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Research Article

Multiple Positive Solutions of Semilinear Elliptic Problems in Exterior Domains

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Assume that q is a positive continuous function in $\mathbb{R}^N$ and satisfies the suitable conditions. We prove that the Dirichlet problem $-\Delta u + u = q(z)|u|^{p-2}u$ admits at least three positive solutions in an exterior domain.

1. Introduction

For $N \geq 3$ and $2 < p < 2^* = 2N/(N-2)$, we consider the semilinear elliptic equations

$$\begin{aligned} & -\Delta u + u = q(z)|u|^{p-2}u \quad \text{in } \Omega, \\ & u \in H_0^1(\Omega), \end{aligned} \tag{1.1}$$

$$\begin{aligned} & -\Delta u + u = q_\infty |u|^{p-2}u \quad \text{in } \Omega, \\ & u \in H_0^1(\Omega), \end{aligned} \tag{1.2}$$

where Ω is an unbounded domain $\mathbb{R}^N$. Let q be a positive continuous function in $\mathbb{R}^N$ and satisfy

$$\lim_{|z|\to\infty} q(z) = q_\infty > 0, \qquad q(z) \not\equiv q_\infty. \tag{q1}$$

Associated with (1.1) and (1.2), we define the functional $a,\ b,\ b^{\infty},\ J$, and J^{∞}, for $u \in H_0^1(\Omega)$

$$
\begin{aligned}
a(u) &= \int_{\Omega} \left(|\nabla u|^2 + u^2 \right) dz = \|u\|_{H^1}^2, \\
b(u) &= \int_{\Omega} q(z) u^p \, dz, \\
b^{\infty}(u) &= \int_{\Omega} q_{\infty} u^p \, dz, \\
J(u) &= \frac{1}{2} a(u) - \frac{1}{p} b(u_+), \\
J^{\infty}(u) &= \frac{1}{2} a(u) - \frac{1}{p} b^{\infty}(u_+),
\end{aligned}
\tag{1.3}
$$

where $u_+ = \max\{u, 0\} \geq 0$. By Rabinowitz [1, Proposition B.10], the functionals a, b, b^{∞}, J, and J^{∞} are of C^2.

It is well known that (1.1) admits infinitely many solutions in a bounded domain. Because of the lack of compactness, it is difficult to deal with this problem in an unbounded domain. Lions [2, 3] proved that if $q(z) \geq q_{\infty} > 0$, then (1.1) has a positive ground state solution in $\mathbb{R}^N$. Bahri and Li [4] proved that there is at least one positive solution of (1.1) in $\mathbb{R}^N$ when $\lim_{|z|\to\infty} q(z) = q_{\infty} > 0$ and $q(z) \geq q_{\infty} - C\exp(-\delta|z|)$ for $\delta > 2$. Zhu [5] has studied the multiplicity of solutions of (1.1) in $\mathbb{R}^N$ as follows. Assume $N \geq 5$, $\lim_{|z|\to\infty} q(z) = q_{\infty}$, $q(z) \geq q_{\infty} > 0$, and there exist positive constants $C,\ \gamma,\ R_0$ such that $q(z) \geq q_{\infty} + C/|z|^{\gamma}$ for $|z| \geq R_0$, then (1.1) has at least two nontrivial solutions (one is positive and the other changes sign). Esteban [6, 7] and Cao [8] have studied the multiplicity of solutions of $-\Delta u + u = q(z)|u|^{p-2}u$ with Neumann condition in an exterior domain $\mathbb{R}^N \setminus \overline{D}$, where D is a $C^{1,1}$ bounded domain in $\mathbb{R}^N$. Hirano [9] proved that if $\|q - q_{\infty}\|_{\infty}$ is sufficiently small and $q(z) \geq q_{\infty}[1 + C\exp(-\delta|z|)]$ for $0 < \delta < 1$, then (1.1) admits at least three nontrivial solutions (one is positive and the other changes sign) in $\mathbb{R}^N$. Recently, under the same conditions, Lin [10] showed that (1.1) admits at least two positive solutions and one nodal solution in an exterior domain. Let $q(z) = a(z) + \mu b(z)$. Wu [11] showed that for sufficiently small μ, if a and b satisfy some hypotheses, then (1.1) has at least three positive solutions in $\mathbb{R}^N$.

In this paper, we consider the multiplicity of positive solutions of (1.1) in an exterior domain. If q satisfies the suitable conditions ($\|q - q_{\infty}\|_{\infty}$ is sufficiently small and $q(z) \geq q_{\infty} + C\exp(-\delta|z|)$ for $0 < \delta < 2$), then we can show that (1.1) admits at least three positive solutions in an exterior domain. First, in Section 3, we use the concentration-compactness argument of Lions [2, 3] to obtain the "ground-state solution" (see Theorem 3.7). In Section 4, we study the idea of category in Adachi-Tanaka [12] and Bahri-Li minimax method to get that there are at least three positive solutions of (1.1) in $\mathbb{R}^N \setminus \overline{D}$ (see Theorems 4.10 and 4.15).

2. Existence of (PS)—Sequences

Let Ω be an unbounded domain in $\mathbb{R}^N$. We define the Palais-Smale (denoted by (PS)) sequences, (PS)-values, and (PS)-conditions in $H_0^1(\Omega)$ for J as follows.

Definition 2.1. (i) For $\beta \in \mathbb{R}$, a sequence $\{u_n\}$ is a $(\mathrm{PS})_\beta$-sequence in $H_0^1(\Omega)$ for J if $J(u_n) = \beta + o_n(1)$ and $J'(u_n) = o_n(1)$ strongly in $H^{-1}(\Omega)$ as $n \to \infty$.

(ii) $\beta \in \mathbb{R}$ is a (PS)-value in $H_0^1(\Omega)$ for J if there is a $(\mathrm{PS})_\beta$-sequence in $H_0^1(\Omega)$ for J.

(iii) J satisfies the $(\mathrm{PS})_\beta$-condition in $H_0^1(\Omega)$ if every $(\mathrm{PS})_\beta$-sequence in $H_0^1(\Omega)$ for J contains a convergent subsequence.

Lemma 2.2. *Let $u \in H_0^1(\Omega)$ be a critical point of J, then u is a nonnegative solution of* (1.1)*. Moreover, if $u \not\equiv 0$, then u is positive in Ω.*

Proof. Suppose that $u \in H_0^1(\Omega)$ satisfies $\langle J'(u), \varphi \rangle = 0$ for any $\varphi \in H_0^1(\Omega)$, that is,

$$\int_\Omega (\nabla u \nabla \varphi + u\varphi) = \int_\Omega q(z) u_+^{p-1} \varphi \quad \text{for any } \varphi \in H_0^1(\Omega). \tag{2.1}$$

Thus, u is a weak solution of $-\Delta u + u = q(z)u_+^{p-1}$ in Ω. Since $q > 0$ in $\mathbb{R}^N$, by the maximum principle, u is nonnegative. If $u \not\equiv 0$, we have that u is positive in Ω. □

Define

$$\alpha(\Omega) = \inf_{u \in \mathbf{M}(\Omega)} J(u), \tag{2.2}$$

where $\mathbf{M}(\Omega) = \{u \in H_0^1(\Omega) \setminus \{0\} \mid a(u) = b(u_+)\}$ and

$$\alpha^\infty(\Omega) = \inf_{u \in \mathbf{M}^\infty(\Omega)} J^\infty(u), \tag{2.3}$$

where $\mathbf{M}^\infty(\Omega) = \{u \in H_0^1(\Omega) \setminus \{0\} \mid a(u) = b^\infty(u_+)\}$.

