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Research Article

On a Mixed Problem for a Constant Coefficient Second-Order System

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The paper is devoted to the study of an initial boundary value problem for a linear second-order differential system with constant coefficients. The first part of the paper is concerned with the existence of the solution to a boundary value problem for the second-order differential system, in the strip $\Omega_A = \mathbf{R}^{d-1} \times (0, A)$, where A is a suitable positive number. The result is proved by means of the same procedure followed in a previous paper to study the related initial value problem. Subsequently, we consider a mixed problem for the second-order constant coefficient system, where the space variable varies in Ω_A and the time-variable belongs to the bounded interval $]0, T[$, with T sufficiently small in order that the operator satisfies suitable energy estimates. We obtain by superposition the existence of a solution $u \in L^2([0,T] \times [0,A], H^3(\mathbf{R}^{d-1}))$, by studying two related mixed problems, whose solutions exist due to the results proved for the Cauchy problem in a previous paper and for the boundary value problem in the first part of this paper.

1. Introduction

Consider the second-order linear differential operator

$$Q[\cdot] = \lambda\partial_t^2 - S\partial_t - \sum_{\alpha=1}^{d} F^{\alpha}\partial_\alpha + \sum_{\alpha=1}^{d-1}\sum_{\beta=1}^{d-1} E^{\alpha,\beta}\partial_\alpha\partial_\beta + E^{d,d}\partial_d^2 - G. \tag{1.1}$$

The coefficients of the operator Q satisfy the following assumptions:

(i) λ is a positive real number;

(ii) for all $\alpha, \beta = 1, \dots, d$, $S, F^{\alpha}, G, E^{\alpha,\beta}$ are $d \times d$ symmetric matrices with real entries;

(iii) for every $v \in \mathbf{C}^d$, $\mathrm{Re}\langle Gv, \overline{v}\rangle \geq c_G\|v\|^2$, where c_G is a positive constant;

(iv) for all $v \in \mathbf{C}^d, \langle Sv, \overline{v} \rangle \in \mathbf{R}, \langle F^d v, \overline{v} \rangle \in \mathbf{R}$; in addition, there exist two positive constants c_S and c_d such that for every $v \in \mathbf{C}^d, \langle Sv, \overline{v} \rangle \geq c_S \|v\|^2$, and $\langle F^d v, \overline{v} \rangle \geq c_d \|v\|^2$;

(v) for every $\alpha = 1, \dots, d$, for all $v \in \mathbf{C}^d, \mathrm{Re}\langle E^{\alpha,\alpha} v, \overline{v} \rangle \geq c_{\alpha,\alpha} \|v\|^2$, with $c_{\alpha,\alpha}$ positive constant.

We will denote by x a point of $\mathbf{R}^d$, by y the first $d-1$ coordinates of x, and by t the time variable.

Let A be a positive real number, and denote by Ω_A the subset of $\mathbf{R}^d$, $\Omega_A = \mathbf{R}^{d-1} \times (0, A)$. In the first section of the paper we will be concerned with the following boundary value problem

$$\begin{gathered} Q[u] = J(x,t), \quad x \in \Omega_A,\ t \in \mathbf{R}, \\ E^{d,d} u(y,0,t) = g(y,t), \quad y \in \mathbf{R}^{d-1},\ t \in \mathbf{R}, \end{gathered} \tag{1.2}$$

where u is the unknown vector-valued function, whereas J and g are given functions, which take values in $\mathbf{R}^d$ and are defined in $\Omega_A \times \mathbf{R}$ and $\mathbf{R}^{d-1} \times \mathbf{R}$, respectively.

Under suitable assumptions on the functions J and g and on the coefficients of the operator Q, we will prove that, in the case where the positive real number A is sufficiently small, there exists a function $u \in L^2_{\mathrm{loc}}(\mathbf{R}^{d-1} \times \mathbf{R}, H^2(]0, A[)) \cap L^2([0, A], H^3(\mathbf{R}^{d-1} \times \mathbf{R}))$, which provides a solution to the boundary value problem (1.2). The existence of the solution is established by means of the techniques applied in [1] to prove that the initial value problem for the system $Q[u] = J$, admits a solution $u \in L^2([0,T], H^3(\mathbf{R}^d))$: the main result of [1] states that if the assumptions (i)–(iv) listed above along with other suitable conditions are fulfilled (see Proposition 3.1), then the adjoint operator of Q satisfies a priori estimates, which allow proving the existence of the solution to the Cauchy problem, through the definition of a suitable functional and a duality argument. As we will explain below, the boundary value problem (1.2) can be regarded exactly as an initial value problem. For this reason, the result of Section 2 does not represent any significant advance with respect to the results proved in [1].

The main novelty of the paper is represented by the study of a mixed problem in the third section. The interest in this kind of problems relies on the fact that they appear frequently as physical models: mixed problems for second-order hyperbolic equations and systems of equations occur in the theory of sound to describe for instance the evolution of the air pressure inside a room where noise is produced, as well as in the electromagnetism to describe the evolution of the electromagnetic field in some region of space (the system of Maxwell equations accounts for this kind of phenomenon).

