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Research Article

Nodal Solutions for a Class of Fourth-Order Two-Point Boundary Value Problems

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We consider the fourth-order two-point boundary value problem $u'''' + Mu = \lambda h(t) f(u)$, $0 < t < 1$, $u(0) = u(1) = u'(0) = u'(1) = 0$, where $\lambda \in \mathbb{R}$ is a parameter, $M \in (-\pi^4, \pi^4/64)$ is given constant, $h \in C([0,1],[0,\infty))$ with $h(t) \not\equiv 0$ on any subinterval of $[0,1]$, $f \in C(\mathbb{R},\mathbb{R})$ satisfies $f(u)u > 0$ for all $u \neq 0$, and $\lim_{u\to-\infty} f(u)/u = 0$, $\lim_{u\to+\infty} f(u)/u = f_{+\infty}$, $\lim_{u\to 0} f(u)/u = f_0$ for some $f_{+\infty}, f_0 \in (0,+\infty)$. By using disconjugate operator theory and bifurcation techniques, we establish existence and multiplicity results of nodal solutions for the above problem.

1. Introduction

The deformations of an elastic beam in equilibrium state with fixed both endpoints can be described by the fourth-order ordinary differential equation boundary value problem

$$\begin{gathered} u'''' = \lambda h(t) f(u), \quad 0 < t < 1, \\ u(0) = u(1) = u'(0) = u'(1) = 0, \end{gathered} \tag{1.1}$$

where $f : \mathbb{R} \to \mathbb{R}$ is continuous, $\lambda \in \mathbb{R}$ is a parameter. Since the problem (1.1) cannot transform into a system of second-order equation, the treatment method of second-order system does not apply to the problem (1.1). Thus, existing literature on the problem (1.1) is limited. In 1984, Agarwal and chow [1] firstly investigated the existence of the solutions of the problem (1.1) by contraction mapping and iterative methods, subsequently, Ma and Wu [2] and Yao [3, 4] studied the existence of positive solutions of this problem by the Krasnosel'skii fixed point theorem on cones and Leray-Schauder fixed point theorem. Especially, when

$h(t) \equiv 0$, Korman [5] investigated the uniqueness of positive solutions of the problem (1.1) by techniques of bifurcation theory. However, the existence of sign-changing solution for this problem have not been discussed.

In this paper, applying disconjugate operator theory and bifurcation techniques, we consider the existence of nodal solution of more general the problem:

$$\begin{gathered} u'''' + Mu = \lambda h(t) f(u), \quad 0 < \mathrm{t} < 1, \\ u(0) = u(1) = u'(0) = u'(1) = 0, \end{gathered} \tag{1.2}$$

under the assumptions:

(H1) $\lambda \in \mathbb{R}$ is a parameter, $M \in (-\pi^4, \pi^4/64)$ is given constant,

(H2) $h \in C([0,1],[0,\infty))$ with $h(t) \not\equiv 0$ on any subinterval of $[0,1]$,

(H3) $f \in C(\mathbb{R},\mathbb{R})$ satisfies $f(u)u > 0$ for all $u \neq 0$, and

$$\lim_{u \to -\infty} \frac{f(u)}{u} = 0, \qquad \lim_{u \to +\infty} \frac{f(u)}{u} = f_{+\infty}, \qquad \lim_{u \to 0} \frac{f(u)}{u} = f_0 \tag{1.3}$$

for some $f_{+\infty}, f_0 \in (0,\infty)$.

