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Research Article

Existence and Nonexistence of Positive Solutions for Singular p-Laplacian Equation in $\mathbb{R}^N$

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We study the existence and nonexistence of solutions for the singular quasilinear problem $-\mathrm{div}(|x|^{-ap}|\nabla u|^{p-2}\nabla u) = h(x)f(u)+\lambda H(x)g(u)$, $x \in \mathbb{R}^N$, $u(x) > 0$, $x \in \mathbb{R}^N$, $\lim_{|x|\to\infty} u(x) = 0$, where $1 < p < N$, $0 \le a < (N-p)/p$ and $f(u)$ and $g(u)$ behave like u^m and u^n with $0 < m \le p-1 < n$ at the origin. We obtain the existence by the upper and lower solution method and the nonexistence by the test function method.

1. Introduction

In this paper, we study through the upper and lower solution method and the test function method the existence and nonexistence of solution to the singular quasilinear elliptic problem

$$\begin{gathered} -\mathrm{div}\left(|x|^{-ap}|\nabla u|^{p-2}\nabla u\right) = h(x)f(u) + \lambda H(x)g(u), \quad x \in \mathbb{R}^N, \\ u(x) > 0, \quad x \in \mathbb{R}^N, \quad \lim_{|x|\to\infty} u(x) = 0 \end{gathered} \tag{1.1}$$

with $1 < p < N$, $0 \le a < (N-p)/p$, $\lambda \ge 0$. $h(x), H(x) : \mathbb{R}^N \to (0,\infty)$ are the locally Hölder continuous functions, not identically zero and $f(u)$ and $g(u)$ are locally Lipschitz continuous functions.

The study of this type of equation in (1.1) is motivated by its various applications, for instance, in fluid mechanics, in Newtonian fluids, in flow through porous media, and in glaciology; see [1]. The equation in (1.1) involves singularities not only in the nonlinearities but also in the differential operator.

Many authors studied this kind of problem for the case $a = 0$; see [2–7]. In these works, the nonlinearities have sublinear and suplinear growth at infinity, and they behave like a function u^k ($k < p-1$, or $k \geq p-1$) at the origin. Roughly speaking, in this case we say that the nonlinearities are concave and convex or "slow diffusion and fast diffusion"; see [8].

When $a = 0$, $f(u) = u^m$, and $g(u) = u^n$, $m < p-1 < n$, by using the lower and upper solution method, Santos in [5] finds a real number $\lambda_0 > 0$, such that the problem (1.1) has at least one solution if $0 \leq \lambda < \lambda_0$.

For $a, \lambda \neq 0$, the existence and multiplicity of solution of singular elliptic equation like (1.1) in a bounded domain Ω with the zero Dirichlet data have been widely studied by many authors, for example, the authors [9–13] and references therein. Assunção et al. in [14] studied the multiplicity of solution for the singular equations in (1.1) with $h(x) = \alpha|x|^{-bm}$, $H(x) = \beta|x|^{-dq}$, $f(u) = |u|^{m-2}u$, and $g(u) = |u|^{q-2}u$ in $\mathbb{R}^N$. Similar consideration can be found in [15–20] and references therein. We note that the variation method is widely used in the above references.

Recently, Chen et al. in [21, 22], by using a variational approach, got some existence of solution for (1.1) with $\lambda = 0$ and $f(u) = u^q$, $q > p-1$. For the case $q < p-1$, $\lambda \geq 0$, the problem for the existence of solution for (1.1) is still open. It seems difficult to consider the case $q < p-1$ by variational method.

The main aim of this work is to study the existence and nonexistence of solution for (1.1), where $f(u)$ is sublinear and $g(u)$ is suplinear. We will use the upper and lower solution method. To the best of our knowledge, there is little information on upper and lower solution method for the problem (1.1). So it is necessary to establish this technique in unbounded domain. To obtain the existence, the assumption $M_\infty < \infty$ (see (2.17) below) is essential. By this, an upper solution for (1.1) is obtained.