The existence results, stated both for the initial value problem in [1] and for the boundary value problem in Section 2 of this paper, turn out to be the backbone in proving the existence of the solution to the following mixed problem in the strip Ω_A, as the time variable t belongs to the bounded interval $[0,T]$, where T is a suitable positive real number:

$$\begin{gathered} Q[w] = J(x,t), \quad x \in \Omega_A,\ t \in [0,T], \\ E^{d,d} w(y,0,t) = g(y,t), \quad y \in \mathbf{R}^{d-1},\ t \in [0,T], \\ w(x,0) = h(x), \quad x \in \Omega_A. \end{gathered} \tag{1.3}$$

We will assume that the vector-valued function J belongs to the space $L^2([0,A], H^2(\mathbf{R}^{d-1}\times\mathbf{R}))$, while, as for the initial data, we suppose that $g\in H^3(\mathbf{R}^{d-1}\times\mathbf{R})$ and $h\in H^3(\mathbf{R}^d)$. Let us notice that in the problem (1.3) an initial value for the unknown vector field u and a Dirichlet boundary condition only are prescribed. Due to the a priori estimates that we will derive for the operator Q, it is not required in problem (1.3), in contrast with classical mixed problems for hyperbolic second-order systems, that the first-order derivatives of u satisfy a prescribed condition at the boundary of the domain $\Omega_A\times\,]0,T[$. This lack of information about the initial value of the first-order derivatives results in a possible nonuniqueness of the solution to (1.3). The existence result of Section 3 will be achieved by means of the definition of two mixed problems related to (1.3): the existence of the solution to the former will be established similarly to the result obtained in [1], while the latter will be studied like the boundary value problem considered in Section 2. Subsequently, thanks to the linearity of the operator Q, a solution to (1.3) belonging to the space $L^2([0,T]\times[0,A], H^3(\mathbf{R}^{d-1}))$ will be determined by superposition of the solutions to the preliminary mixed problems.

2. Boundary Value Problem

By adopting the same strategy of [1] to prove the existence of the solution to the initial value problem, let us determine the existence of a solution to (1.2) through a duality argument, by proving energy estimates.

Let us denote by $Q^\star$ the adjoint operator of Q:

$$Q^\star[u]=\lambda\partial_t^2u+S\partial_tu+\sum_{\alpha=1}^{d}F^\alpha\partial_\alpha u+\sum_{\alpha=1}^{d-1}\sum_{\beta=1}^{d-1}E^{\alpha,\beta}\partial_\alpha\partial_\beta u+E^{d,d}\partial_d^2u-Gu. \tag{2.1}$$

For all $\alpha,\beta=1,\ldots,d-1$, let $C^G, C^S, C^\alpha, C^{\alpha,\beta}$ be the norms of the matrices $G, S, F^\alpha, E^{\alpha,\beta}$, respectively.

Proposition 2.1. *Consider the operator Q defined in* (1.1) *and the corresponding adjoint $Q^\star$. Let the conditions (i)–(iv), listed in the Introduction, be fulfilled. In addition, assume the sums $\eta_d=c_d/2-C^{d,d}/2$, $\eta_1=c_G-C^S/2-\sum_{\alpha=1}^{d-1}C^\alpha/2-1/2$, $\eta_2=\lambda-C^S/2$, and for every $\alpha=1,\ldots,d-1$, $\eta_\alpha=c_{\alpha,\alpha}-C^\alpha/2-\sum_{\beta\neq\alpha,\beta=1}^{d-1}C^{\alpha,\beta}$ to be positive real numbers. Moreover, denote by δ the sum $\delta=\sum_{\alpha=1}^{d-1}C^\alpha/2+C^G/2+C^S/2+C^d+1/2$, and suppose that, as long as the positive number A is sufficiently small,* $\min\{\eta_1,\eta_\alpha,\eta_2:\alpha=1,\ldots,d-1\}-\delta e^{\delta A}>0$.

Define the linear space

$$\begin{aligned}\Sigma_A=\Big\{&\phi:\mathbf{R}^d\times\mathbf{R}\longrightarrow\mathbf{R}^d:\phi\in C^2\Big(\mathbf{R}^{d+1}\Big),\\ &\operatorname{supp}\|\phi\|\ \textit{compact subset of}\ \mathbf{R}^{d-1}\times\Big]-\infty,A\Big[\times\mathbf{R};\ \forall y,\forall t,\ \phi(y,0,t)=0\Big\}.\end{aligned} \tag{2.2}$$

Then, for all functions $\phi\in\Sigma_A$, the following estimates hold:

$$\|\phi\|^2_{L^2([0,A],H^{-2}(\mathbf{R}^{d-1}\times\mathbf{R}))}\leq C_1\|Q^\star[\phi]\|^2_{L^2([0,A],H^{-3}(\mathbf{R}^{d-1}\times\mathbf{R}))}, \tag{2.3}$$

$$\|\partial_d \phi(0)\|^2_{H^{-3}(\mathbf{R}^{d-1}\times\mathbf{R})} \le C_2 e^{\delta A} \|Q^\star[\phi]\|^2_{L^2([0,A],H^{-3}(\mathbf{R}^{d-1}\times\mathbf{R}))}, \tag{2.4}$$

where C_1, C_2 are suitable positive constants.