However, in order to use bifurcation technique to study the nodal solutions of the problem (1.2), we firstly need to prove that the generalized eigenvalue problem

$$\begin{gathered} u'''' + Mu = \mu \widetilde{h}(t) u, \quad 0 < t < 1, \\ u(0) = u(1) = u'(0) = u'(1) = 0 \end{gathered} \tag{1.4}$$

(where $\widetilde{h}$ satisfies (H2)) has an infinite number of positive eigenvalues

$$\mu_1 < \mu_2 < \cdots < \mu_k < \mu_{k+1} < \cdots, \tag{1.5}$$

and each eigenvalue corresponding an essential unique eigenfunction ψ_k which has exactly $k-1$ simple zeros in $(0,1)$ and is positive near 0. Fortunately, Elias [6] developed a theory on the eigenvalue problem

$$\begin{gathered} \mathcal{L}y + \lambda \widehat{h}(t) y = 0, \\ (\mathcal{L}_i y)(a) = 0, \quad i \in \{i_1, \dots, i_k\}, \\ (\mathcal{L}_j y)(b) = 0, \quad j \in \{j_1, \dots, j_{n-k}\}, \end{gathered} \tag{1.6}$$

where

$$\begin{gathered} \mathcal{L}_0 y = \rho_0 y, \\ \mathcal{L}_i y = \rho_i (\mathcal{L}_{i-1} y)', \quad i = 1, \dots, n, \\ \mathcal{L} y = \mathcal{L}_n y, \end{gathered} \tag{1.7}$$

and $\rho_i \in C^{n-i}[a,b]$ with $\rho_i > 0$ $(i = 0,1,\dots,n)$ on $[a,b]$. $\mathcal{L}_0 y,\dots,\mathcal{L}_{n-1}y$ are called the *quasi-derivatives* of $y(t)$. To apply Elias's theory, we have to prove that (1.4) can be rewritten to the form of (1.6), that is, the linear operator

$$L[u] := u'''' + Mu \tag{1.8}$$

has a factorization of the form

$$L[u] = l_4\Big(l_3\Big(l_2\big(l_1(l_0u)'\big)'\Big)'\Big)' \tag{1.9}$$

on $[0,1]$, where $l_i \in C^{4-i}[0,1]$ with $l_i > 0$ on $[0,1]$, and $u(0) = u(1) = u'(0) = u'(1) = 0$ if and only if

$$(l_0u)(0) = (l_0u)(1) = (l_1u)(0) = (l_1u)(1) = 0. \tag{1.10}$$

This can be achieved under (H1) by using *disconjugacy theory* in [7].

The rest of paper is arranged as follows: in Section 2, we state some disconjugacy theory which can be used in this paper, and then show that (H1) implies the equation

$$L[u] = 0 \tag{1.11}$$

is disconjugacy on $[0,1]$, moreover, we establish some preliminary properties on the eigenvalues and eigenfunctions of the generalized eigenvalue problem (1.4). Finally in Section 3, we state and prove our main result.

Remark 1.1. For other results on the existence and multiplicity of positive solutions and nodal solutions for boundary value problems of ordinary differential equations based on bifurcation techniques, see Ma [8–12], An and Ma [13], Yang [14] and their references.

2. Preliminary Results

Let

$$\mathbf{L}[y] = y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_n(x)y = 0 \tag{2.1}$$

be nth-order linear differential equation whose coefficients $p_k(\cdot)$ $(k = 1,\dots,n)$ are continuous on an interval I.

Definition 2.1 (see [7, Definition 0.2, page 2]). Equation (2.1) is said to be disconjugate on an interval I if no nontrivial solution has n zeros on I, multiple zeros being counted according to their multiplicity.

Lemma 2.2 (see [7, Theorem 0.7, page 3]). *Equation* (2.1) *is disconjugate on a compact interval I if and only if there exists a basis of solutions $y_0, \dots, y_{n-1}$ such that*

$$\mathcal{W}_k := \mathcal{W}_k(y_0, \dots, y_{k-1}) = \begin{vmatrix} y_0 & \cdots & y_{k-1} \\ \vdots & & \vdots \\ y_0^{(k-1)} & \cdots & y_{k-1}^{(k-1)} \end{vmatrix} > 0 \quad (k = 1, \dots, n) \tag{2.2}$$