We also obtain a sufficient condition on $h(x)$, $H(x)$ to guarantee the nonexistence of nontrivial solution for the problem (2.21). (see Theorem 2.5 below). It must be particularly pointed out that our primary interest is in the mixed case in which $0 < m \leq p-1 < n$ with $H(x)$ satisfying

$$H_\infty = \int_0^\infty \left(s^{1-N+ap} \int_0^s t^{N-1} \overline{H}(t)\, dt \right)^{1/(p-1)} ds < \infty, \quad \overline{H}(t) = \max_{|x|=t} H(x), \tag{1.2}$$

while $h(x)$ satisfies

$$h_\infty = \int_0^\infty \left(s^{1-N+ap} \int_0^s t^{N-1} \overline{h}(t)\, dt \right)^{1/(p-1)} ds = \infty, \quad \overline{h}(t) = \max_{|x|=t} h(x). \tag{1.3}$$

This paper is organized as follows. In Section 2, we state the main results and present some preliminaries which will be used in what follows. We also introduce the precise hypotheses under which our problem is studied. In Section 3, we give the proof of some lemmas and the existence. The proof of nonexistence is given in Section 4.

2. Preliminaries and Main Results

Let us now introduce some weighted Sobolev spaces and their norms. Let Ω be a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$. If $r \in \mathbb{R}^1$ and $p \geq 1$, we define $L^p(\Omega, |x|^{-r})$ as being the subspace of $L^p(\Omega)$ of the Lebesgue measurable function $u : \Omega \to \mathbb{R}^1$, satisfying

$$\|u\|_{p,r} := \|u\|_{L^p(\Omega,|x|^{-r})} = \left(\int_\Omega |x|^{-r}|u|^p dx\right)^{1/p} < \infty. \tag{2.1}$$

If $1 < p < N$ and $-\infty < a < (N-p)/p$, we define $W^{1,p}(\Omega, |x|^{-ap})$ (resp., $W_0^{1,p}(\Omega, |x|^{-ap})$ as being the closure of $C^\infty(\Omega)$ (resp., $C_0^\infty(\Omega)$) with respect to the norm defined by

$$\|u\| := \left(\int_\Omega |x|^{-ap}|\nabla u|^p dx\right)^{1/p}. \tag{2.2}$$

For the weighted Sobolev space $W^{1,p}(\Omega, |x|^{-ap})$, we have the following compact imbedding theorem which is an extension of the classical Rellich-Kondrachov compact theorem.

Theorem 2.1 ((compact imbedding theorem) [13]). *Suppose that $\Omega \subset \mathbb{R}^N$ is an open bounded domain with C^1 boundary and $0 \in \Omega$, $1 < p < N$, $-\infty < a < (N-p)/p$. Then, the imbedding $W^{1,p}(\Omega, |x|^{-ap}) \hookrightarrow L^q(\Omega, |x|^{-r})$ is compact if $1 \leq q < Np/(N-p)$, $r < (1+a)q + N(1-q/p)$.*

We now consider the existence of positive solutions for problem (1.1). Our main tool will be the upper and lower solution method. This method, in the bounded domain situation, has been used by many authors, for instance, [10, 12, 13]. But for the unbounded domain, we need to establish this method and then to construct an upper solution and a lower solution for (1.1). We now give the definitions of upper and lower solutions.

Definition 2.2 (see [10, 12]). A function $\underline{u} \in W^{1,p}(\mathbb{R}^N, |x|^{-ap}) \cap L^\infty(\mathbb{R}^N)$ is said to be a weak lower solution of the equation

$$-\operatorname{div}\left(|x|^{-ap}|\nabla u|^{p-2}\nabla u\right) = F(x,u), \quad x \in \mathbb{R}^N \tag{2.3}$$

if

$$-\operatorname{div}\left(|x|^{-ap}\left|\nabla \underline{u}\right|^{p-2}\nabla \underline{u}\right) \leq F(x,\underline{u}), \quad x \in \mathbb{R}^N \tag{2.4}$$

or

$$\int_{\mathbb{R}^N} |x|^{-ap}\left|\nabla \underline{u}\right|^{p-2}\nabla \underline{u}\nabla\phi dx \leq \int_{\mathbb{R}^N} F(x,\underline{u})\phi dx, \tag{2.5}$$

for any $\phi \in C_0^1(\mathbb{R}^N)$, $\phi \geq 0$.

Similarly, a function $\overline{u} \in W^{1,p}(\mathbb{R}^N, |x|^{-ap}) \cap L^\infty(\mathbb{R}^N)$ is said to be a weak upper solution of (2.3) if

$$-\operatorname{div}\left(|x|^{-ap}|\nabla \overline{u}|^{p-2}\nabla \overline{u}\right) \geq F(x, \overline{u}), \quad x \in \mathbb{R}^N \tag{2.6}$$

or

$$\int_{\mathbb{R}^N} |x|^{-ap}|\nabla \overline{u}|^{p-2}\nabla \overline{u}\nabla \phi dx \geq \int_{\mathbb{R}^N} F(x, \overline{u})\phi dx, \tag{2.7}$$

for any $\phi \in C_0^1(\mathbb{R}^N)$ and $\phi \geq 0$ in $\mathbb{R}^N$.