Proof. Consider a vector function $\phi \in C^2(\mathbf{R}^{d+1})$, with compact support in the subset $\mathbf{R}^{d-1}\times\,]-\infty, A] \times \mathbf{R}$. Applying the Fourier transform with respect to both the tangential variable y and the time variable t, we can obtain a priori estimates, which, by substituting x_d for t and carrying out similar calculations, turn out to be like the estimates obtained in [1] for the initial value problem. Subsequently, assuming the function ϕ belongs to Σ_A, we deduce the a priori estimates (2.3) and (2.4) for the adjoint operator $Q^\star$.

Due to (2.3) and (2.4), by means of a duality argument, we can prove the existence result for the solution to the boundary value problem (1.2). □

Proposition 2.2. *Consider the boundary value problem* (1.2), *and let the assumptions of Proposition 2.1 be satisfied. Furthermore, suppose that $J \in L^2([0,A], H^2(\mathbf{R}^{d-1}\times\mathbf{R}))$ and $g \in H^3(\mathbf{R}^{d-1}\times\mathbf{R})$. Then, the boundary value problem* (1.2) *has a solution $u \in L^2_{\text{loc}}(\mathbf{R}^{d-1}\times\mathbf{R}, H^2(]0,A[)) \cap L^2([0,A], H^3(\mathbf{R}^{d-1}\times\mathbf{R}))$.*

Proof. The result can be proved by means of the same tools used to establish the existence result for the initial value problem in [1]. For the sake of completeness, let us sketch the proof. Because of the a priori estimate (2.3), the operator $Q^\star$ is one-to-one on the linear space Σ_A. Let $\phi \in \Sigma_A$, and define the following linear functional on the space $Q^\star(\Sigma_A)$:

$$\mathcal{L}(Q^\star[\phi]) = \int_0^A \langle J, \phi\rangle_{H^2,H^{-2}} dx_d - \langle g, \partial_{x_d}\phi(0)\rangle_{H^3,H^{-3}}. \tag{2.5}$$

The functional $\mathcal{L}$ turns out to be well defined, since $Q^\star$ is injective on Σ_A. Moreover, $\mathcal{L}$ is continuous with respect to the norm of the space $L^2([0,A], H^{-3}(\mathbf{R}^{d-1}\times\mathbf{R}))$, because of the energy estimates proved in Proposition 2.1. Due to the Hahn-Banach Theorem, the functional $\mathcal{L}$ can be extended to the space $L^2([0,A], H^{-3}(\mathbf{R}^{d-1}\times\mathbf{R}))$. Let us denote by $\mathcal{M}$ this functional.

By the Riesz Theorem, there exists a function u, which belongs to the dual space $L^2([0,A], H^3(\mathbf{R}^{d-1}\times\mathbf{R}))$, so that for every $v \in L^2([0,A], H^{-3}(\mathbf{R}^{d-1}\times\mathbf{R}))$, $\mathcal{M}(v) = \int_0^A \langle u, v\rangle_{H^3,H^{-3}} dx_d$. In particular, in the case where $\phi \in \Sigma_A$,

$$\int_0^A \langle u, Q^\star[\phi]\rangle_{H^3,H^{-3}} dx_d = \int_0^A \langle J, \phi\rangle_{H^2,H^{-2}} dx_d - \langle g, \partial_{x_d}\phi(0)\rangle_{H^3,H^{-3}}. \tag{2.6}$$

Hence, since for every $\phi \in C_0^\infty(\mathbf{R}^{d-1}\times\,]0,A[\,\times\mathbf{R})$

$$\int_0^A \langle u, Q^\star[\phi]\rangle_{H^3,H^{-3}} dx_d = \int_0^A \langle J, \phi\rangle_{H^2,H^{-2}} dx_d, \tag{2.7}$$

the function u turns out to be a solution of the system (1.2) in the sense of distributions.

In order to prove that the boundary condition is satisfied, by adopting the same strategy followed in [1], we have to study the regularity of the solution u. For this purpose, let us extend the functions u and J by zero outside the interval $[0, A]$. By means of an approximation argument, we construct a sequence of smooth functions $v_n \in C^\infty(\mathbf{R}^d \times \mathbf{R}) \cap L^2(\mathbf{R}, H^3(\mathbf{R}^{d-1} \times \mathbf{R}))$, so that $(v_n)_n$ turns out to be convergent to the function u, with respect to the norm of the space $L^2(\mathbf{R}, H^3(\mathbf{R}^{d-1} \times \mathbf{R}))$. We define the approximating sequence in such a way for all $n \in \mathbf{N}$, v_n vanishes outside a compact neighbourhood of $[0, A]$, for example, $[-1, A+1]$.