on I. A disconjugate operator $\mathbf{L}[y] = y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_n(x)y$ can be written as

$$\mathbf{L}[y] \equiv \rho_n D\big(\rho_{(n-1)} \cdots D\big(\rho_1 D(\rho_0 y)\big) \cdots\big), \quad D \equiv \frac{d}{dx}, \tag{2.3}$$

where $\rho_0 \in C^{n-k}(I)$ $(k = 0, 1, \dots, n)$ and

$$\rho_0 = \frac{1}{\mathcal{W}_1}, \qquad \rho_1 = \frac{\mathcal{W}_1^2}{\mathcal{W}_2}, \qquad \rho_k = \frac{\mathcal{W}_k^2}{\mathcal{W}_{k-1} \cdot \mathcal{W}_{k+1}}, \quad k = 2, \dots, n-1, \tag{2.4}$$

and $\rho_0\rho_1 \cdots \rho_n \equiv 1$.

Lemma 2.3 (see [7, Theorem 0.13, page 9]). *Green's function $G(x, \delta)$ of the disconjugate Equation* (2.3) *and the two-point boundary value conditions*

$$\begin{aligned} y^{(i)}(a) &= 0, \quad i = 0, \dots, k-1, \\ y^{(i)}(b) &= 0, \quad i = 0, \dots, n-k-1 \end{aligned} \tag{2.5}$$

satisfies

$$(-1)^{n-k} G(x, \delta) > 0, \quad \forall (x, \delta) \in (a, b) \times (a, b). \tag{2.6}$$

Now using Lemmas 2.2 and 2.3, we will prove some preliminary results.

Theorem 2.4. *Let (H1) hold. Then*

(i) $L[u] = 0$ *is disconjugate on* $[0, 1]$, *and* $L[u]$ *has a factorization*

$$L[u] = \rho_4\Big(\rho_3\Big(\rho_2\big(\rho_1(\rho_0 u)'\big)'\Big)'\Big)', \tag{2.7}$$

where $\rho_k \in C^{4-k}[0, 1]$ with $\rho_k > 0$ $(k = 0, 1, 2, 3, 4)$.

(ii) $u(0) = u(1) = u'(0) = u'(1) = 0$ *if and only if*

$$(L_0 u)(0) = (L_1 u)(0) = (L_0 u)(1) = (L_1 u)(1) = 0, \tag{2.8}$$

where

$$L_0 u = \rho_0 u, \qquad L_i u = \rho_i (L_{i-1} u)', \quad i = 1,2,3,4. \tag{2.9}$$

Proof. We divide the proof into three cases.

Case 1. $M = 0$. The case is obvious.

Case 2. $M \in (-\pi^4, 0)$.
In the case, take

$$u_0(t) = e^{-mt}, \qquad u_1(t) = e^{mt}, \qquad u_2(t) = -\sin m(t+\sigma), \qquad u_3(t) = \cos m(t+\sigma), \tag{2.10}$$

where $m = \sqrt[4]{-M}$, σ is a positive constant. Clearly, $m \in (0, \pi)$ and then

$$\sin m(t+\sigma) > 0, \quad t \in [0,1]. \tag{2.11}$$

It is easy to check that $u_0(t)$, $u_1(t)$, $u_2(t)$, $u_3(t)$ form a basis of solutions of $L[u] = 0$. By simple computation, we have

$$\mathcal{W}_1 = e^{-mt}, \qquad \mathcal{W}_2 = 2m, \qquad \mathcal{W}_3 = 4m^3 \sin m(t+\sigma), \qquad \mathcal{W}_4 = 8m^6. \tag{2.12}$$

Clearly, $\mathcal{W}_k > 0$, $(k = 1,2,3,4)$ on $[0,1]$.

