the supremum norm $\|f\| = \sup\{|f(x)| : x \in \mathbf{R}\}$. Compared with $\mathcal{AP}(\mathbf{R})$, $\mathcal{SO}(\mathbf{R})$ is a quite large space (see [17–20]). What we are interested in $\mathcal{SO}(\mathbf{R})$ is based on the belief that $\mathcal{SO}(\mathbf{R})$ certainly has a variety of applications in many mathematical areas too. In [15], we studied slowly oscillating solutions of a parabolic inverse problem for Cauchy problems. In this paper, we devote such solutions for a type of boundary value problem.

Set $J \in \{\mathbf{R}, \mathbf{R}^n\}$. Let $\mathcal{C}(J)$ (resp., $\mathcal{C}(J \times \Omega)$, where $\Omega \subset \mathbf{R}^m$) denote the C^*-algebra of bounded continuous complex-valued functions on J (resp., $J \times \Omega$) with the supremum norm. For $f \in \mathcal{C}(J)$ (resp., $\mathcal{C}(J \times \Omega)$) and $s \in J$, the translate of f by s is the function $R_s f(t) = f(t+s)$ (resp., $R_s f(t, Z) = f(t+s, Z)$, $(t, Z) \in J \times \Omega$).

Definition 1.1. (1) A function $f \in \mathcal{C}(J)$ is called slowly oscillating if for every $\tau \in J$, $R_\tau f - f \in C_0(J)$, the space of the functions vanishing at infinity. Denote by $\mathcal{SO}(J)$ the set of all such functions.

(2) A function $f \in \mathcal{C}(J \times \Omega)$ is said to be slowly oscillating in $t \in J$ and uniform on compact subsets of Ω if $f(\cdot, Z) \in \mathcal{SO}(J)$ for each $Z \in \Omega$ and is uniformly continuous on

$J \times K$ for any compact subset $K \subset \Omega$. Denote by $\mathcal{SO}(J \times \Omega)$ the set of all such functions. For convenience, such functions are also called uniformly slowly oscillating functions.

(3) Let X be a Banach space, and let $C(J, X)$ be the space of bounded continuous functions from J to X. If we replace $C(J)$ in (1) by $C(J, X)$, then we get the definition of $\mathcal{SO}(J, X)$.

As in [17], we always assume that $f \in \mathcal{SO}(J)$ is uniformly continuous.

The following two propositions come from [15, Section 1].

Proposition 1.2. *Let $f \in \mathcal{SO}(J)$ $(\mathcal{SO}(J \times \Omega))$ be such that $\partial f / \partial x_i$ is uniformly continuous on J. Then $\partial f / \partial x_i \in \mathcal{SO}(J)$ $(\mathcal{SO}(J \times \Omega))$.*

For $H = (h_1, h_2, \ldots, h_n) \in C(\mathbf{R})^n$, suppose that $H(t) \in \Omega$ for all $t \in \mathbf{R}$. Define $H \times \iota \to \Omega \times \mathbf{R}$ by

$$H \times \iota(t) = (h_1(t), h_2(t), \ldots, h_n(t), t) \quad (t \in \mathbf{R}). \tag{1.1}$$

The following proposition shows that the composite is also slowly oscillating.

Proposition 1.3. *Let $f \in \mathcal{SO}(\mathbf{R} \times \Omega)$. If $H \in \mathcal{SO}(\mathbf{R})^n$ and $H(t) \in \Omega$ for all $t \in \mathbf{R}$, then $f \circ (H \times \iota) \in \mathcal{SO}(\mathbf{R})$.*

In the sequel, we will use the notations: $\mathbf{R}_T^m = \mathbf{R}^m \times (0, T)$, $\|F\|_T = \sup\{|F(x,t)| : x \in \mathbf{R}^n,\ 0 \le t \le T\}$. $F \in \mathcal{SO}(\mathbf{R}^n \times \mathbf{R}_T^m)$ means that $F(x^{(1)}, x^{(2)}, t)$ is slowly oscillating in $x^{(1)} \in \mathbf{R}^n$ and uniformly on $(x^{(2)}, t) \in \mathbf{R}_T^m$; $F \in \mathcal{SO}(\mathbf{R}^n \times \mathbf{R}^m)$ means that $F(x^{(1)}, x^{(2)})$ is slowly oscillating in $x^{(1)} \in \mathbf{R}^n$ and uniformly on $x^{(2)} \in \mathbf{R}^m$.