Lemma 2.3. *Let $\beta \in \mathbb{R}$ and let $\{u_n\}$ be a $(PS)_\beta$-sequence in $H_0^1(\Omega)$ for J. Then,*

(i) *$\{u_n\}$ is a bounded sequence in $H_0^1(\Omega)$,*

(ii) *$a(u_n) = b(u_n^+) + o_n(1) = (2p/(p-2))\beta + o_n(1)$ as $n \to \infty$ and $\beta \geq 0$.*

By Chen et al. [13] and Chen and Wang [14], we have the following lemmas.

Lemma 2.4. (*i*) *For each $u \in H_0^1(\Omega) \setminus \{0\}$ with $u_+ \not\equiv 0$, there exists the unique number $s_u > 0$ such that $s_u u \in \mathbf{M}(\Omega)$ and $\sup_{s \geq 0} J(su) = J(s_u u)$.*

(*ii*) *Let $\beta > 0$ and $\{u_n\}$ a sequence in $H_0^1(\Omega) \setminus \{0\}$ for J such that $u_n \not\equiv 0$, $J(u_n) = \beta + o_n(1)$ and $a(u_n) = b(u_n^+) + o_n(1)$. Then, there is a sequence $\{s_n\}$ in $\mathbb{R}^+$ such that $s_n = 1 + o_n(1)$, $\{s_n u_n\}$ in $\mathbf{M}(\Omega)$ and $J(s_n u_n) = \beta + o_n(1)$ as $n \to \infty$.*

Lemma 2.5. *There exists a positive constant c such that $\|u\|_{H^1} \geq c > 0$ for each $u \in \mathbf{M}(\Omega)$. Moreover, $\alpha(\Omega) > 0$.*

Lemma 2.6. *Let $\Omega_1 \subsetneq \Omega_2$. If J satisfies the $(PS)_{\alpha(\Omega_1)}$-condition or $\alpha(\Omega_1)$ is a critical value, then $\alpha(\Omega_2) < \alpha(\Omega_1)$.*

Proof. See Chen et al. [13] or Lin et al. [15]. □

Remark 2.7. The above definitions and lemmas hold not only for J^∞ and $\mathbf{M}^\infty(\Omega)$ but also for $\alpha^\infty(\Omega)$.

Lemma 2.8. *Every minimizing sequence $\{u_n\}$ in $\mathbf{M}^\infty(\Omega)$ of $\alpha^\infty(\Omega)$ is a $(PS)_{\alpha^\infty(\Omega)}$-sequence in $H_0^1(\Omega)$ for J. Moreover, $\alpha^\infty(\Omega)$ is a (PS)-value.*

3. Existence of Ground State Solution

From now on, let $\Omega = \mathbb{R}^N \setminus \overline{D}$ be an exterior domain, where D is a $C^{1,1}$ bounded domain in $\mathbb{R}^N$. By Lions [2, 3], Struwe [16], and Lien et al. [17], we have the following decomposition lemmas.

Lemma 3.1 (Palais-Smale Decomposition Lemma for J). *Assume that q is a positive continuous function in $\mathbb{R}^N$ and $\lim_{|z|\to\infty} q(z) = q_\infty > 0$. Let $\{u_n\}$ be a $(PS)_\beta$-sequence in $H_0^1(\Omega)$ for J. Then, there are a subsequence $\{u_n\}$, a nonnegative integer l, sequences $\{z_n^i\}_{n=1}^\infty$ in $\mathbb{R}^N$, functions u in $H_0^1(\Omega)$, and $w^i \neq 0$ in $H^1(\mathbb{R}^N)$ for $1 \le i \le l$ such that*

$$\begin{gathered}
\left|z_n^i - z_n^j\right| \longrightarrow \infty \quad \text{for } 1 \le i,\ j \le l,\ i \neq j, \\
-\Delta u + u = q(z)|u|^{p-2}u \quad \text{in } \Omega, \\
-\Delta w^i + w^i = q_\infty \left|w^i\right|^{p-2} w^i \quad \text{in } \mathbb{R}^N, \\
u_n = u + \sum_{i=1}^{l} w^i\left(\cdot - z_n^i\right) + o_n(1) \quad \text{strongly in } H^1\left(\mathbb{R}^N\right), \\
J(u_n) = J(u) + \sum_{i=1}^{l} J^\infty\left(w^i\right) + o_n(1).
\end{gathered} \tag{3.1}$$

Lemma 3.2 (Palais-Smale Decomposition Lemma for J^∞). *Let $\{u_n\}$ be a $(PS)_\beta$-sequence in $H_0^1(\Omega)$ for J^∞. Then, there are a subsequence $\{u_n\}$, a nonnegative integer l, sequences $\{z_n^i\}_{n=1}^\infty$ in $\mathbb{R}^N$, functions u in $H_0^1(\Omega)$, and $w^i \neq 0$ in $H^1(\mathbb{R}^N)$ for $1 \le i \le l$ such that*

$$\begin{gathered}
\left|z_n^i - z_n^j\right| \longrightarrow \infty \quad \text{for } 1 \le i,\ j \le l,\ i \neq j, \\
-\Delta u + u = q_\infty |u|^{p-2}u_+ \quad \text{in } \Omega, \\
-\Delta w^i + w^i = q_\infty \left|w^i\right|^{p-2} w_+^i \quad \text{in } \mathbb{R}^N, \\
u_n = u + \sum_{i=1}^{l} w^i\left(\cdot - z_n^i\right) + o_n(1) \quad \text{strongly in } H^1\left(\mathbb{R}^N\right), \\
J^\infty(u_n) = J^\infty(u) + \sum_{i=1}^{l} J^\infty\left(w^i\right) + o_n(1).
\end{gathered} \tag{3.2}$$

Lemma 3.3. *(i)* $\alpha^\infty(\Omega) = \alpha^\infty(\mathbb{R}^N)$ *(denoted by* α^∞*).*

(ii) Let $\{u_n\} \subset \mathbf{M}(\Omega)$ *be a* $(PS)_\beta$*-sequence in* $H_0^1(\Omega)$ *for* J *with* $0 < \beta < \alpha^\infty$*. Then, there exist a subsequence* $\{u_n\}$ *and a nonzero* $u_0 \in H_0^1(\Omega)$ *such that* $u_n \to u_0$ *strongly in* $H_0^1(\Omega)$*, that is,* J *satisfies the* $(PS)_\beta$*-condition in* $H_0^1(\Omega)$*. Moreover,* u_0 *is a positive solution of* (1.1) *such that* $J(u_0) = \beta$*.*

Proof. (i) Since Ω is an exterior domain, by Lien et al. [17], Ω is a ball-up domain (for any $r > 0$, there exists $z \in \Omega$ such that $B^N(z; r) \subset \Omega$) and $\alpha^\infty(\Omega) = \alpha^\infty(\mathbb{R}^N)$.

(ii) Since $\{u_n\} \subset \mathbf{M}(\Omega)$ is a $(\mathrm{PS})_\beta$-sequence in $H_0^1(\Omega)$ for J with $0 < \beta < \alpha^\infty$, by Lemma 2.3, $\{u_n\}$ is bounded. Thus, there exist a subsequence $\{u_n\}$ and $u_0 \in H_0^1(\Omega)$ such that $u_n \rightharpoonup u_0$ weakly in $H_0^1(\Omega)$. It is easy to check that u_0 is a solution of (1.1). Applying Palais-Smale Decomposition Lemma 3.1, we get

$$\alpha^\infty > \beta = J(u_n) \geq l\alpha^\infty. \tag{3.3}$$

Then, $l = 0$ and $u_0 \neq 0$. Hence, $u_n \to u_0$ strongly in $H_0^1(\Omega)$ and $J(u_0) = \beta$. Moreover, by Lemma 2.2, u_0 is positive in Ω. □

It is well known that there is the unique (up to translation), positive, smooth, and radially symmetric solution w of (1.2) in $\mathbb{R}^N$ such that $J^\infty(w) = \alpha^\infty$. (See Bahri and Lions [18], Gidas et al. [19, 20] and Kwong [21]). Recall the facts