By Lemma 2.2, $L[u] = 0$ is disconjugate on $[0,1]$, and $L[u]$ has a factorization

$$u'''' + Mu = \frac{2m^3}{\sin m(t+\sigma)} \left(\frac{\sin^2 m(t+\sigma)}{m} \left(\frac{e^{mt}}{m \sin m(t+\sigma)} \left(\frac{1}{2me^{2mt}} \left(e^{mt} u\right)' \right)' \right)' \right)', \tag{2.13}$$

and accordingly

$$L_0 u = \rho_0 u = e^{mt} u, \qquad L_1 u = \rho_1 (L_0 u)' = \frac{mu + u'}{2me^{mt}}. \tag{2.14}$$

Using (2.14), we conclude that $u(0) = u(1) = u'(0) = u'(1) = 0$ is equivalent to (2.8).

Case 3. $M \in (0, \pi^4/64)$.
In the case, take

$$u_0(t) = e^{-mt} \cos mt, \qquad u_1(t) = e^{-mt} \sin mt, \qquad u_2(t) = e^{mt} \cos mt, \qquad u_3(t) = e^{mt} \sin mt, \tag{2.15}$$

where $m = (\sqrt{2}/2)\sqrt[4]{M}$.

It is easy to check that $u_0(t)$, $u_1(t)$, $u_2(t)$, $u_3(t)$ form a basis of solutions of $L[u] = 0$. By simple computation, we have

$$\mathcal{W}_1 = \frac{\cos mt}{e^{mt}}, \qquad \mathcal{W}_2 = \frac{m}{e^{2mt}}, \qquad \mathcal{W}_3 = \frac{4a^3(\cos mt - \sin mt)}{e^{mt}}, \qquad \mathcal{W}_4 = 32m^6. \tag{2.16}$$

From $M \in (0, \pi^4/64)$ and $m = (\sqrt{2}/2)\sqrt[4]{M}$, we have $0 < m < \pi/4$, so $\mathcal{W}_k > 0, (k = 1,2,3,4)$ on $[0,1]$.

By Lemma 2.2, $L[u] = 0$ is disconjugate on $[0,1]$, and $L[u]$ has a factorization

$$\begin{aligned} &u'''' + Mu \\ &= \frac{8m^3 e^{mt}}{\cos mt - \sin mt} \\ &\times \left(\frac{(\cos mt - \sin mt)^2}{2m} \left(\frac{1}{4me^{2mt} \cos mt(\cos mt - \sin mt)} \left(\frac{\cos^2 mt}{m} \left(\frac{e^{mt}}{\cos mt} u \right)' \right)' \right)' \right)', \end{aligned} \tag{2.17}$$

and accordingly

$$\begin{aligned} L_0 u = \rho_0 u = \frac{e^{mt}}{\cos mt} u, \\ L_1 u = \rho_1 (L_0 u)' = e^{mt}(\cos mt + \sin mt)u + \frac{e^{mt}\cos mt}{m} u'. \end{aligned} \tag{2.18}$$

Using (2.18), we conclude that $u(0) = u(1) = u'(0) = u'(1) = 0$ is equivalent to (2.8).

This completes the proof of the theorem. □

Theorem 2.5. *Let (H1) hold and $\widetilde{h}$ satisfy (H2). Then*

(i) *Equation* (1.4) *has an infinite number of positive eigenvalues*

$$\mu_1 < \mu_2 < \cdots < \mu_k < \mu_{k+1} < \cdots . \tag{2.19}$$

(ii) $\mu_k \to \infty$ *as* $k \to \infty$.

(iii) *To each eigenvalue there corresponding an essential unique eigenfunction* φ_k *which has exactly* $k-1$ *simple zeros in* $(0,1)$ *and is positive near 0.*

(iv) *Given an arbitrary subinterval of* $[0,1]$*, then an eigenfunction which belongs to a sufficiently large eigenvalue change its sign in that subinterval.*

(v) *For each* $k \in \mathbb{N}$*, the algebraic multiplicity of* μ_k *is 1.*

Proof. (i)–(iv) are immediate consequences of Elias [6, Theorems 1–5] and Theorem 2.4. we only prove (v).