Let

$$Z(x, t; \xi, s) = \frac{1}{\left(2\sqrt{\pi(t-s)}\right)^{n+m}} \exp\left\{-\frac{\sum (x_i - \xi_i)^2}{4(t-s)}\right\} \quad (x, \xi \in \mathbf{R}^{n+m}) \tag{1.2}$$

be the fundamental solution of the heat equation [21].

2. A Type of Boundary Value Problem

We will keep the notation in Section 1 and at the same time introduce the following new notation:

$$\begin{gathered} x = (x_1, x_2, \ldots, x_{n-1}), \qquad \xi = (\xi_1, \xi_2, \ldots, \xi_{n-1}), \\ X = (x, x_n), \qquad \zeta = (\xi, \xi_n), \qquad D^n = \{X \in \mathbf{R}^n : x_n > 0\}. \end{gathered} \tag{2.1}$$

In this section, we always assume the following: f, $f_{x_n x_n} \in \mathcal{SO}(\mathbf{R}^{n-1} \times \overline{D_{T_0}})$, $h(x,t) \ge \text{const} > 0$, h, $(\Delta h - h_t) \in \mathcal{SO}(\overline{\mathbf{R}_{T_0}^{n-1}})$, φ, $\varphi_{x_n x_n} \in \mathcal{SO}(\mathbf{R}^{n-1} \times D)$, $\varphi \in C^3(\mathbf{R}^{n-1} \times D)$, and g, $(\Delta g - g_t) \in \mathcal{SO}(\overline{\mathbf{R}_{T_0}^{n-1}})$.

Let

$$G(X,t;\zeta,\tau) = Z(X,t;\xi,\xi_n,\tau) + Z(X,t;\xi,-\xi_n,\tau) \tag{2.2}$$

be Green's function for the boundary value problems [22, 23].

The following estimates are easily obtained:

$$\begin{gathered}\left\| \int_0^t ds \int_{D^n} G(X,t;\zeta,s)d\zeta \right\| \le m_1(T), \\ \left\| \int_0^t ds \int_{\mathbf{R}^{n-1}} Z(X,t;\xi,0,s)d\xi \right\| \le m_2(T), \\ \left\| \int_0^t ds \int_{\mathbf{R}^n} \frac{\partial Z(X,t;\zeta,s)}{\partial x_n} d\zeta \right\| \le m_3(T),\end{gathered} \tag{2.3}$$

where $m_i(T)$ $(i = 1,2,3)$ are positive and increasing for $T \ge 0$ and $m_i(T) \to 0$ as $T \to 0$.

To show the main results of this section, the following lemmas are needed. The first lemma is Lemma 3.1 on page 15 in [24].

Lemma 2.1. *Let φ, ϕ, and χ be real, continuous functions on $[0,T]$ with $\chi \ge 0$. If*

$$\varphi(t) \le \phi(t) + \int_0^t \chi(s)\varphi(s)ds \quad (t \in [0,T]), \tag{2.4}$$

then

$$\varphi(t) \le \phi(t) + \int_0^t \chi(s)\phi(s) \exp\left\{ \int_s^t \chi(\rho)d\rho \right\} ds \quad (t \in [0,T]). \tag{2.5}$$

Lemma 2.2. *Let φ be a continuous function on $[0,T]$. If ϕ, χ_1, and χ_2 are nondecreasing and nonnegative on $[0,T]$ and*

$$\varphi(t) \le \phi(t) + \chi_1(t) \int_0^t \varphi(s)ds + \chi_2(t) \int_0^t \frac{\varphi(s)}{\sqrt{t-s}} ds \quad (t \in [0,T]), \tag{2.6}$$

then

$$\varphi(t) \le \phi(t) \left[1 + t\chi_1(t) + 2\sqrt{t}\chi_2(t) \right] e^{t\chi(t)}, \tag{2.7}$$

where

$$\chi(t) = t\chi_1^2(t) + 4\sqrt{t}\chi_1(t)\chi_2(t) + \pi\chi_2^2(t). \tag{2.8}$$