(i) for any $\varepsilon > 0$, there exist constants C_0, $C_0' > 0$ such that for all $z \in \mathbb{R}^N$

$$w(z) \leq C_0 \exp(-|z|), \quad |\nabla w(z)| \leq C_0' \exp(-(1-\varepsilon)|z|), \tag{3.4}$$

(ii) for any $\varepsilon > 0$, there exists a constant $C_\varepsilon > 0$ such that

$$w(z) \geq C_\varepsilon \exp(-(1+\varepsilon)|z|) \quad \forall z \in \mathbb{R}^N. \tag{3.5}$$

Suppose $D \subset B^N(0; R) = \{z \in \mathbb{R}^N \mid |z| < R\}$ for some $R > 0$. Let $\psi_R : \mathbb{R}^N \to [0,1]$ be a C^∞-function on $\mathbb{R}^N$ such that $0 \leq \psi_R \leq 1$, $|\nabla \psi_R| \leq c$ and

$$\psi_R(z) = \begin{cases} 1 & \text{for } |z| \geq R+1, \\ 0 & \text{for } |z| \leq R. \end{cases} \tag{3.6}$$

We define

$$w_{\overline{z}}(z) = \psi_R(z) w(z - \overline{z}) \qquad \text{for } \overline{z} \in \mathbb{R}^N. \tag{3.7}$$

Clearly, $w_{\overline{z}}(z) \in H_0^1(\Omega)$.

We need the following lemmas to prove that $\sup_{t \geq 0} J(t w_{\overline{z}}) < \alpha^\infty$ for sufficiently large $|\overline{z}|$.

Lemma 3.4. *Let E be a domain in $\mathbb{R}^N$. If $f : E \to \mathbb{R}$ satisfies*

$$\int_E \left| f(z)e^{\sigma|z|} \right| dz < \infty \quad \text{for some } \sigma > 0, \tag{3.8}$$

then

$$\left(\int_E f(z)e^{-\sigma|z-\overline{z}|} dz \right) e^{\sigma|\overline{z}|} = \int_E f(z)e^{\sigma(\langle z,\overline{z}\rangle/|\overline{z}|)} dz + o(1) \quad \text{as } |\overline{z}| \longrightarrow \infty. \tag{3.9}$$

Proof. Since $\sigma|\overline{z}| \le \sigma|z| + \sigma|z-\overline{z}|$, we have

$$\left| f(z)e^{-\sigma|z-\overline{z}|}e^{\sigma|\overline{z}|} \right| \le \left| f(z)e^{\sigma|z|} \right|. \tag{3.10}$$

Since $-\sigma|z-\overline{z}| + \sigma|\overline{z}| = \sigma(\langle z,\overline{z}\rangle/|\overline{z}|) + o(1)$ as $|\overline{z}| \to \infty$, then the lemma follows from the Lebesque-dominated convergence theorem. □

Next, assume that q is a positive continuous function in $\mathbb{R}^N$ and satisfies (q1) and

$$q(z) \ge q_\infty + C\exp(-\delta|z|) \quad \text{for some } C > 0 \text{ and } 0 < \delta < 2. \tag{q2}$$

Then, we have the following lemmas.

Lemma 3.5. (*i*) *There exists a number $t_0 > 0$ such that for $0 \le t < t_0$ and each $w_{\overline{z}} \in H_0^1(\Omega)$, we have*

$$J(tw_{\overline{z}}) < \alpha^\infty. \tag{3.11}$$

There exists a number $t_1 > 0$ such that for any $t > t_1$ and $|\overline{z}| \ge R+2$, we have

$$J(tw_{\overline{z}}) < 0. \tag{3.12}$$

Proof. (i) Since $\alpha^\infty > 0 = J(0)$, J is continuous in $H_0^1(\Omega)$ and $\{w_{\overline{z}}\}$ is bounded in $H_0^1(\Omega)$, then there exists $t_0 > 0$ such that for $0 \le t < t_0$ and each $w_{\overline{z}} \in H_0^1(\Omega)$

$$J(tw_{\overline{z}}) < \alpha^\infty. \tag{3.13}$$

For $|\overline{z}| \geq R+2$: since $0 \leq \psi_R \leq 1$, $|\nabla\psi_R| \leq c$ and $q(z) \gneqq q_\infty$, we have that

$$\begin{aligned}
J(tw_{\overline{z}}) &= \frac{t^2}{2}\int_\Omega \left[\left|\nabla\left(\psi_R(z)w(z-\overline{z})\right)\right|^2 + \left(\psi_R(z)w(z-\overline{z})\right)^2\right]dz \\
&\quad - \frac{t^2}{p}\int_\Omega q(z)\left(\psi_R(z)w(z-\overline{z})\right)^p dz \\
&\leq \frac{t^2}{2}\int_{\mathbb{R}^N}\left[\left|(\nabla\psi_R)w(z-\overline{z}) + \psi_R\nabla w(z-\overline{z})\right|^2 + w(z-\overline{z})^2\right]dz \\
&\quad - \frac{t^p}{p}\int_{B(\overline{z};1)} q_\infty w(z-\overline{z})^p dz \ \left(\because \psi_R(z) = 1 \text{ for } z \in B(\overline{z};1)\right) \\
&\leq \frac{t^2}{2}\int_{\mathbb{R}^N}\left\{\left[cw(z) + |\nabla w(z)|\right]^2 + w(z)^2\right\}dz - \frac{t^p}{p}\int_{B(0;1)} q_\infty w(z)^p dz.
\end{aligned} \tag{3.14}$$

Hence, there exists $t_1 > 0$ such that

$$J(tw_{\overline{z}}) < 0 \quad \text{for any } t > t_1,\ |\overline{z}| \geq R+2. \tag{3.15}$$

□

Lemma 3.6. *There exists a number $R_1 > R+2 > 0$ such that for any $|\overline{z}| \geq R_1$, we obtain*

$$\sup_{t\geq 0} J(tw_{\overline{z}}) < \alpha^\infty. \tag{3.16}$$

Proof. Applying the above lemma, we only need to show that there exists a number $R_1 > R+2 > 0$ such that for any $|\overline{z}| \geq R_1$,

$$\sup_{t_0\leq t\leq t_1} J(tw_{\overline{z}}) < \alpha^\infty. \tag{3.17}$$

For $t_0 \leq t \leq t_1$, since

$$\left|\nabla\left(\psi_R w(z-\overline{z})\right)\right|^2 = \left|\nabla\psi_R\right|^2 w(z-\overline{z})^2 + \psi_R^2|\nabla w(z-\overline{z})|^2 + 2\psi_R w(z-\overline{z})\nabla\psi_R\nabla w(z-\overline{z}), \tag{3.18}$$

then we have

$$\begin{aligned}
J(tw_{\overline{z}}) &= \frac{t^2}{2}\int_{\mathbb{R}^N}\left\{\left|\nabla\left(\psi_R(z)w(z-\overline{z})\right)\right|^2+\left[\left(\psi_R(z)w(z-\overline{z})\right)\right]^2\right\}dz \\
&\quad-\frac{t^p}{p}\int_{\mathbb{R}^N}q(z)\left[\psi_R(z)w(z-\overline{z})\right]^p dz \quad (\because \text{ the defination of } \psi_R) \\
&\le \frac{t^2}{2}\int_{\mathbb{R}^N}\left[|\nabla w(z-\overline{z})|^2+w(z-\overline{z})^2\right]dz-\frac{t^p}{p}\int_{\mathbb{R}^N}q_\infty w(z-\overline{z})^p dz \\
&\quad+\frac{t^2}{2}\int_{\mathbb{R}^N}\left[\left|\nabla\psi_R\right|^2 w(z-\overline{z})^2+2\psi_R w(z-\overline{z})\nabla\psi_R\nabla w(z-\overline{z})\right]dz \\
&\quad-\frac{t^p}{p}\int_{\mathbb{R}^N}\left[q(z)\psi_R^p w(z-\overline{z})^p-q_\infty w(z-\overline{z})^p\right]dz \quad (\because (3.18) \text{ and } 0\le\psi_R\le 1) \\
&\le \alpha^\infty+\frac{t_1^2}{2}\int_{\mathbb{R}^N}\left[\left|\nabla\psi_R\right|^2 w(z-\overline{z})^2+2|w(z-\overline{z})|\left|\nabla\psi_R\right||\nabla w(z-\overline{z})|\right]dz \\
&\quad-\frac{t_0^p}{p}\int_{\{|z|\ge R+1\}}(q(z)-q_\infty)w(z-\overline{z})^p dz \\
&\quad+\frac{t_1^p}{p}\int_{\{|z|\le R+1\}}q_\infty w(z-\overline{z})^p dz \quad \left(\because \sup_{t\ge 0}J^\infty(tw)=\alpha^\infty \text{ and the defination of } \psi_R\right).
\end{aligned} \tag{3.19}$$