Moreover, let us denote by P the differential operator $P[\cdot] = \lambda\partial_t^2 - S\partial_t - \sum_{\alpha=1}^{d-1} F^\alpha \partial_\alpha + \sum_{\alpha=1}^{d-1}\sum_{\beta=1}^{d-1} E^{\alpha,\beta}\partial_\alpha\partial_\beta - G$. Therefore, $Q[\cdot] = E^{d,d}\partial_d\partial_d - F^d\partial_d + P[\cdot]$. Set for all $n \in \mathbf{N}$, $J_n = E^{d,d}\partial_d\partial_d v_n - F^d\partial_d v_n + P[v_n]$. Let $\phi \in C_0^\infty(\mathbf{R}^d \times]-1, A[)$. Thus,

$$\int_{-1}^{A+1}\int_{\mathbf{R}^{d-1}\times\mathbf{R}} \left\langle E^{d,d}\partial_d^2 v_n - F^d\partial_d v_n, \phi\right\rangle dy\, dt\, dx_d = \int_{-1}^{A+1}\int_{\mathbf{R}^{d-1}\times\mathbf{R}} \langle -P[v_n] + J_n, \phi\rangle dy\, dt\, dx_d. \tag{2.8}$$

Since the sequence $(v_n)_n$ is convergent to the function u with respect to the norm of $L^2(\mathbf{R}, H^3(\mathbf{R}^{d-1} \times \mathbf{R}))$, integrating by parts, we obtain

$$\begin{aligned}
\lim_{n\to\infty}&\int_{-1}^{A+1}\int_{\mathbf{R}^{d-1}\times\mathbf{R}} \langle J_n, \phi\rangle dy\, dt\, dx_d \\
&= \int_{-1}^{A+1}\int_{\mathbf{R}^{d-1}\times\mathbf{R}} \left(\langle P[u], \phi\rangle + \left\langle E^{d,d}u, \partial_d^2\phi\right\rangle + \left\langle F^d u, \partial_d\phi\right\rangle\right) dy\, dt\, dx_d \\
&= \int_{-1}^{A+1}\int_{\mathbf{R}^{d-1}\times\mathbf{R}} \langle J, \phi\rangle dy\, dt\, dx_d.
\end{aligned} \tag{2.9}$$

As a result, the sequence of functions $(J_n)_n$ is weakly convergent to the function J in the space $L^2(\mathbf{R}^{d-1} \times [-1, A+1] \times \mathbf{R})$. Hence, the sequence $(J_n)_n$ turns out to be bounded in $L^2(\mathbf{R}^{d-1} \times [-1, A+1] \times \mathbf{R})$.

Let us consider again the system $E^{d,d}\partial_d\partial_d v_n - F^d\partial_d v_n = -P[v_n] + J_n$, for every $n \in \mathbf{N}$.

By means of integration on the interval $[-1, a]$, with $a < A+1$, we obtain

$$E^{d,d}\partial_d v_n(a) - F^d v_n(a) = \int_{-1}^{a} (-P[v_n] + J_n)d\sigma. \tag{2.10}$$

Let us estimate the L^2-norm of the r.h.s. of (2.10). We deduce that

$$\int_{-1}^{A+1}\int_{\mathbf{R}^{d-1}\times\mathbf{R}} \left(\int_{-1}^{a}(-P[v_n] + J_n)d\sigma\right)^2 dy\, dt\, da \le (A+2)^2\text{const}. \tag{2.11}$$

Since there exists a suitable constant such that, for all $n \in \mathbf{N}$, $\|F^d v_n\|_{L^2(\mathbf{R}^{d-1}\times[-1,A+1]\times\mathbf{R})} \le$ const., the sequence $(\partial_d v_n)_n$ turns out to be bounded in $L^2(\mathbf{R}^{d-1} \times [-1, A+1] \times \mathbf{R})$. Thus, the sequence of functions $(\partial_{x_d}^2 v_n)_n$ also turns out to be bounded in $L^2(\mathbf{R}^{d-1} \times [-1, A+1] \times \mathbf{R})$.

Let $\alpha = 1, \ldots, d-1$. Differentiating with respect to x_α or to t both members of (2.10), we have

$$E^{d,d}\partial_\alpha\partial_d v_n(a) - F^d\partial_\alpha v_n(a) = \int_{-1}^{a} \partial_\alpha(-P[v_n] + J_n)d\sigma. \tag{2.12}$$

The sequence $(\partial_\alpha v_n)_n$ is convergent to $\partial_\alpha u$ in $L^2(\mathbf{R}^{d-1}\times]-1, A+1[\times\mathbf{R})$.

Due to the convergence of $(v_n)_n$ to u in $L^2(\mathbf{R}, H^3(\mathbf{R}^{d-1} \times \mathbf{R}))$, the sequence $(\int_{-1}^{a} P[v_n]d\sigma)_n$ turns out to converge to $\int_{-1}^{a} P[u]d\sigma$ in $L^2(\mathbf{R}^{d-1}\times]-1, A+1[\times\mathbf{R})$.

Furthermore, the sequence $(\partial_\alpha(\int_{-1}^{a} P[v_n]d\sigma))_n$ is weakly convergent in $L^2(\mathbf{R}^{d-1}\times]-1, A+1[\times\mathbf{R})$. Therefore, it is bounded in the space $L^2(\mathbf{R}^{d-1}\times]-1, A+1[\times\mathbf{R})$. Similarly, the sequence $(\int_{-1}^{a} \partial_\alpha J_n d\sigma)_n$ also turns out to be bounded in $L^2(\mathbf{R}^{d-1}\times]-1, A+1[\times\mathbf{R})$. Hence, $(\partial_\alpha\partial_d v_n)_n$ is bounded in $L^2(\mathbf{R}^{d-1}\times]-1, A+1[\times\mathbf{R})$. Since the sequences of functions $(\partial_d^2 v_n)_n$ and $(\partial_\alpha\partial_d v_n)_n$ are bounded in $L^2(\mathbf{R}^{d-1}\times]-1, A+1[\times\mathbf{R})$, the sequence $(\partial_d v_n)_n$ turns out to satisfy the assumptions of the Riesz-Fréchet-Kolmogorov theorem.