Proof. Replacing $\varphi(s)$ in the two integrals of (2.6) by the expression on the right hand side in (2.6), changing the integral order of the resulting inequality and making use of the monotonicity of ϕ, χ_1 and χ_2, one gets

$$\varphi(t) \le \phi(t)\left[1 + t\chi_1(t) + 2\sqrt{t}\chi_2(t)\right] + \left[t\chi_1^2(t) + 4\sqrt{t}\chi_1(t)\chi_2(t) + \pi\chi_2^2(t)\right]\int_0^t \varphi(s)ds. \tag{2.9}$$

Apply Lemma 2.1 to get the conclusion. □

Lemma 2.3. *Let $F(X,t) \in \mathcal{SO}(\overline{D_T^n})$, $\phi(x,t), q(x,t) \in \mathcal{SO}(\overline{\mathbf{R}_T^{n-1}})$, and $\varphi \in \mathcal{SO}(D^n)$. Then the problem*

$$\begin{gathered} u_t - \Delta u + qu = F(X,t), \quad (X,t) \in D_T^n, \\ u(X,0) = \varphi(X), \quad X \in D^n, \\ u_{x_n}(x,0,t) = \phi(x,t), \quad (x,t) \in \mathbf{R}_T^{n-1} \end{gathered} \tag{2.10}$$

has a unique solution u, and u is in $\mathcal{SO}(\overline{D_T^n})$ and satisfies

$$\|u\|_T \le K(T)\left[T\|F\|_T + \|\varphi\| + \frac{\sqrt{T}}{2}\|\phi\|_T\right], \tag{2.11}$$

where $K(T) = 2(1 + T\|q\|_T e^{T\|q\|_T})$.

One sees that $K(T)$ depends on $\|q\|_T$ only and is bounded near zero.

Proof. The existence and uniqueness of the solution comes from Theorem 5.3 on page 320 in [25].

As in [22, 23], the solution u can be written as

$$\begin{aligned} u(X,t) = &\int_{D^n} \varphi(\zeta)G(X,t;\zeta,0)d\zeta + \int_0^t ds \int_{D^n} F(\zeta,s)G(X,t;\zeta,s)d\zeta \\ &- \int_0^t ds \int_{D^n} q(\xi,s)u(\zeta,s)G(X,t;\zeta,s)d\zeta - 2\int_0^t ds \int_{\mathbf{R}^{n-1}} \phi(\xi,s)Z(X,t;\xi,0,s)d\xi \\ = &\, v(x,t) - \int_0^t ds \int_{D^n} q(\xi,s)u(\zeta,s)G(X,t;\zeta,s)d\zeta. \end{aligned} \tag{2.12}$$

So,

$$\|u\|_t \le 2\|\varphi\| + 2\int_0^t \|F\|_s ds + 2\int_0^t \frac{\|\phi\|_s}{\sqrt{t-s}}ds + 2\int_0^t \|q\|_s \|u\|_s ds. \tag{2.13}$$

By Lemma 2.1, one gets the desired inequality.

Now we show that $u \in \mathcal{SO}(\overline{D^n_T})$. As in the proofs of Lemmas 2.1 and 2.3 in [15], one gets $v \in \mathcal{SO}(\overline{D^n_T})$. For $x, \tau \in \mathbf{R}^{n-1}$ with $|x| \geq A > 0$,

$$\begin{aligned}
&u(x+\tau, x_n, t) - u(x, x_n, t)\\
&= v(x+\tau, x_n, t) - v(x, x_n, t) - \int_0^t ds \int_{D^n} q(\xi, s) u(\zeta, s)[G(x+\tau, x_n, t; \zeta, s) - G(x, x_n, t; \zeta, s)] d\zeta\\
&= v(x+\tau, x_n, t) - v(x, x_n, t)\\
&\quad - \int_0^t ds \int_{D^n} [q(x+\tau+\xi, s) u(x+\tau+\xi, x_n+\xi_n, s) - q(x+\xi, s) u(x+\xi, x_n+\xi_n, s)] G(\theta, t; \zeta, s) d\zeta\\
&= v(x+\tau, x_n, t) - v(x, x_n, t)\\
&\quad - \int_0^t ds \int_{D^n} [q(x+\tau+\xi, s) - q(x+\xi, s)] u(x+\tau+\xi, x_n+\xi_n, s) G(\theta, t; \zeta, s) d\zeta\\
&\quad - \int_0^t ds \int_{D^n} [u(x+\tau+\xi, x_n+\xi_n, s) - u(x+\xi, x_n+\xi_n, s)] q(x+\xi, s) G(\theta, t; \zeta, s) d\zeta.
\end{aligned} \tag{2.14}$$