Since the support of $\nabla\psi_R$ is bounded, then

$$\begin{aligned}
&\int_{\operatorname{supp}(\nabla\psi_R)}\left|\nabla\psi_R\right|^2 w(z-\overline{z})^2 dz\le C_1\exp(-2|\overline{z}|), \\
&\int_{\operatorname{supp}(\nabla\psi_R)}|w(z-\overline{z})|\left|\nabla\psi_R\right||\nabla w(z-\overline{z})|dz\le C_2\exp(-(2-\varepsilon)|\overline{z}|).
\end{aligned} \tag{3.20}$$

Similarly, we have

$$\int_{\{|z|\le R+1\}}q_\infty w(z-\overline{z})^p dz\le C_3\exp(-p|\overline{z}|). \tag{3.21}$$

Since $q(z) \geq q_\infty + C\exp(-\delta|z|)$ for some $0 < \delta < 2$, by Lemma 3.4, there exists $R_1' > R + 2 > 0$ such that for any $|\overline{z}| > R_1'$

$$\int_{\{|z|\leq R+1\}} (q(z) - q_\infty) w(z - \overline{z})^p dz \geq C_\varepsilon' \exp(-\min\{\delta, p(1+\varepsilon)\}|\overline{z}|) \geq C_\varepsilon' \exp(-\delta|\overline{z}|). \tag{3.22}$$

Choosing $0 < \varepsilon < 2 - \delta$ and using (3.20)–(3.22), there exists $R_1 > R_1'$ such that for $|\overline{z}| \geq R_1$, we have

$$\sup_{t_0 \leq t \leq t_1} J(tw_{\overline{z}}) < \alpha^\infty, \tag{3.23}$$

that is, $\sup_{t\geq 0} J(tw_{\overline{z}}) < \alpha^\infty$. □

Using the Ekeland variational principle (or see Stuart [22]), there is a $(\mathrm{PS})_{\alpha(\Omega)}$-sequence $\{u_n\} \subset \mathbf{M}(\Omega)$ for J. Then, we apply Lemma 3.3(ii) to obtain the existence of positive ground state solution of (1.1) in Ω.

Theorem 3.7. *Assume that q is a positive continuous function in $\mathbb{R}^N$ and satisfies* (q1) *and* (q2). *Then, there exists at least one positive ground state solution u_0 of* (1.1) *in* Ω.

Proof. Since $w_{\overline{z}} \in H_0^1(\Omega)$, by Lemma 2.4(i), there exists $s_{\overline{z}} > 0$ such that $s_{\overline{z}} w_{\overline{z}} \in \mathbf{M}(\Omega)$. Thus, by Lemma 3.6, $\alpha(\Omega) \leq J(s_{\overline{z}} w_{\overline{z}}) \leq \sup_{t\geq 0} J(tw_{\overline{z}}) < \alpha^\infty$ for $|\overline{z}| \geq R_1$. Using the Ekeland variational principle, there is a $(\mathrm{PS})_{\alpha(\Omega)}$-sequence $\{u_n\} \subset \mathbf{M}(\Omega)$ for J. Apply Lemma 3.3(ii), there exists at least one positive solution u_0 of (1.1) in Ω such that $J(u_0) = \alpha(\Omega)$. □

4. Existence of Multiple Solutions

In this section, we use two methods to obtain the existence of multiple positive solutions of (1.1) in an exterior domain. Part I: we study the idea of category to prove Theorem 4.10. Part II: we study the Bahri-Li minimax method to prove Theorem 4.15.

Lemma 4.1. *Assume that q is a positive continuous function in $\mathbb{R}^N$. If q satisfies* (q1), (q2) *and* $(m/2)q_\infty \gneqq q(z)$ *where $m > 2$, then there exists $m_0 > 2$ such that for $m \leq m_0$, we obtain that* $2\alpha(\Omega) > \alpha^\infty$.

Proof. Since $q(z) \gneqq q_\infty$, by Lions [2, 3], let $w_0 \in H^1(\mathbb{R}^N)$ be a positive solution of $-\Delta w_0 + w_0 = q(z)|w_0|^{p-2}w_0$ in $\mathbb{R}^N$ and $J(w_0) = \alpha(\mathbb{R}^N)$. By Lemma 2.4(i) and Remark 2.7, there exists $s_0 > 0$ such that $s_0 w_0 \in \mathbf{M}^\infty(\mathbb{R}^N)$ and $J^\infty(s_0 w_0) \geq \alpha^\infty$ and

$$\int_{\mathbb{R}^N} \left[|\nabla(s_0 w_0)|^2 + (s_0 w_0)^2\right] dz = \int_{\mathbb{R}^N} q_\infty (s_0 w_0)^p dz \geq \frac{2p}{p-2}\alpha^\infty. \tag{4.1}$$

Moreover, we have

$$1=\frac{\int_{\mathbb{R}^N}|\nabla w_0|^2+w_0^2}{\int_{\mathbb{R}^N}q(z)w_0^p}<\frac{\int_{\mathbb{R}^N}|\nabla w_0|^2+w_0^2}{\int_{\mathbb{R}^N}q_\infty w_0^p}=s_0^{p-2}<\frac{\int_{\mathbb{R}^N}(m/2)q_\infty w_0^p}{\int_{\mathbb{R}^N}q_\infty w_0^p}=\frac{m}{2}. \tag{4.2}$$

Hence, using the above inequalities, we get

$$\begin{aligned}\alpha\left(\mathbb{R}^N\right)&=J(w_0)=\sup_{s\geq 0}J(sw_0)>J(s_0w_0)\\&=J^\infty(s_0w_0)-\frac{1}{p}\int_{\mathbb{R}^N}(q(z)-q_\infty)(s_0w_0)^p dz\\&\geq \alpha^\infty-\frac{1}{p}\left(\frac{m}{2}-1\right)\int_{\mathbb{R}^N}q_\infty(s_0w_0)^p dz\\&=\alpha^\infty-\frac{s_0^2}{p}\left(\frac{m}{2}-1\right)\int_{\mathbb{R}^N}\left(|\nabla w_0|^2+w_0^2\right)dz\\&>\alpha^\infty-\frac{1}{p}\left(\frac{m}{2}-1\right)\left(\frac{m}{2}\right)^{2/(p-2)}\frac{2p}{p-2}\alpha\left(\mathbb{R}^N\right),\end{aligned} \tag{4.3}$$

that is, $[1+((m-2)/(p-2))(m/2)^{2/(p-2)}]\alpha(\mathbb{R}^N)>\alpha^\infty$. Choose some $m_0>2$ such that for $2<m\leq m_0$, then $2\alpha(\mathbb{R}^N)>\alpha^\infty$. By Lemma 2.6 and Theorem 3.7, $2\alpha(\Omega)>2\alpha(\mathbb{R}^N)>\alpha^\infty$. □