Thus the function u admits a first-order weak derivative with respect to x_d in $L^2_{\text{loc}}(\mathbf{R}^{d-1}\times]-1, A+1[\times\mathbf{R})$. Therefore the function u belongs to the space $L^2_{\text{loc}}(\mathbf{R}^{d-1} \times \mathbf{R}, H^1(]0, A[)) \cap L^2([0, A], H^3(\mathbf{R}^{d-1} \times \mathbf{R}))$.

If we introduce a new variable, the system (1.2) may be reduced to a first-order system with respect to the variable x_d. Let us denote by U the vector function $U = (u, E^{d,d}\partial_d u)^T$ and by $\mathcal{C}_1$ and $\mathcal{C}_2$ the following differential operators

$$\mathcal{C}_1 = -\sum_{\alpha=1}^{d-1}\sum_{\beta=1}^{d-1} E^{\alpha,\beta}\partial_\alpha\partial_\beta - \lambda\partial_t^2 + \sum_{\alpha=1}^{d-1} F^\alpha\partial_\alpha + S\partial_t + G; \qquad \mathcal{C}_2 = F^d E^{d,d^{-1}}. \tag{2.13}$$

Thus, the system (1.2) can be rewritten as

$$\partial_d\begin{pmatrix} u \\ E^{d,d}\partial_d u\end{pmatrix} - \begin{pmatrix} 0_d & E^{d,d^{-1}} \\ \mathcal{C}_1 & \mathcal{C}_2\end{pmatrix}\begin{pmatrix} u \\ E^{d,d}\partial_d u\end{pmatrix} = \begin{pmatrix} 0_d \\ J\end{pmatrix}. \tag{2.14}$$

By setting $\mathcal{C} = \begin{pmatrix} 0_d & E^{d,d^{-1}}I \\ \mathcal{C}_1 & \mathcal{C}_2\end{pmatrix}$ and $\mathcal{F} = \begin{pmatrix} 0_d \\ J\end{pmatrix}$, the system (1.2) becomes $\partial_d U = \mathcal{C}U + \mathcal{F}$.

Because of the regularity properties of the function u, $\mathcal{C}_1 u$ and $\mathcal{C}_2 u$ turn out to belong to $L^2([0, A], L^2_{\text{loc}}(\mathbf{R}^{d-1} \times \mathbf{R}))$.

Multiplying both members of the system by any function of the space $C_0^\infty(\mathbf{R}^{d-1}\times]0, A[\times\mathbf{R})$, we prove that the vector function U has a weak partial derivative with respect to the variable x_d. Thus $u \in L^2_{\text{loc}}(\mathbf{R}^{d-1} \times \mathbf{R}, H^2(]0, A[)) \cap L^2([0, A], H^3(\mathbf{R}^{d-1} \times \mathbf{R}))$ and $Q[u] = J$, a.e. in $\mathbf{R}^{d-1}\times]0, A[\times\mathbf{R}$.

Since u belongs to $L^2_{\text{loc}}(\mathbf{R}^{d-1} \times \mathbf{R}, H^2(]0, A[))$, the traces of u and $\partial_d u$ are well-defined on the hyperplane $x_d = 0$, and belong to $L^2_{\text{loc}}(\mathbf{R}^{d-1} \times \mathbf{R}, H^{3/2}(]0, A[))$ and $L^2_{\text{loc}}(\mathbf{R}^{d-1} \times \mathbf{R}, H^{1/2}(]0, A[))$, respectively.

Let us consider a function $\phi \in C_0^\infty(\mathbf{R}^{d-1}\times]-\infty, A[\times\mathbf{R})$, which, in a neighbourhood of $x_d = 0$ has the form $\phi(y, x_d, t) = x_d\psi(y, t)$, with $\psi \in C_0^\infty(\mathbf{R}^{d-1} \times \mathbf{R})$. Thus,

$$\mathcal{L}(Q^\star[\phi]) = \int_0^A \langle J, \phi\rangle_{H^2,H^{-2}} dx_d - \langle g, \partial_{x_d}\phi(0)\rangle_{H^3,H^{-3}} = \int_0^A \langle u, Q^\star[\phi]\rangle_{H^3,H^{-3}} dx_d. \tag{2.15}$$

Integrating by parts,

$$\begin{aligned} &\int_{\mathbf{R}^{d-1}}\int_0^A\int_{\mathbf{R}} \langle Q[u], \phi\rangle dy\, dx_d dt - \int_{\mathbf{R}^{d-1}}\int_{\mathbf{R}} \left\langle E^{d,d}u(0), \psi\right\rangle dy\, dt \\ &\quad = \int_{\mathbf{R}^{d-1}}\int_0^A\int_{\mathbf{R}} \langle J, \phi\rangle dy\, dx_d\, dt - \int_{\mathbf{R}^{d-1}}\int_{\mathbf{R}} \langle g, \psi\rangle dy\, dt. \end{aligned} \tag{2.16}$$