Let

$$\widehat{L}u := u'''' + Mu, \quad u \in D\left(\widehat{L}\right) \tag{2.20}$$

with

$$D\left(\widehat{L}\right) := \left\{u \in C^4[0,1] \mid u(0) = u(1) = u'(0) = u'(1) = 0\right\}. \tag{2.21}$$

To show (v), it is enough to prove

$$\ker\left(\widehat{L} - \mu_k h(\cdot)\right)^2 = \ker\left(\widehat{L} - \mu_k h(\cdot)\right). \tag{2.22}$$

Clearly

$$\ker\left(\widehat{L} - \mu_k h(\cdot)\right)^2 \supseteq \ker\left(\widehat{L} - \mu_k h(\cdot)\right). \tag{2.23}$$

Suppose on the contrary that the algebraic multiplicity of μ_k is greater than 1. Then there exists $u \in \ker(\widehat{L} - \mu_k h(\cdot))^2 \setminus \ker(\widehat{L} - \mu_k h(\cdot))$, and subsequently

$$\widehat{L}u - \mu_k h(x)u = q\psi_k \tag{2.24}$$

for some $q \neq 0$. Multiplying both sides of (2.24) by $\psi_k(x)$ and integrating from 0 to 1, we deduce that

$$0 = q\int_0^1 [\psi_k(x)]^2 dx, \tag{2.25}$$

which is a contradiction! □

Theorem 2.6 (Maximum principle). *Let (H1) hold. Let* $e \in C[0,1]$ *with* $e \geq 0$ *on* $[0,1]$ *and* $e \not\equiv 0$ *in* $[0,1]$. *If* $u \in C^4\ [0,1]$ *satisfies*

$$\begin{gathered} u'''' + Mu = e(t), \\ u(0) = u(1) = u'(0) = u'(1) = 0. \end{gathered} \tag{2.26}$$

Then $u > 0$ *on* $(0,1)$.

Proof. When $M \in (-\pi^4, \pi^4/64)$, the homogeneous problem

$$\begin{gathered} u'''' + Mu = 0, \\ u(0) = u(1) = u'(0) = u'(1) = 0 \end{gathered} \tag{2.27}$$

has only trivial solution. So the boundary value problem (2.26) has a unique solution which may be represented in the form

$$u(t) = \int_0^1 G(t,s)e(s)ds, \tag{2.28}$$

where $G(t,s)$ is Green's function.

By Theorem 2.4 and Lemma 2.3 (take $n = 4,\ k = 2$), we have

$$(-1)^{4-2}G(t,s) > 0, \quad \forall (t,s) \in (0,1) \times (0,1), \tag{2.29}$$

that is, $G(t,s) > 0,$ for all $(t,s) \in (0,1) \times (0,1)$.

Using (2.28), when $e \geq 0$ on $[0,1]$ with $e \not\equiv 0$ in $[0,1]$, then $u > 0$ on $(0,1)$. □

3. Statement of the Results

Theorem 3.1. *Let (H1), (H2), and (H3) hold. Assume that for some* $k \in \mathbb{N}$*,*

$$\lambda > \frac{\mu_k}{f_0}. \tag{3.1}$$

Then there are at least $2k - 1$ *nontrivial solutions of the problem* (1.2)*. In fact, there exist solutions* $w_1, \ldots, w_k$*, such that for* $1 \leq j \leq k$*,* w_j *has exactly* $j-1$ *simple zeros on the open interval* $(0,1)$ *and* $w_j''(0) < 0$ *and there exist solutions* $z_2, \ldots, z_k$*, such that for* $2 \leq j \leq k$*,* z_j *has exactly* $j-1$ *simple zeros on the open interval* $(0,1)$ *and* $z_j''(0) > 0$*.*