Note that

$$\begin{gathered}
\left| \int_0^t ds \int_{D^n} [q(x+\tau+\xi, s) - q(x+\xi, s)] u(x+\tau+\xi, x_n+\xi_n, s) G(\theta, t; \zeta, s) d\zeta \right| \leq B \cdot \operatorname{dist}_A(R_\tau q - q)_t\\
\left| \int_{D^n} q(\xi, s) G(\theta, t; \zeta, s) d\zeta \right| \leq B \|q\|_s,
\end{gathered} \tag{2.15}$$

where B is a constant and

$$\operatorname{dist}_A(R_\tau q, q)_t = \sup_{s \in [0,t], |x| \geq A} |q(x+\tau, s) - q(x, s)|. \tag{2.16}$$

So,

$$\operatorname{dist}_A(R_\tau u, u)_t \leq \operatorname{dist}_A(R_\tau v, v)_t + B \cdot \operatorname{dist}_A(R_\tau q, q)_t + B \int_0^t \operatorname{dist}_A(R_\tau u, u)_s \|q\|_s ds. \tag{2.17}$$

By Lemma 2.1, one has

$$\operatorname{dist}_A(R_\tau u, u)_t \leq m[\operatorname{dist}_A(R_\tau v, v)_t + B \cdot \operatorname{dist}_A(R_\tau q, q)_t], \tag{2.18}$$

where m is a constant. Since v and q are slowly oscillating, the right-hand sides of the inequality above approaches zero as $A \to \infty$. This means that $u \in \mathcal{SO}(\overline{D^n_T})$. The proof is complete. □

Consider the following problem.

Problem 1. Find functions $u \in \mathcal{SO}(\mathbf{R}^{n-1} \times \overline{D_T})$ *and* $q \in \mathcal{SO}(\overline{\mathbf{R}_T^{n-1}})$ *such that*

$$u_t - \Delta u + q(x,t)u = f(X,t), \quad (X,t) \in D_T^n, \tag{2.19}$$

$$u(X,0) = \varphi(X), \quad X \in D^n, \tag{2.20}$$

$$u_{x_n}(x,0,t) = g(x,t), \quad (x,t) \in \mathbf{R}_T^{n-1}, \tag{2.21}$$

$$u(x,a,t) = h(x,t), \quad (x,t) \in \mathbf{R}_T^{n-1}, \; a \in (0,\infty). \tag{2.22}$$

One sees that

$$h(x,0) = \varphi(x,a), \quad \varphi_{x_n}(x,0) = g(x,0), \quad x \in \mathbf{R}^{n-1}, \tag{2.23}$$

$$\begin{gathered} h_t(x,0) = u_t|_{x_n=a,t=0} = \left[\Delta u - qu + f(X,t)\right]_{x_n=a,t=0} = \Delta\varphi(X)\big|_{x_n=a} - q(x,0)\varphi(x,a) + f(x,a,0), \\ g_t(x,0) = u_{tx_n}|_{x_n=0,t=0} = \Delta\varphi_{x_n}(X)\big|_{x_n=0} - q(x,0)\varphi_{x_n}(x,0) + f_{x_n}(x,0,0). \end{gathered} \tag{2.24}$$

It follows from (2.24) that

$$\begin{aligned} &\varphi_{x_n}(x,0)\Delta\varphi(X)\big|_{x_n=a} + f(x,a,0)\varphi_{x_n}(x,0) - h_t(x,0)\varphi_{x_n}(x,0) \\ &\quad = \varphi(x,a)\Delta\varphi_{x_n}(X)\big|_{x_n=0} + f_{x_n}(x,0,0)\varphi(x,a) - g_t(x,0)\varphi(x,a). \end{aligned} \tag{2.25}$$

Let $V(X,t) = u_{x_n}(X,t)$, and let $W(X,t) = V_{x_n}(X,t)$. We have the following two additional problems for V and W, respectively.