Lemma 4.2. *There exists a number $\delta_0>0$ such that if $u\in\mathbf{M}^\infty(\Omega)$ and $J^\infty(u)\leq\alpha^\infty+\delta_0$, then*

$$\int_{\mathbb{R}^N}\frac{z}{|z|}\left(|\nabla u|^2+u^2\right)dz\neq\overrightarrow{0}. \tag{4.4}$$

Proof. On the contrary, there exists a sequence $\{u_n\}$ in $\mathbf{M}^\infty(\Omega)$ such that $J^\infty(u_n)=\alpha^\infty+o_n(1)$ as $n\to\infty$ and

$$\int_{\mathbb{R}^N}\frac{z}{|z|}\left(|\nabla u_n|^2+u_n^2\right)dz=\overrightarrow{0}\quad\forall n. \tag{4.5}$$

By Lemma 2.8, $\{u_n\}$ is a $(\mathrm{PS})_{\alpha^\infty}$-sequence in $H_0^1(\Omega)$ for J^∞. Since $\alpha^\infty(\Omega)=\alpha^\infty(\mathbb{R}^N)$, Lien et al. [17] proved that (1.2) does not have any ground state solution in an exterior domain, that is, $\inf_{v\in\mathbf{M}^\infty(\Omega)}J^\infty(v)=\alpha^\infty(\Omega)$ is not achieved. Applying the Palais-Smale Decomposition Lemma 3.2, we have that there exists a sequence $\{z_n\}$ in $\mathbb{R}^N$ such that $|z_n|\to\infty$ as $n\to\infty$ and

$$u_n(z)=w(z-z_n)+o_n(1)\quad\text{strongly in } H^1\left(\mathbb{R}^N\right), \tag{4.6}$$

where w is the positive solution of (1.2) in $\mathbb{R}^N$. Suppose the subsequence $z_n/|z_n| \to z_0$ as $n \to \infty$, where z_0 is a unit vector in $\mathbb{R}^N$. Then, by the Lebesgue dominated convergence theorem, we have

$$\begin{aligned}\vec{0} &= \int_{\mathbb{R}^N} \frac{z}{|z|}\left(|\nabla u_n|^2 + u_n^2\right)dz \\ &= \int_{\mathbb{R}^N} \frac{z+z_n}{|z+z_n|}\left(|\nabla w|^2 + w^2\right)dz + o_n(1) \\ &= \left(\frac{2p}{p-2}\right)\alpha^\infty z_0 + o_n(1),\end{aligned} \tag{4.7}$$

which is a contradiction. □

Using the results of Lemma 2.4(i), let $K(u) = J(s_u u) = \sup_{s\geq 0} J(su)$ for each $u \in H_0^1(\Omega) \setminus \{0\}$ with $u_+ \not\equiv 0$. For $c \in \mathbb{R}$, we denote

$$[K \leq c] = \{u \in \Sigma \mid K(u) \leq c\}, \tag{4.8}$$

where $\Sigma = \{u \in H_0^1(\Omega) \mid u_+ \not\equiv 0 \text{ and } \|u\|_{H^1} = 1\}$. Then, we have the following lemma.

Lemma 4.3. *(i) $K \in C^1(\Sigma, \mathbb{R})$ and*

$$\langle K'(u), \varphi\rangle = s_u \langle J'(s_u u), \varphi\rangle \tag{4.9}$$

for all $\varphi \in T_u\Sigma = \{\varphi \in H_0^1(\Omega) \mid \langle \varphi, u\rangle = 0\}$.

(ii) $u \in \Sigma$ is a critical point of $K(u)$ if and only if $s_u u \in H_0^1(\Omega)$ is a critical point of J.

Proof. (i) For $u \in \Sigma$, it is easy to check that

$$\begin{aligned}\frac{d}{ds} J(su)\big|_{s=s_u} &= 0, \\ \frac{d^2}{ds^2} J(su)\big|_{s=s_u} &= a(u) - (p-1)s_u^{p-2} b(u_+) = (2-p)a(u) < 0.\end{aligned} \tag{4.10}$$

Then, using the implicit function theorem to obtain that $s_u \in C^1(\Sigma, (0,\infty))$. Therefore, $K(u) = J(s_u u) \in C^1(\Sigma, \mathbb{R})$. Since $s_u u \in \mathbf{M}(\Omega)$, we can get $\langle J'(s_u u), u\rangle = 0$. Thus,

$$\begin{aligned}\langle K'(u), \varphi\rangle &= \langle J'(s_u u), s_u \varphi\rangle + \langle J'(s_u u), \langle s_u', \varphi\rangle u\rangle \\ &= s_u \langle J'(s_u u), \varphi\rangle \quad \forall \varphi \in T_u\Sigma.\end{aligned} \tag{4.11}$$

(ii) By (i), $K'(u) = 0$ if and only if $\langle J'(s_u u), \varphi\rangle = 0$ for all $\varphi \in T_u\Sigma$. Since $H_0^1(\Omega)$ is a Hilbert space and $\langle J'(s_u u), u\rangle = 0$, so it is equivalent to $J'(s_u u) = 0$ in $H^{-1}(\Omega)$. □

Lemma 4.4. *Assume that q is a positive continuous function in $\mathbb{R}^N$ and satisfies* (q1) *and for $m > 2$ and $0 < \delta < 2$*

$$\frac{m}{2}q_\infty \gneqq q(z) \geq q_\infty + C\exp(-\delta|z|) \quad \text{where } 0 < C \leq \frac{m-2}{2}q_\infty. \tag{4.12}$$

We have that there exists a number $m_0 \geq m_1 > 2$ (m_0 is defined in Lemma 4.1) such that if $m \leq m_1$, then

$$\int_{\mathbb{R}^N} \frac{z}{|z|}\left(|\nabla u|^2 + u^2\right)dz \neq \overrightarrow{0} \quad \text{for any } u \in [K < \alpha^\infty]. \tag{4.13}$$

Proof. By the assumptions of q, Lemmas 2.4(i) and 3.6, the set $[K < \alpha^\infty]$ is nonempty. For any $u \in [K < \alpha^\infty]$, $u \in \Sigma$, $s_u u \in \mathbf{M}(\Omega)$ and $J(s_u u) < \alpha^\infty$, we get $J(s_u u) \geq \alpha(\Omega)$ and

$$\frac{2p}{p-2}\alpha(\Omega) \leq s_u^2 = s_u^p \int_\Omega q(z)u_+^p dz < \frac{2p}{p-2}\alpha^\infty. \tag{4.14}$$

Since $2\alpha(\Omega) > \alpha^\infty$ (by Lemma 4.1), then we have

$$\begin{aligned}\frac{p}{p-2}\alpha^\infty &< \frac{2p}{p-2}\alpha(\Omega) \leq s_u^p \|q\|_\infty \int_\Omega u_+^p dz \\ &< \left(\frac{2p}{p-2}\alpha^\infty\right)^{p/2} \|q\|_\infty \int_\Omega u_+^p dz.\end{aligned} \tag{4.15}$$

By Lemma 4.2 (i) and Remark 2.7, there exists $t_\infty > 0$ such that $t_\infty u \in \mathbf{M}^\infty(\Omega)$, then by (4.15), we have

$$t_\infty^2 = t_\infty^p \int_\Omega q_\infty u_+^p dz > t_\infty^p q_\infty \left(\frac{p-2}{2p\alpha^\infty}\right)^{(p-2)/2} \frac{1}{mq_\infty}, \tag{4.16}$$

that is,

$$m^{1/(p-2)}\sqrt{\frac{2p\alpha^\infty}{p-2}} > t_\infty. \tag{4.17}$$

Since $u \in [K < \alpha^\infty]$ and by the definitions of J and J_∞,

$$\begin{aligned}\alpha^\infty > J(s_u u) &= \sup_{s\geq 0} J(su) \geq J(t_\infty u) \\ &= \frac{1}{2}a(t_\infty u) - \frac{1}{p}\int_\Omega q(z)t_\infty^p u_+^p dz \\ &= J^\infty(t_\infty u) - \frac{1}{p}\int_\Omega (q(z) - q_\infty)t_\infty^p u_+^p dz.\end{aligned} \tag{4.18}$$