Hence, we obtain $E^{d,d}u(0) = g$, a.e. in $\mathbf{R}^{d-1} \times \mathbf{R}$. □

3. Mixed Problem

This section deals with the study of the initial boundary value problem (1.3). We will prove the existence of the solution after solving two auxiliary problems: first we will determine the solution of an initial value problem, by means of the techniques developed in [1]; next, we will find the solution of a suitable boundary value problem, in accordance with the results stated in the previous section. Since the operator Q is linear, the solution to the mixed problem (1.3) will be determined by superposition. As a matter of fact, both auxiliary problems are mixed problems, but, as we will explain below, the solution of the former will be found as in the case of initial value problems, whereas the latter may be studied in the framework of boundary value problems.

Let us define the first problem as follows:

$$\begin{aligned} Q[v] &= 0, \quad x \in \Omega_A,\ t \in [0, T], \\ E^{d,d}v(y, 0, t) &= 0, \quad y \in \mathbf{R}^{d-1},\ t \in [0, T], \\ v(x, 0) &= h(x), \quad x \in \Omega_A, \end{aligned} \tag{3.1}$$

where $h \in H^3(\mathbf{R}^d)$.

We consider the Cauchy problem

$$\begin{aligned} Q[v] &= 0, \quad x \in \mathbf{R}^d,\ t \in [0, T], \\ v(x, 0) &= h(x), \quad x \in \mathbf{R}^d, \end{aligned} \tag{3.2}$$

and determine the solution by means of a duality argument through the procedure followed in [1]. For this purpose, we have to assume conditions on the coefficients of the operator Q in order for energy estimates to be satisfied. Furthermore, let us define the linear space

$$\begin{aligned}\mathcal{F}_T = \Big\{\phi : \mathbf{R}^d \times \mathbf{R} \longrightarrow \mathbf{R}^d : \phi \in C^2\Big(\mathbf{R}^{d-1} \times [0,+\infty[\times [0,T]\Big),\\ \operatorname{supp}\|\phi\| \text{ compact subset of } \mathbf{R}^{d-1} \times [0,+\infty[\times]-\infty,T[;\ \forall x,\ \phi(x,0) = 0\Big\};\end{aligned} \tag{3.3}$$

and quote from [1] the following result.

Proposition 3.1. *Consider the operator Q defined in* (1.1) *and the corresponding adjoint $Q^\star$. Assume the conditions (i)–(iv) listed in the Introduction to be fulfilled. In addition, let the sums $c_S/2 - \lambda/2$, $\overline{C}_1 = c_G - \sum_{\alpha=1}^{d} C^\alpha/2 - 1/2$, and for every $\alpha = 1,\ldots,d$, $\overline{C}_2 = \min_{\alpha=1,\ldots,d}\{c_{\alpha,\alpha} - C^\alpha/2 - \sum_{\beta\neq\alpha,\beta=1}^{d} C^{\alpha,\beta}\}$, be positive real numbers. Moreover, we denote by $\widetilde{C}$ the sum $\widetilde{C} = \sum_{\alpha=1}^{d} C^\alpha/2 + C^G/2 + C^S + 1/2$ and suppose $\min\{\overline{C}_1, \overline{C}_2\} - \widetilde{C}e^{\widetilde{C}T}$ is positive, provided that T is small enough.*

Then, the operator $Q^\star$ satisfies the following estimates:

(a) *for every $u \in \mathcal{F}_T$,*

$$\|u\|^2_{L^2([0,T],H^{-2}(\mathbf{R}^d))} + \lambda\|\partial_t u\|^2_{L^2([0,T],H^{-3}(\mathbf{R}^d))} \leq c_1\|Q^\star[u]\|^2_{L^2([0,T],H^{-3}(\mathbf{R}^d))}; \tag{3.4}$$

(b) *for all $u \in \mathcal{F}_T$,*

$$\|\partial_t u(0)\|^2_{H^{-3}(\mathbf{R}^d)} \leq c_2 e^{\widetilde{C}T}\|Q^\star[u]\|^2_{L^2([0,T],H^{-3}(\mathbf{R}^d))}, \tag{3.5}$$

with c_1, c_2 being positive constants that are independent of T.

By taking into account the energy estimates of Proposition 3.1, we establish the following existence result.

Proposition 3.2. *Consider the initial boundary value problem* (3.1), *and let the assumptions of Proposition 3.1 be satisfied. If the function $h \in H^3(\mathbf{R}^d)$, then the problem* (3.1) *has a solution $v \in L^2([0,T],H^3(\mathbf{R}^d)) \cap L^2_{\text{loc}}(\mathbf{R}^d, H^2(]0,T[))$.*

Proof. Let us define on $Q^\star(\mathcal{F}_T)$ the linear functional

$$\Lambda(Q^\star[\phi]) = -\lambda\langle h, \partial_t\phi(0)\rangle_{H^3,H^{-3}}, \tag{3.6}$$

where $\phi \in \mathcal{F}_T$.