Problem 2. Find functions $V \in \mathcal{SO}(\mathbf{R}^{n-1} \times \overline{D_T})$ *and* $q \in \mathcal{SO}(\overline{\mathbf{R}_T^{n-1}})$ *such that*

$$V_t - \Delta V + q(x,t)V = f_{x_n}(X,t), \quad (X,t) \in D_T^n, \tag{2.26}$$

$$V(X,0) = \varphi_{x_n}(X), \quad X \in D^n, \tag{2.27}$$

$$V(x,0,t) = g(x,t), \quad (x,t) \in \mathbf{R}_T^{n-1}, \tag{2.28}$$

$$V_{x_n}(x,a,t) = h_t - \Delta h + qh - f(x,a,t), \quad (x,t) \in \mathbf{R}_T^{n-1}. \tag{2.29}$$

Problem 3. Find functions $W \in \mathcal{SO}(\mathbf{R}^{n-1} \times \overline{D_T})$ *and* $q \in \mathcal{SO}(\overline{\mathbf{R}_T^{n-1}})$ *such that*

$$W_t - \Delta W + q(x,t)W = f_{x_n x_n}(X,t), \quad (X,t) \in D_T^n, \tag{2.30}$$

$$W(X,0) = \varphi_{x_n x_n}(X), \quad X \in D^n, \tag{2.31}$$

$$W_{x_n}(x,0,t) = g_t - \Delta g + qg - f_{x_n}(x,0,t), \quad (x,t) \in \mathbf{R}_T^{n-1}, \tag{2.32}$$

$$W(x,a,t) = h_t - \Delta h + hq - f(x,a,t), \quad (x,t) \in \mathbf{R}_T^{n-1}. \tag{2.33}$$

Lemma 2.4. *Problems 1, 2, and 3 are equivalent to each other.*

Proof. The existence and uniqueness of the solution (V, q) of Problem 2 can be easily obtained from that of the solution (u, q) of Problem 1. Conversely, let (V, q) be the solution of Problem 2. We show that Problem 1 has a unique solution (u, q). The uniqueness comes from the uniqueness of (2.19)–(2.21). For the existence, let

$$u(X,t) = \int_a^{x_n} V(x,y,t)dy + h(x,t). \tag{2.34}$$

Obviously, $u(X,t) \in \mathcal{SO}(\mathbf{R}^{n-1} \times \overline{D_T})$ and satisfies (2.22). Also u satisfies (2.21) because $u_{x_n}(x,0,t) = V(x,0,t) = g(x,t)$. By (2.23) and (2.27), one sees that (2.20) is true. Finally, we show that u satisfies (2.19) and therefore, along with q, constitutes a solution of Problem 1. In fact,

$$\begin{aligned} u_t - \Delta u + qu &= h_t - \Delta h + qh + \int_a^{x_n} \left[V_t(x,y,t) - \Delta V(x,y,t) + qV(x,y,t)\right] dy \\ &\quad + \int_a^{x_n} \frac{\partial^2}{\partial y^2} V(x,y,t)dy - \frac{\partial^2}{\partial x_{n^2}} \int_a^{x_n} V(x,y,t)dy \\ &= h_t - \Delta h + qh + f(X,t) - f(x,a,t) + V_{x_n}(X,t) - V_{x_n}(x,a,t) - V_{x_n}(X,t) \\ &= f(X,t). \quad (\text{by } (2.29)) \end{aligned} \tag{2.35}$$

Thus, we have shown the equivalence of Problems 1 and 2. Replacing (2.34) by the function

$$V(X,t) = \int_a^{x_n} W(x,y,t)dy + g(x,t), \tag{2.36}$$

the equivalence of Problems 2 and 3 can be proved similarly. The proof is complete. □