Let $Y = C[0,1]$ *with the norm* $\|u\|_\infty = \max_{t\in[0,1]}|u(t)|$*. Let*

$$E = \left\{u \in C^2[0,1] \mid u(0) = u(1) = u'(0) = u'(1) = 0\right\} \tag{3.2}$$

with the norm $\|u\|_E = \max\{\|u\|_\infty, \|u'\|_\infty, \|u''\|_\infty\}$*. Then* $\widehat{L}^{-1} : Y \to E$ *is completely continuous, here* $\widehat{L}$ *is given as in* (2.20)*.*

Let $\zeta, \xi \in C(\mathbb{R}, \mathbb{R})$ *be such that*

$$f(u) = f_0 u + \zeta(u), \qquad f(u) = f_{+\infty}u^+ + \xi(u), \tag{3.3}$$

here $u^+ = \max\{u, 0\}$*. Clearly*

$$\lim_{|u|\to 0}\frac{\zeta(u)}{u} = 0, \qquad \lim_{|u|\to\infty}\frac{\xi(u)}{u} = 0. \tag{3.4}$$

Let

$$\widetilde{\xi}(u) = \max_{0\leq|s|\leq u}|\xi(s)|, \tag{3.5}$$

then $\widetilde{\xi}$ *is nondecreasing and*

$$\lim_{u\to\infty}\frac{\widetilde{\xi}(u)}{u}=0. \tag{3.6}$$

Let us consider

$$\widehat{L}u=\lambda h(x)f_0u+\lambda h(x)\zeta(u) \tag{3.7}$$

as a bifurcation problem from the trivial solution $u\equiv 0$.

Equation (3.7) *can be converted to the equivalent equation*

$$u(x)=\lambda\widehat{L}^{-1}\left[h(\cdot)f_0u(\cdot)\right](x)+\lambda\widehat{L}^{-1}[h(\cdot)\zeta(u(\cdot))](x). \tag{3.8}$$

Further we note that $\|\widehat{L}^{-1}[h(\cdot)\zeta(u(\cdot))]\|_E=o(\|u\|_E)$ *for* u *near 0 in* E.

In what follows, we use the terminology of Rabinowitz [15].

Let $\mathbf{E}=\mathbb{R}\times E$ *under the product topology. Let* S_k^+ *denote the set of function in* E *which have exactly* $k-1$ *interior nodal (i.e., nondegenerate) zeros in* $(0,1)$ *and are positive near* $t=0$, *set* $S_k^-=-S_k^+$, *and* $S_k=S_k^+\cup S_k^-$. *They are disjoint and open in* E. *Finally, let* $\Phi_k^\pm=\mathbb{R}\times S_k^\pm$ *and* $\Phi_k=\mathbb{R}\times S_k$.

The results of Rabinowitz [13] for (3.8) *can be stated as follows: for each integer* $k\geq 1$, $\nu=\{+,-\}$, *there exists a continuum* $\mathcal{C}_k^\nu\subseteq\Phi_k^\nu$ *of solutions of* (3.8), *joining* $(\mu_k/f_0,0)$ *to infinity in* Φ_k^ν. *Moreover,* $\mathcal{C}_k^\nu\setminus(\mu_k/f_0,0)\subset\Phi_k^\nu$.

Notice that we have used the fact that if u *is a nontrivial solution of* (3.7), *then all zeros of* u *on* $(0,1)$ *are simply under (H1), (H2), and (H3).*

In fact, (3.7) *can be rewritten to*

$$\widehat{L}u=\lambda\widehat{h}(t)u, \tag{3.9}$$

where

$$\widehat{h}(t)=\begin{cases}h(t)\dfrac{f(u(t))}{u(t)}, & u(t)\neq 0,\\ h(t)f_0, & u(t)=0,\end{cases} \tag{3.10}$$

clearly $\widehat{h}(t)$ *satisfies (H2). So Theorem 2.5(iii) yields that all zeros of* u *on* $(0,1)$ *are simple.*