Through the procedure followed in [1], we can prove there exists a function $v \in L^2_{\text{loc}}(\mathbf{R}^d, H^2(]0,T[)) \cap L^2([0,T],H^3(\mathbf{R}^d))$, such that for every $\phi \in \mathcal{F}_T$, $\int_0^T \langle v, Q^\star[\phi]\rangle_{H^3,H^{-3}}dt = -\lambda\langle h, \partial_t\phi(0)\rangle_{H^3,H^{-3}}$.

In addition, due to the results proved in [1], $Q[v] = 0$, a.e. $x \in \mathbf{R}^d$, $t \in]0,T[$, and $v(x,0) = h(x)$, a.e. $x \in \mathbf{R}^d$.

Moreover, since for all $t \in [0,T]$, the function $v(\cdot,t) \in H^3(\overline{\Omega}_A)$, the trace of v on the boundary of Ω_A belongs to $L^2([0,T], H^{3-1/2}(\mathbf{R}^{d-1}))$. Let us determine the trace of v on $x_d = 0$. Let ψ be a function of the space $C^2(\mathbf{R}^{d-1} \times [0,+\infty[\times [0,T])$, such that $\operatorname{supp}\|\psi\|$ is a compact subset of $(\mathbf{R}^{d-1} \times [0,A[\times]0,T[)$. Therefore, $\psi \in \mathcal{F}_T$ and $\int_0^T \langle v, Q^\star[\psi]\rangle_{H^3,H^{-3}} dt = 0$. By integrating by parts,

$$\int_0^T \int_{\Omega_A} \langle Q[v], \psi\rangle dt\, dx - \int_0^T \int_{\mathbf{R}^{d-1}} \left(\left\langle E^{d,d} v(0), \partial_d\ \psi(0) \right\rangle + \left\langle F^d v(0), \psi(0) \right\rangle - \left\langle E^{d,d} \partial_d v(0), \psi(0) \right\rangle \right) dy\, dt = 0. \tag{3.7}$$

Thus,

$$\int_0^T \int_{\mathbf{R}^{d-1}} \left(\left\langle E^{d,d} v(0), \partial_d\ \psi(0) \right\rangle + \left\langle F^d v(0), \psi(0) \right\rangle - \left\langle E^{d,d} \partial_d v(0), \psi(0) \right\rangle \right) dy\, dt = 0. \tag{3.8}$$

Consider a vector function ψ, which, as long as x_d is nonnegative and sufficiently small, has the form $\psi(y, x_d, t) = x_d \chi(y,t)$, with $\chi \in C_0^\infty(\mathbf{R}^{d-1} \times]0,T[)$. Hence, we deduce by means of a standard argument that $E^{d,d} v(0) = 0$, a.e. in $\mathbf{R}^{d-1} \times]0,T[$. □

Let us define now the second auxiliary initial boundary value problem in order to obtain by superposition a solution to (1.3):

$$\begin{gathered} Q[w] = J, \quad x \in \Omega_A,\ t > 0, \\ E^{d,d} w(y,0,t) = g, \quad y \in \mathbf{R}^{d-1},\ t > 0, \\ w(x,0) = 0, \quad x \in \Omega_A. \end{gathered} \tag{3.9}$$

We assume that $J \in L^2([0,A], H^2(\mathbf{R}^{d-1} \times \mathbf{R}))$, whereas $g \in H^3(\mathbf{R}^{d-1} \times \mathbf{R})$.

The existence of the solution to (3.9) can be proved by means of the duality argument and a procedure similar to the previous problem.

Proposition 3.3. *Consider the initial boundary value problem* (3.9), *and let the assumptions of Proposition 2.1 be satisfied. If* $J \in L^2([0,A], H^2(\mathbf{R}^{d-1} \times \mathbf{R}))$ *and* $g \in H^3(\mathbf{R}^{d-1} \times \mathbf{R})$, *then there exists a function* $w \in L^2([0,A], H^3(\mathbf{R}^{d-1} \times \mathbf{R})) \cap L^2_{\mathrm{loc}}(\mathbf{R}^{d-1} \times \mathbf{R}, H^2(]0,A[))$, *which provides a solution to* (3.9).

Proof. We consider the boundary value problem

$$\begin{gathered} Q[w] = J, \quad x \in \Omega_A,\ t > 0, \\ E^{d,d} w(y,0,t) = g, \quad y \in \mathbf{R}^{d-1},\ t > 0. \end{gathered} \tag{3.10}$$

Since the operator Q satisfies the assumptions of Proposition 2.1, energy estimates can be proved for the adjoint operator $Q^\star$. Let us define the linear space

$$\mathcal{G}_A = \Big\{\phi : \mathbf{R}^d \times \mathbf{R} \longrightarrow \mathbf{R}^d : \phi \in C^2\Big(\mathbf{R}^{d-1} \times [0,A] \times [0,+\infty[\Big),$$
$$\operatorname{supp}\|\phi\| \text{ compact subset of } \mathbf{R}^{d-1} \times [0,A[\times [0,+\infty[;\ \forall y,\ \forall t,\ \phi(y,0,t) = 0\Big\}. \tag{3.11}$$

For every $\phi \in \mathcal{G}_A$, consider the following functional:

$$\Delta(Q^\star[\phi]) = \int_0^A \langle J,\phi\rangle_{H^2,H^{-2}} dx_d - \langle g,\partial_d\phi(0)\rangle_{H^3,H^{-3}}. \tag{3.12}$$