From (4.17) and (4.18), we have

$$\begin{aligned} J^{\infty}(t_{\infty}u) &< \alpha^{\infty} + \frac{1}{p}\int_{\Omega}(q(z)-q_{\infty})t_{\infty}u_{+}^{p}dz \\ &\leq \alpha^{\infty} + \frac{1}{pq_{\infty}}\left(\frac{m-2}{2}\right)q_{\infty}t_{\infty}^{2} \\ &< \alpha^{\infty} + \frac{m-2}{p-2}m^{2/(p-2)}\alpha^{\infty}. \end{aligned} \tag{4.19}$$

Hence, there exists $m_0 \geq m_1 > 2$ such that if $2 < m < m_1$, then

$$J^{\infty}(t_{\infty}u) \leq \alpha^{\infty} + \delta_0, \quad \text{where } t_{\infty}u \in \mathbf{M}^{\infty}(\Omega). \tag{4.20}$$

By Lemma 4.2, we obtain

$$\int_{\mathbb{R}^N}\frac{z}{|z|}\left[|\nabla(t_{\infty}u)|^2 + (t_{\infty}u)^2\right]dz \neq \overrightarrow{0}, \tag{4.21}$$

or

$$\int_{\mathbb{R}^N}\frac{z}{|z|}\left(|\nabla u|^2 + u^2\right)dz \neq \overrightarrow{0}. \tag{4.22}$$

□

We try to show that for a sufficiently small $\sigma > 0$

$$\operatorname{cat}([K \leq \alpha^{\infty} - \sigma]) \geq 2. \tag{4.23}$$

To prove (4.23), we need some preliminaries. Recall the definition of Lusternik-Schnirelman category.

Definition 4.5. (i) For a topological space X, we say a nonempty, closed subset $A \subset X$ is contractible to a point in X if and only if there exists a continuous mapping

$$\eta : [0,1] \times A \longrightarrow X \tag{4.24}$$

such that for some $x_0 \in X$ and

$$\begin{aligned} \eta(0,x) &= x \quad \forall x \in A, \\ \eta(1,x) &= x_0 \quad \forall x \in A. \end{aligned} \tag{4.25}$$

(ii) We define

$$\operatorname{cat}(X) = \min\left\{ k \in \mathbb{N} \mid \text{there exist closed subsets } A_1, \dots, A_k \subset X \text{ such that} \right.$$
$$\left. A_j \text{ is contractible to a point in } X \text{ for all } j \text{ and } \bigcup_{j=1}^{k} A_j = X \right\}. \tag{4.26}$$

When there do not exist finitely many closed subsets $A_1, \dots, A_k \subset X$ such that A_j is contractible to a point in X for all j and $\bigcup_{j=1}^{k} A_j = X$, we say $\operatorname{cat}(X) = \infty$.

We need the following two lemmas.

Lemma 4.6. *Suppose that X is a Hilbert manifold and $\Psi \in C^1(X, \mathbb{R})$. Assume that there are $c_0 \in \mathbb{R}$ and $k \in \mathbb{N}$,*

(i) $\Psi(x)$ satisfies the $(PS)_c$-condition for $c \le c_0$,

(ii) $\operatorname{cat}(\{x \in X \mid \Psi(x) \le c_0\}) \ge k$.

Then, $\Psi(x)$ has at least k critical points in $\{x \in X; \Psi(x) \le c_0\}$.

Proof. See Ambrosetti [23, Theorem 2.3]. □

Lemma 4.7. *Let $N \ge 1$, $S^{N-1} = \{z \in \mathbb{R}^N \mid |z| = 1\}$, and let X be a topological space. Suppose that there are two continuous maps*

$$F : S^{N-1} \longrightarrow X, \qquad G : X \longrightarrow S^{N-1} \tag{4.27}$$

such that $G \circ F$ is homotopic to the identity map of S^{N-1}, that is, there exists a continuous map $\zeta : [0,1] \times S^{N-1} \to S^{N-1}$ such that

$$\zeta(0, z) = (G \circ F)(z) \quad \textit{for each } z \in S^{N-1},$$
$$\zeta(1, z) = z \quad \textit{for each } z \in S^{N-1}. \tag{4.28}$$

Then,

$$\operatorname{cat}(X) \ge 2. \tag{4.29}$$

Proof. See Adachi and Tanaka [12, Lemma 2.5]. □

From the result of Lemma 4.4, for $2 < m \le m_1$, let q satisfy the condition

$$\frac{m}{2} q_\infty \gneqq q(z) \ge q_\infty + C \exp(-\delta |z|) \quad \text{where } 0 < C \le \frac{m-2}{2} q_\infty \text{ and } 0 < \delta < 2. \tag{q_2'}$$

In this section, assume that q is a positive continuous function in $\mathbb{R}^N$ and satisfies (q1), and (q_2'). Let $\tilde{z} \in S^{N-1}$ and $w_n(z) = \psi_R(z) w(z - n\tilde{z}) \in H_0^1(\Omega)$ for each $n \in \mathbb{N}$. By Lemma 2.4(i), there exist unique numbers $(n,\tilde{z}) > 0$ such that $s(n,\tilde{z})w_n \in \mathbf{M}(\Omega)$. We define a map $F_n : S^{N-1} \to H_0^1(\Omega)$ by

$$F_n(\tilde{z})(z) = \frac{s(n,\tilde{z})w_n(z)}{\|s(n,\tilde{z})w_n(z)\|_{H^1}} \quad \text{for } \tilde{z} \in S^{N-1}. \tag{4.30}$$

Then, we have the following lemma.

Lemma 4.8. *There are $n_0 \in \mathbb{N}$ and a sequence $\{\sigma_n\}$ in $\mathbb{R}^+$ such that*

$$F_n\left(S^{N-1}\right) \subset [K \le \alpha^\infty - \sigma_n] \quad \text{for each } n \ge n_0. \tag{4.31}$$

Proof. Since there exists a unique number $s(n,\tilde{z}) > 0$ such that $s(n,\tilde{z})w_n \in \mathbf{M}(\Omega)$, and by the definition of K, then we obtain that there exists $t_n > 0$ such that

$$K\left(\frac{s(n,\tilde{z})w_n(z)}{\|s(n,\tilde{z})w_n(z)\|_{H^1}}\right) = J\left(t_n\frac{s(n,\tilde{z})w_n(z)}{\|s(n,\tilde{z})w_n(z)\|_{H^1}}\right), \tag{4.32}$$

where $t_n = \|s(n,\tilde{z})w_n(z)\|_{H^1}$. By Lemma 3.6, there is $n_0 \in \mathbb{N}$ such that $J(s(n,\tilde{z})w_n) \le \sup_{t\ge 0} J(tw_n) < \alpha^\infty$ for each $n \ge n_0$. Thus, the conclusion holds. □

Applying Lemma 4.4, we obtain

$$\int_{\mathbb{R}^N} \frac{z}{|z|}\left(|\nabla u|^2 + u^2\right)dz \ne \overrightarrow{0} \quad \text{for any } u \in [K \in \alpha^\infty]. \tag{4.33}$$

Now, we define

$$G : [K < \alpha^\infty] \longrightarrow S^{N-1} \tag{4.34}$$

by

$$G(u) = \frac{\int_{\mathbb{R}^N}(z/|z|)\left(|\nabla u|^2 + |u|^2\right)dz}{\left|\int_{\mathbb{R}^N}(z/|z|)\left(|\nabla u|^2 + |u|^2\right)dz\right|}. \tag{4.35}$$

Lemma 4.9. *For each $n \ge n_0$, the map*

$$G \circ F_n : S^{N-1} \longrightarrow S^{N-1} \tag{4.36}$$

is homotopic to the identity.