By Lemma 2.4, to solve Problem 1, we only need to solve Problem 3. By (2.30)–(2.32), we have the integral equation about W:

$$\begin{aligned} W(X,t) &= \int_{D^n} \varphi_{\xi_n\xi_n}(\zeta) G(X,t;\zeta,0)d\zeta + \int_0^t ds \int_{D^n} f_{\xi_n\xi_n}(\zeta,s) G(X,t;\zeta,s)d\zeta \\ &\quad - \int_0^t ds \int_{D^n} q(\xi,s) W(\zeta,s) G(X,t;\zeta,s)d\zeta \\ &\quad - 2\int_0^t ds \int_{\mathbf{R}^{n-1}} \left[g_s - \Delta g + qg - f_{\xi_n}(\xi,0,s)\right] Z(X,t;\xi,0,s)d\xi. \end{aligned} \tag{2.37}$$

Rewrite (2.33) as

$$q = Lq = h^{-1}(x,t)\left[\Delta h - h_t + f(x,a,t) + W(x,a,t)\right], \tag{2.38}$$

where W is determined by (2.37).

One can directly test that Problem 3 is equivalent to (2.37)-(2.38).

Note that for a given $q(x,t) \in \mathcal{SO}(\overline{\mathbf{R}_T^{n-1}})$, Lemma 2.3 shows that (2.30)–(2.32) (or equivalently, (2.37)) have a unique solution $W \in \mathcal{SO}(\mathbf{R}^{n-1} \times \overline{D_T})$. Thus, (2.38) does define an operator L. Therefore, we only need to show that the integral (2.38) has a unique solution q and $q \in \mathcal{SO}(\overline{\mathbf{R}_T^{n-1}})$. That is, L has a fixed point in $\mathcal{SO}(\overline{\mathbf{R}_T^{n-1}})$. Let

$$\begin{aligned}&\left\{ \left\| \Delta h - h_t + f(x,a,t) \right\|_{T_0} + 2\left\| \varphi_{\xi_n\xi_n} \right\| + \left\| \int_0^t ds \int_{D^n} f_{\xi_n\xi_n}(\zeta,s) G(x,a,t;\zeta,s) d\zeta \right\|_{T_0} \right. \\ &\left. +2\left\| \int_0^t ds \int_{\mathbf{R}^{n-1}} \left[\Delta g - g_s + f_{\xi_n}(\xi,0,s) \right] Z(x,a,t;\xi,0,s) d\xi \right\|_{T_0} \right\} \left\| h^{-1} \right\|_{T_0} = \frac{M}{2}.\end{aligned} \tag{2.39}$$

Set $B(M,T) = \{q \in \mathcal{SO}(\overline{\mathbf{R}_T^{n-1}}) : \|q\|_T \le M\}$, where $T \le T_0$. If $q \in B(M,t)$, then, by Lemma 2.3, $W(X,t)$ is in $\mathcal{SO}(\mathbf{R}^{n-1} \times \overline{D_T})$, and so, by (2.38), Lq is in $\mathcal{SO}(\overline{\mathbf{R}_T^{n-1}})$ with

$$\|Lq\|_T \le \frac{M}{2} + \left\| h^{-1} \right\|_{T_0} \left[2m_2(T) \|g\|_{T_0} + m_1(T) \|W\|_T \right] M. \tag{2.40}$$

Equation (2.37) gives the estimate

$$\begin{aligned}\|W\|_T \le{}& \left\| 2\varphi\xi_n\xi_n \right\| + 2m_2(T_0) \left\| g_t - \Delta g - f_{x_n}(x,0,t) \right\|_{T_0} + 2Mm_2(T_0) \|g\|_{T_0} \\ &+ m_1(T_0) \left\| f_{x_nx_n} \right\|_{T_0} + Mm_1(T) \|W\|_T.\end{aligned} \tag{2.41}$$

Choose $t_0 < T_0$ such that when $T \le t_0$, one has $1 < 2(1 - Mm_1(T))$. It follows that

$$\|W\|_T \le 2\left\{ 2\left\| \varphi_{x_nx_n} \right\| + 2m_2(T_0) \left\| g_t - \Delta g - f_{x_n}(x,0,t) \right\|_{T_0} + 2Mm_2(T_0) \|g\|_{T_0} + m_1(T_0) \left\| f_{x_nx_n} \right\|_{T_0} \right\}. \tag{2.42}$$

Choose $T_1 \le t_0$ such that when $T \le T_1$, one has

$$\begin{aligned}&2\left\| h^{-1} \right\|_{T_0} \left\{ m_2(T) \|g\|_{T_0} + m_1(T) \right. \\ &\left. \times \left(2\left\| \varphi_{x_nx_n} \right\| + 2m_2(T_0) \left\| g_t - \Delta g - f_{x_n}(x,0,t) \right\|_{T_0} + 2Mm_2(T_0) \|g\|_{T_0} + m_1(T_0) \left\| f_{x_nx_n} \right\| \right) \right\} < \frac{1}{2},\end{aligned} \tag{2.43}$$

and therefore, $\|Lq\|_T \le M$.