As proved in the previous section, the functional turns out to be well defined and continuous as a consequence of the energy estimates. Furthermore, the functional can be extended to the space $L^2([0,A],H^{-3}(\mathbf{R}^{d-1} \times \mathbf{R}))$, and there exists a function $w \in L^2([0,A],H^3(\mathbf{R}^{d-1} \times \mathbf{R}))$, so that for every $\phi \in \mathcal{G}_A$, $\int_0^A \langle J,\phi\rangle_{H^2,H^{-2}} dx_d - \langle g,\partial_d\phi(0)\rangle_{H^3,H^{-3}} = \int_0^A \langle w,Q^\star[\phi]\rangle_{H^2,H^{-2}} dx_d$. After studying the regularity properties of the function w as in the previous section and in [1], we can prove that $w \in L^2_{\text{loc}}(\mathbf{R}^{d-1} \times \mathbf{R},H^2(]0,A[)) \cap L^2([0,A],H^3(\mathbf{R}^{d-1} \times \mathbf{R}))$, $Q[w] = J$, a.e. in $\mathbf{R}^{d-1}\times\,]0,A[\, \times\mathbf{R}$, and w satisfies the boundary condition.

The function w will be a solution to the mixed problem (3.9) after proving that the initial condition is satisfied. First of all, we have to remark that, since $w \in L^2([0,A],H^3(\mathbf{R}^{d-1}\times \mathbf{R}))$, for all $x_d \in [0,A]$, the trace of $w(\cdot,x_d,\cdot)$ on $\mathbf{R}^{d-1} \times \{0\}$ turns out to belong to the space $H^2(\mathbf{R}^{d-1})$. Thus, $w(\cdot,0) \in L^2([0,A],H^2(\mathbf{R}^{d-1}))$. Consider a function $\phi \in \mathcal{G}_A$, such that $\operatorname{supp}\|\phi\| \subset (\mathbf{R}^{d-1}\times\,]0,A[\, \times [0,+\infty[)$. Hence, $\int_0^A \langle J,\phi\rangle_{H^2,H^{-2}} dx_d = \int_0^A \langle w,Q^\star[\phi]\rangle_{H^2,H^{-2}} dx_d$.

By integrating by parts, we have

$$\int_0^A \int_{\mathbf{R}^{d-1}} \int_0^\infty \langle Q[w],\phi\rangle dt\, dy\, dx_d - \int_0^A \int_{\mathbf{R}^{d-1}} \lambda\langle w(0),\partial_t\phi(0)\rangle dy\, dx_d$$
$$+ \int_0^A \int_{\mathbf{R}^{d-1}} \lambda\langle \partial_t w(0),\phi(0)\rangle dy\, dx_d = \int_0^A \int_{\mathbf{R}^{d-1}} \int_0^\infty \langle J,\phi\rangle dt\, dy\, dx_d, \tag{3.13}$$

whence

$$\int_0^A \int_{\mathbf{R}^{d-1}} \lambda\langle w(0),\partial_t\phi(0)\rangle dx_d\, dy - \int_0^A \int_{\mathbf{R}^{d-1}} \lambda\langle \partial_t w(0),\phi(0)\rangle dx_d\, dy = 0. \tag{3.14}$$

By means of a suitable choice of the function ϕ, we prove that $w(x,0) = 0$, a.e. in $\mathbf{R}^{d-1}\times\,]0,A[$. Therefore, the function w turns out to be a solution to (3.9). □

Finally, both the solution v to the auxiliary problem (3.1) and the solution w to (3.9) belong to the space $L^2([0,T] \times [0,A],H^3(\mathbf{R}^{d-1}))$. Denote by u the sum $u = w + v$. The function $u \in L^2([0,T] \times [0,A],H^3(\mathbf{R}^{d-1}))$, and, due to the previous results, u has second-order partial derivatives with respect to t and x_d, which belong to $L^2_{\mathrm{loc}}(\mathbf{R}^{d-1}\times]0,A[\times]0,T[)$. Hence, u turns out to be a solution to the initial boundary value problem (2.3). To avoid inconsistencies in the auxiliary mixed problems (3.1) and (3.9) as well as in (1.3), we have to require that the data g and h satisfy compatibility conditions: if g and h are smooth functions up to the boundary, we assume that, for every $y \in \mathbf{R}^{d-1}$, $h(y,0) = g(y,0) = 0$.

Let us state now the main result.

Theorem 3.4. *Consider the initial boundary value problem* (1.3). *Suppose that the hypotheses of Propositions 3.2 and 3.3 are satisfied. If* $J \in L^2([0,A], H^2(\mathbf{R}^{d-1} \times \mathbf{R}))$, $g \in H^3(\mathbf{R}^{d-1} \times \mathbf{R})$, *and* $h \in H^3(\mathbf{R}^d)$, *then there exists a function* $u \in L^2([0,T]\times[0,A], H^3(\mathbf{R}^{d-1}))$, *which provides a solution to* (1.3).

References

[1] R. Cavazzoni, "Initial value problem for a constant coefficient second order system," submitted.