Proof. Define

$$\zeta_n(\theta,\tilde{z}) : [0,1] \times S^{N-1} \longrightarrow S^{N-1} \tag{4.37}$$

by

$$\zeta_n(\theta,\tilde{z}) = \begin{cases} G\left(\dfrac{(1-2\theta)s(n,\tilde{z})\psi_R w(z-n\tilde{z}) + 2\theta\psi_R w(z-n\tilde{z})}{\left\|(1-2\theta)s(n,\tilde{z})\psi_R w(z-n\tilde{z}) + 2\theta\psi_R w(z-n\tilde{z})\right\|_{H^1}}\right) & \text{for } \theta \in \left[0,\dfrac{1}{2}\right), \\ G\left(\dfrac{\psi_R w(z-(n/2(1-\theta))\tilde{z})}{\left\|\psi_R w(z-(n/2(1-\theta))\tilde{z})\right\|_{H^1}}\right) & \text{for } \theta \in \left[\dfrac{1}{2},1\right), \\ \tilde{z} & \text{for } \theta = 1. \end{cases} \tag{4.38}$$

We need to show that $\lim_{\theta\to 1^-} \zeta_n(\theta,\tilde{z}) = \tilde{z}$ and

$$\lim_{\theta\to 1/2^-} \zeta_{\rm n}(\theta,\tilde{z}) = G\left(\frac{\psi_R w(z-n\tilde{z})}{\left\|\psi_R w(z-n\tilde{z})\right\|_{\rm H^1}}\right). \tag{4.39}$$

(a) $\lim_{\theta\to 1^-}\zeta_n(\theta,\tilde{z}) = \tilde{z}$: for $1/2 < \theta < 1$, since

$$\begin{aligned} &\int_{\mathbb{R}^N} \frac{z}{|z|}\left(\left|\nabla\left[\psi_R w\left(z-\frac{n}{2(1-\theta)}\tilde{z}\right)\right]\right|^2 + \psi_R^2 w\left(z-\frac{n}{2(1-\theta)}\tilde{z}\right)^2\right)dz \\ &\quad= \int_{\mathbb{R}^N} \frac{z+(n/2(1-\theta))\tilde{z}}{|z+(n/2(1-\theta))\tilde{z}|}\left(|\nabla w(z)|^2 + w(z)^2\right)dz + o(1) \\ &\quad= \left(\frac{2p}{p-2}\right)\alpha^\infty\tilde{z} + o(1) \quad \text{as } \theta \longrightarrow 1^-, \end{aligned} \tag{4.40}$$

and $\|\psi_R w(z-(n/2(1-\theta))\tilde{z})\|_{H^1}^2 = (2p/(p-2))\alpha^\infty + o(1)$ as $\theta \to 1^-$, then $\lim_{\theta\to 1^-} \zeta_n(\theta,\tilde{z}) = \tilde{z}$.

(b) By the continuity of G, it is easy to check that

$$\lim_{\theta\to 1/2^-} \zeta_n(\theta,\tilde{z}) = G\left(\frac{\psi_R w(z-n\tilde{z})}{\left\|\psi_R w(z-n\tilde{z})\right\|_{H^1}}\right). \tag{4.41}$$

Thus, $\zeta_n(\theta,\tilde{z}) \in C([0,1]\times S^{N-1}, S^{N-1})$ and

$$\begin{aligned} &\zeta_n(0,\tilde{z}) = G(F_n(\tilde{z})) \quad \forall\tilde{z}\in S^{N-1}, \\ &\zeta_n(1,\tilde{z}) = \tilde{z} \quad \forall\tilde{z}\in \mathrm{S}^{\mathrm{N}-1}, \end{aligned} \tag{4.42}$$

provided $n \geq n_0$. This completes the proof. □

Theorem 4.10. *Assume that q is a positive continuous function in $\mathbb{R}^N$ and satisfies* (q1) *and* (q_2'). *Then, $J(u)$ has at least two critical points in*

$$[K < \alpha^\infty], \tag{4.43}$$

and there exists at least two positive solutions of (1.1) *in* Ω.

Proof. Applying Lemmas 4.7 and 4.9, we have for $n \geq n_0$

$$\operatorname{cat}([K \leq \alpha^\infty - \sigma_n]) \geq 2. \tag{4.44}$$

Next, we need to show that K satisfies the $(\mathrm{PS})_\beta$-condition for $0 < \beta \leq \alpha^\infty - \sigma_\mathrm{n}$. Let $\{u_n\} \subset \Sigma$ satisfiy $K(u_n) = \beta + o_n(1)$ and

$$\begin{aligned} \|K'(u_n)\|_{T^{-1}_{u_n}\Sigma} &= \sup\{\langle K'(u_n), \varphi\rangle \mid \varphi \in T_{u_n}\Sigma \text{ and } \|\varphi\|_{H^1} = 1\} \\ &= o_n(1) \quad \text{as } n \longrightarrow \infty. \end{aligned} \tag{4.45}$$

Since $K(u_n) = J(s_n u_n) = \beta + o_n(1)$ as $n \to \infty$ and $s_n u_n \in \mathbf{M}(\Omega)$, then

$$s_n^2 = \frac{2p}{p-2}\beta + o_n(1). \tag{4.46}$$

Using (4.9) and $\langle J'(s_n u_n), u_n\rangle = 0$ to obtain that

$$\|J'(s_n u_n)\|_{H^{-1}} = o_n(1) \quad \text{as } n \longrightarrow \infty. \tag{4.47}$$

Hence, $\{s_n u_n\} \subset \mathbf{M}(\Omega)$ is a $(\mathrm{PS})_\beta$-sequence for J. By Lemma 3.3(ii), K satisfies the $(\mathrm{PS})_\beta$-condition for $0 < \beta \leq \alpha^\infty - \sigma_\mathrm{n}$. Now, we apply Lemma 4.6 to get that K has at least two critical points in $[K < \alpha^\infty]$. Moreover, by Lemmas 4.3(ii) and 2.2, there are at least two positive solutions of (1.1) in Ω. □

Recall that there exist a unique $s_u > 0$ and a unique $s_u^\infty > 0$ such that $s_u u \in \mathbf{M}(\Omega)$ and $s_u^\infty u \in \mathbf{M}^\infty(\Omega)$. Then, we have the following results.

Lemma 4.11. *For each $u \in \Sigma$, we have that*

$$\left(\frac{p-m}{p-2}\right) J^\infty(s_u^\infty u) \leq J(s_u u) \leq J^\infty(s_u^\infty u), \quad \textit{where } m > 2. \tag{4.48}$$

Proof. Since $(m/2)q_\infty \gneqq q(z) \gneqq q_\infty$, where $m > 2$, we obtain that for each $u \in \Sigma$ and

$$J(s_u u) \le J^\infty(s_u u) \le \sup_{s\ge 0} J^\infty(su) = J^\infty(s_u^\infty u),$$

$$\begin{aligned} J(s_u u) &= \sup_{s\ge 0} J(su) \ge J(s_u^\infty u) = \frac{1}{2}\|s_u^\infty u\|_{H^1}^2 - \frac{1}{p}\int_\Omega q(z)(s_u^\infty u_+)^p dz \\ &\ge \frac{1}{2}\int_\Omega q_\infty (s_u^\infty u_+)^p dz - \frac{1}{p}\int_\Omega \frac{m}{2} q_\infty (s_u^\infty u_+)^p dz \\ &= \left(\frac{1}{2} - \frac{m}{2p}\right)\int_\Omega q_\infty (s_u^\infty u_+)^p dz = \left(\frac{p-m}{p-2}\right) J^\infty(s_u^\infty u). \end{aligned} \tag{4.49}$$