Let $q_1,q_2 \in B(M,T)$. By (2.38), $\|Lq_1 - Lq_2\|_T \leq \|h^{-1}\|_T\|W_1 - W_2\|_T$. Note that the function $W = W_1 - W_2$ is the solution of the problem

$$\begin{gathered} W_t - \Delta W + qW = W_2(q_2 - q_1), \quad (X,t) \in D_T^n, \\ W(X,0) = 0, \quad X \in D^n, \\ W_{x_n}(x,0,t) = (q_2 - q_1)g(x,t), \quad (x,t) \in \mathbf{R}_T^{n-1}. \end{gathered} \tag{2.44}$$

So, by Lemma 2.3, one has

$$\|W\|_T \leq K(T)\left(\frac{\sqrt{T}}{2}\|q_1 - q_2\|_T\|g\|_T + T\|q_1 - q_2\|_T\|W_2\|_T\right). \tag{2.45}$$

Choose $T_2 < t_0$ such that for $T \leq T_2$, $\|h^{-1}\|_{T_0}\|W_1 - W_2\|_T \leq (1/2)\|q_1 - q_2\|_T$. Now, set $T \leq \min\{T_1, T_2\}$. Then L is a contraction from $B(M,T)$ into itself, and therefore, has a unique fixed point. Thus, we have shown.

Theorem 2.5. *Let functions f, g, h, and φ be as above. Then, for small T, Problem 3 has a unique solution (W,q) in $\mathbf{R}_T^n$ with $W \in \mathcal{SO}(\mathbf{R}^{n-1} \times \overline{D_T})$ and $q \in \mathcal{SO}(\overline{\mathbf{R}_T^{n-1}})$.*

Let (W^i, q_i) $(i = 1.2)$ be the solutions of Problem 3 in D_T^n for the functions f^i, g^i, h^i, and φ^i. Set $h^0 = h^1 - h^2$, $f^0 = f^1 - f^2$, $\varphi^0 = \varphi^1 - \varphi^2$, and $g^0 = g^1 - g^2$. For the stability of the solution, we have the following.

Theorem 2.6. *For $0 \leq t \leq T$, one has*

$$\begin{aligned} \|q_1 - q_2\|_t \leq{} & c_1\left\|h^0\right\|_t + c_2\left\|g^0\right\|_t + c_3\left\|f^0_{x_nx_n}\right\|_t + c_4\left\|\varphi^0_{x_nx_n}\right\| + c_5\left\|h^0_t - \Delta h^0 - f^0(x,a,t)\right\|_t \\ & + c_6\left\|g^0_t - \Delta g^0 - f^0_{x_n}(x,0,t)\right\|_t, \end{aligned} \tag{2.46}$$

where c_i $(1 \leq i \leq 6)$ depends on t, $\|h_1^{-1}\|_t$, $\|g^1\|_t$, $\|f^1_{x_nx_n}\|_t$, $\|\varphi^1_{x_nx_n}\|$, $\|q_1\|_t$, $\|q_2\|_t$, and $\|g^1_t - \Delta g^1 - f^1_{x_n}(x,0,t)\|_t$.