Proof of Theorem 3.1. We only need to show that

$$\begin{gathered}\mathcal{C}_j^-\cap(\{\lambda\times E\})\neq\emptyset,\quad j=1,2,\ldots,k,\\ \mathcal{C}_j^+\cap(\{\lambda\times E\})\neq\emptyset,\quad j=2,\ldots,k.\end{gathered} \tag{3.11}$$

Suppose on the contrary that

$$\mathcal{C}_i^{\iota} \cap (\{\lambda \times E\}) = \emptyset, \quad \text{for some } (i,\iota) \in \Gamma, \tag{3.12}$$

where

$$\Gamma := \{(j,\nu) \mid j \in \{2,\dots,k\} \text{ as } \nu = +,\ j \in \{1,2,\dots,k\} \text{ as } \nu = -\}. \tag{3.13}$$

Since $\mathcal{C}_i^{\iota}$ joins $(\eta_i/f_0, 0)$ to infinity in Φ_i^{ν} and $(\lambda, u) = (0,0)$ is the unique solutions of $(3.7)_{\lambda=0}$ in E, there exists a sequence $\{(\chi_m, u_m)\} \subset \mathcal{C}_i^{\iota}$ such that $\chi_m \in (0,\lambda)$ and $\|u_m\|_E \to \infty$ as $m \to \infty$. We may assume that $\chi_m \to \overline{\chi} \in [0,\lambda]$ as $m \to \infty$. Let $v_m = u_m/\|u_m\|_E$, $m \ge 1$. From the fact

$$\widehat{L}u_m(x) = \chi_m\left[h(x)f_{+\infty}\right](u_m)^{+}(x) + \chi_m h(x)\xi(u_m)(x), \tag{3.14}$$

we have that

$$v_m(x) = \chi_m \widehat{L}^{-1}\left(\left[h(\cdot)f_{+\infty}\right](v_m)^{+}\right)(x) + \chi_m \widehat{L}^{-1}\left[h(\cdot)\frac{\xi(u_m)(x)}{\|u_m\|_E}\right](x). \tag{3.15}$$

Furthermore, since $\widehat{L}^{-1}|_E : E \to E$ is completely continuous, we may assume that there exist $v \in E$ with $\|v\|_E = 1$ such that $\|v_m - v\|_E \to 0$ as $m \to \infty$.

Since

$$\frac{|\xi(u_m)|}{\|u\|_E} \le \frac{\xi(\|u_m\|_\infty)}{\|u\|_E} \le \frac{\xi(\|u_m\|_E)}{\|u\|_E}, \tag{3.16}$$

we have from (3.15) and (3.6) that

$$v = \overline{\chi}\widehat{L}^{-1}\left[h(\cdot)f_{+\infty}\right]v^{+}, \tag{3.17}$$

that is,

$$\begin{gathered} v'''' + Mv = \overline{\chi}h(x)f_{+\infty}v^{+}, \\ v(0) = v(1) = v''(0) = v''(1) = 0. \end{gathered} \tag{3.18}$$

By (H2), (H3), and (3.17) and the fact that $\|v\|_E = 1$, we conclude that $\overline{\chi}h(x)f_{+\infty}v^{+} \not\equiv 0$, and consequently

$$\overline{\chi} > 0, \qquad v^{+} \not\equiv 0. \tag{3.19}$$

By Theorem 2.6, we know that $v(x) > 0$ in $(0,1)$. This means $\overline{\chi}f_{+\infty}$ is the first eigenvalue of $\widehat{L}u = \eta h(t)u$ and v is the corresponding eigenfunction. Hence $v \in S_1^{+}$. Since S_1^{+} is open and $\|v_m - v\|_E \to 0$, we have that $v_m \in S_1^{+}$ for m large. But this contradict the assumption that $(\chi_m, v_m) \in C_i^{\iota}$ and $(i,\iota) \in \Gamma$, so (3.12) is wrong, which completes the proof. □

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