□

Let

$$\begin{aligned} K(u) &= \max_{s\ge 0} J(su) = J(s_u u) > 0, \\ K^\infty(u) &= \max_{s\ge 0} J^\infty(su) = J(s_u^\infty u) > 0, \end{aligned} \tag{4.50}$$

where $s_u u \in \mathbf{M}(\Omega)$ and $s_u^\infty u \in \mathbf{M}^\infty(\Omega)$. Bahri-Li's minimax argument [4] also works for K. Let

$$\Gamma = \left\{ g \in C\left(\overline{B_r(0)},\ \Sigma\right) \big| g|_{\partial B_r(0)} = \frac{\varphi_R(z) w(z-y)}{\|\varphi_R(z) w(z-y)\|_{H^1}} \right\} \quad \text{for large } r = |y|. \tag{4.51}$$

Then, we define

$$\begin{aligned} \gamma(\Omega) &= \inf_{g\in\Gamma} \sup_{y\in \overline{B_r(0)}} K(g(y)), \\ \gamma^\infty(\Omega) &= \inf_{g\in\Gamma} \sup_{y\in \overline{B_r(0)}} K^\infty(g(y)). \end{aligned} \tag{4.52}$$

Lemma 4.12. $\alpha^\infty < \gamma^\infty(\Omega) < 2\alpha^\infty$.

Proof. Bahri and Li [4] proved that (1.2) admits at least one positive solution u in Ω and $J^\infty(u) = \gamma^\infty(\Omega) < 2\alpha^\infty$. Lien et al. [17] proved that (1.2) does not have any positive ground state solution in Ω and $\alpha^\infty(\Omega) = \alpha^\infty(\mathbb{R}^N) = \alpha^\infty$. Hence, $\alpha^\infty < \gamma^\infty(\Omega) < 2\alpha^\infty$. □

The following minimax lemma is given in Shi [24] to unify the mountain pass lemma of Ambrosetti and Rabinowitz [25] and the saddle point theorem of Rabinowitz [26].

Lemma 4.13. *Let V be a compact metric space, $V_0 \subset V$ a closed set, X a Banach space, $\chi \in C(V_0, X)$ and let us define the complete metric space M by*

$$M = \{ g \in C(V, X) \mid g(s) = \chi(s) \ \textit{if}\ s \in V_0 \} \tag{4.53}$$

with the usual distance d. Let $\varphi \in C^1(X, \mathbb{R})$ and let us define

$$c = \inf_{g \in M} \max_{s \in V} \varphi(g(s)), \qquad c_1 = \max_{\chi(V_0)} \varphi. \tag{4.54}$$

If $c > c_1$, then for each $\varepsilon > 0$ and each $g \in M$ such that

$$\max_{s \in V} \varphi(g(s)) \leq c + \varepsilon, \tag{4.55}$$

there exists $v \in X$ such that

$$\begin{gathered} c - \varepsilon \leq \varphi(v) \leq \max_{s \in V} \varphi(g(s)), \\ \operatorname{dist}(v, g(V)) \leq \varepsilon^{1/2}, \\ \|\varphi'(v)\| \leq \varepsilon^{1/2}. \end{gathered} \tag{4.56}$$

Lemma 4.14. *Assume that q is a positive continuous function in $\mathbb{R}^N$. If q satisfies* (q1) *and* (q2). *Let $\{u_n\} \subset \mathbf{M}(\Omega)$ be a $(PS)_\beta$-sequence in $H_0^1(\Omega)$ for J with $\alpha^\infty < \beta < \alpha^\infty + \alpha(\Omega)$. Then, there exist a subsequence $\{u_n\}$ and a nonzero $u_0 \in H_0^1(\Omega)$ such that $u_n \to u_0$ strongly in $H_0^1(\Omega)$, that is, J satisfies the $(PS)_\beta$-condition in $H_0^1(\Omega)$. Moreover, u_0 is a positive solution of* (1.1) *such that $J(u_0) = \beta$.*

Proof. The proof is similar to Lemma 3.3(ii). Applying Palais-Smale Decomposition Lemma 3.1, we get

$$\alpha^\infty + \alpha(\Omega) > \beta = J(u_n) \geq l\alpha^\infty + \alpha(\Omega) \ (\text{or } \geq l\alpha^\infty). \tag{4.57}$$

Since w is the unique (up to translation), positive solution of (1.2) in $\mathbb{R}^N$ and $J^\infty(w) = \alpha^\infty > \alpha(\Omega)$, then $l = 0$ and $u_0 \neq 0$. Hence, $u_n \to u_0$ strongly in $H_0^1(\Omega)$ and $J(u_0) = \beta$. Moreover, by Lemma 2.2, u_0 is positive in Ω. □

Theorem 4.15. *Assume that q is a positive continuous function in $\mathbb{R}^N$. If q satisfies* (q1) *and there exists a number $m' > 2$ such that for any $2 < m \leq m'$,*

$$\frac{m}{2} q_\infty \gneqq q(z) \geq q_\infty + C\exp(-\delta|z|), \quad \text{where } 0 < C \leq \frac{m-2}{2} q_\infty \text{ and } 0 < \delta < 2, \tag{$q_{2'}'$}$$

then (1.1) *admits at least three positive solutions in Ω.*

Proof. Applying Lemma 4.11(iii) to obtain

$$\begin{gathered} \left(\frac{p-m}{p-2}\right)\alpha^\infty \leq \alpha(\Omega) \leq \alpha^\infty, \\ \left(\frac{p-m}{p-2}\right)\gamma^\infty(\Omega) \leq \gamma(\Omega) \leq \gamma^\infty(\Omega). \end{gathered} \tag{4.58}$$

Since $\alpha^\infty < \gamma^\infty(\Omega) < 2\alpha^\infty$, given $0 < \varepsilon < (2\alpha^\infty - \gamma^\infty(\Omega))/2$, there is a number $\min\{m_1, p\} \geq m_2 > 2$ such that for any $2 < m \leq m_2$, we have

$$\gamma^\infty(\Omega) < \alpha^\infty + \alpha(\Omega) \leq 2\alpha^\infty. \tag{4.59}$$

Choosing some $\min\{m_2, p\} \geq m' > 2$ such that for any $2 < m \leq m'$, we get

$$\alpha^\infty < \gamma(\Omega) \leq \gamma^\infty(\Omega) < \alpha^\infty + \alpha(\Omega) \leq 2\alpha^\infty. \tag{4.60}$$

By Lemma 3.6, for any $t \geq 0$, we have

$$J\left(t\psi_R(z)w(z-y)\right) \leq \alpha^\infty + o(1) \quad \text{as } |y| \longrightarrow \infty. \tag{4.61}$$

Then,

$$\begin{aligned} K\left(\frac{\psi_R(z)w(z-y)}{\|\psi_R(z)w(z-y)\|_{H^1}}\right) &= J\left(\frac{t_y\psi_R(z)w(z-y)}{\|\psi_R(z)w(z-y)\|_{H^1}}\right) \\ &\leq \alpha^\infty + o(1) \quad \text{as } |y| \longrightarrow \infty, \end{aligned} \tag{4.62}$$

that is, $\gamma(\Omega) > K(\psi_R(z)w(z-y)/\|\psi_R(z)w(z-y)\|_{H^1})$ for large $r = |y|$. Applying Lemma 4.3 and the minimax Lemma 4.13 to obtain that $\gamma(\Omega)$ is a (PS)-value in $H_0^1(\Omega)$ for J. Hence, by Lemmas 2.2 and 4.14, we have that there exists a positive solution u of (1.1) in Ω such that $J(u) = \gamma(\Omega)$. From the result of Theorem 4.10, (1.1) admits at least three positive solutions in Ω. □

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