Proof. By (2.33),

$$q_1 - q_2 = \left(h^1\right)^{-1}\left[\Delta h^0 - h^0_t + f^0(x,a,t) - q_2h^0 + W_1 - W_2\right]. \tag{2.47}$$

So,

$$\|q_1 - q_2\|_t \leq \left\|\left(h^1\right)^{-1}\right\|_t\left[\left\|\Delta h^0 - h^0_t + f^0(x,a,t)\right\|_t + \|q_2\|_t\left\|h^0\right\|_t + \|W_1 - W_2\|_t\right]. \tag{2.48}$$

Note that the function $W = W_1 - W_2$ is the solution of the problem

$$\begin{gathered} W_t - \Delta W + q_2 W = f^0_{x_n x_n} - W_1(q_1 - q_2), \quad (X,t) \in D^n_T, \\ W(X,0) = \varphi^0_{x_n x_n}(X), \quad X \in D^n, \\ W_{x_n}(x,0,t) = g^0_t - \Delta g^0 + q_2 g^0 - f^0_{x_n}(x,0,t) + (q_1 - q_2) g^1, \quad (x,t) \in \mathbf{R}^{n-1}_T. \end{gathered} \tag{2.49}$$

Using a formula similar to (2.37) and Lemma 2.2 for the function W, one gets

$$\begin{aligned} \|W\|_t \le \Bigg\{ & t\left\| f^0_{x_n x_n}\right\|_t + \left\|\varphi^0_{x_n x_n}\right\| + 2\sqrt{\frac{t}{\pi}}\|q_2\|_t\left\|g^0\right\|_t + 2\sqrt{\frac{t}{\pi}}\left\|g^0_t - \Delta g^0 - f^0_{x_n}(x,0,t)\right\|_t \\ & + \|W_1\|_t \int_0^t \|q_1 - q_2\|_s ds + \frac{\|g^1\|_t}{\sqrt{\pi}} \int_0^t \frac{\|q_1 - q_2\|_s}{\sqrt{(t-s)}} \Bigg\} \exp\left\{ \int_0^t \|q_2\|_\rho d\rho\, ds \right\}. \end{aligned} \tag{2.50}$$

Applying Lemma 2.2 and (2.48), one gets the desired conclusion with

$$\begin{aligned} c_1 &= \phi(t)\left\|\left(h^1\right)^{-1}\right\|_t \|q_2\|_t, \\ c_2 &= 2\phi(t)\sqrt{\frac{t}{\pi}}\left\|\left(h^1\right)^{-1}\right\|_t \|q_2\|_t \exp\left\{\int_0^t \|q_2\|_s ds\right\}, \\ c_3 &= t\phi(t)\left\|\left(h^1\right)^{-1}\right\|_t \exp\left\{\int_0^t \|q_2\|_s ds\right\}, \\ c_4 &= \phi(t)\left\|\left(h^1\right)^{-1}\right\|_t \exp\left\{\int_0^t \|q_2\|_s ds\right\}, \\ c_5 &= \phi(t)\left\|\left(h^1\right)^{-1}\right\|_t, \\ c_6 &= 2\phi(t)\sqrt{\frac{t}{\pi}}\left\|\left(h^1\right)^{-1}\right\|_t \exp\left\{\int_0^t \|q_2\|_s ds\right\}, \end{aligned} \tag{2.51}$$

where

$$\begin{aligned} \phi(t) &= \left(1 + t\chi_1(t) + 2\sqrt{t}\chi_2(t)\right) e^{t\chi(t)}, \\ \chi(t) &= t\chi_1^2(t) + 4\sqrt{t}\chi_1(t)\chi_2(t) + \pi\chi_2^2(t), \\ \chi_1(t) &= \left\|\left(h^1\right)^{-1}\right\|_t \Phi(t) \exp\left\{\int_0^t \|q_2\|_s ds\right\}, \\ \chi_2(t) &= \pi^{-1/2}\left\|\left(h^1\right)^{-1}\right\|_t \left\|g^1\right\|_t \exp\left\{\int_0^t \|q_2\|_s ds\right\} \end{aligned} \tag{2.52}$$

and $\Phi(t)$ is majorant of $\|W_1\|_t$. One can specially assume that

$$\Phi(t) = \left(\left\| \varphi^1_{x_n x_n} \right\| + t \left\| f^1_{x_n x_n} \right\|_t + \int_0^t \frac{\left\| g^1_s - \Delta g^1 - f^1_{x_n}(x, 0, s) \right\|}{\sqrt{\pi(t-s)}} ds \right) \exp\left\{ \int_s^t \|q_2\|_s ds \right\}. \quad (2.53)$$

The proof is complete. □

Corollary 2.7. *Under the conditions in Theorem 2.6, the solution of Problem 3 is unique.*

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