A function $u \in W^{1,p}(\mathbb{R}^N, |x|^{-ap}) \cap L^\infty(\mathbb{R}^N)$ is said to be a weak solution of (2.3) if and only if u is a weak lower solution and weak upper solution of (2.3).

A function $v \in W^{1,p}(\Omega, |x|^{-ap}) \cap L^\infty(\Omega)$ is said to be less than or equal to $w \in W^{1,p}(\Omega, |x|^{-ap}) \cap L^\infty(\Omega)$ on $\partial\Omega$ if $\max\{0, v - w\} \in W_0^{1,p}(\Omega, |x|^{-ap})$.

If $1 < p < N$ and $-\infty < a < (N-p)/p$, we define the weighted Sobolev space $W^{1,p}(\mathbb{R}^N, |x|^{-ap})$ as being the closure of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm $\|\cdot\|$ defined by

$$\|u\| = \left(\int_{\mathbb{R}^N} |x|^{-ap}|\nabla u|^p dx\right)^{1/p}. \tag{2.8}$$

The following lemma will be basic in our approach.

Lemma 2.3. *Let $F(x,u)$ be Lipschitz continuous and nondecreasing in u and locally Hölder continuous in x. Moreover, assume that there exist the functions $\underline{u}, \overline{u} \in W^{1,p}(\mathbb{R}^N, |x|^{-ap}) \cap L^\infty(\mathbb{R}^N)$ such that*

$$\begin{gathered}
-\operatorname{div}\left(|x|^{-ap}|\nabla \underline{u}|^{p-2}\nabla \underline{u}\right) \leq F(x, \underline{u}), \quad x \in \mathbb{R}^N, \\
-\operatorname{div}\left(|x|^{-ap}|\nabla \overline{u}|^{p-2}\nabla \overline{u}\right) \geq F(x, \overline{u}), \quad x \in \mathbb{R}^N, \\
\underline{u}(x) \leq \overline{u}(x), \quad \text{a.e. in } \mathbb{R}^N.
\end{gathered} \tag{2.9}$$

Then, there exist a minimal weak solution $V_0(x)$ and a maximal weak solution $U_0(x)$ of (2.3) *satisfying*

$$\underline{u}(x) \leq V_0(x) \leq U_0(x) \leq \overline{u}(x), \quad x \in \mathbb{R}^N \tag{2.10}$$

and $V_0(x), U_0(x) \in W^{1,p}(\mathbb{R}^N, |x|^{-ap}) \cap L^\infty(\mathbb{R}^N)$.

Proof. Denote $B_k = \{x \in \mathbb{R}^N \mid |x| < k\}$, $k = 1, 2, \ldots$. Let $\overline{u}, \underline{u}$ be a pair of upper and lower solutions of (2.3) with $\underline{u}(x) \le \overline{u}(x)$, a.e. in $\mathbb{R}^N$. We consider the boundary value problem

$$\begin{gathered} -\operatorname{div}\left(|x|^{-ap}|\nabla u|^{p-2}\nabla u\right) = F(x,u), \quad x \in B_k, \\ u(x) = \overline{u}(x), \quad x \in \partial B_k. \end{gathered} \tag{2.11}$$

By Theorem 1.1 in [10], one concludes that there exists $u_k(x) \in W^{1,p}(B_k, |x|^{-ap}) \cap L^\infty(B_k)$ which is a weak solution of (2.11) with $\underline{u}(x) \le u_k(x) \le \overline{u}(x)$ a.e. in B_k for $k = 1, 2, \ldots$.

We define its extension by

$$U_k(x) = \begin{cases} u_k(x), & x \in \overline{B}_k, \\ u_k(x) = \overline{u}(x), & x \in B_k^c = \mathbb{R}^N \setminus \overline{B}_k. \end{cases} \tag{2.12}$$

Similarly, let $v_k(x)$ be a weak solution of the boundary value problem

$$\begin{gathered} -\operatorname{div}\left(|x|^{-ap}|\nabla v_k|^{p-2}\nabla v_k\right) = F(x,v_k), \quad x \in B_k, \\ v_k(x) = \underline{u}(x), \quad x \in \partial B_k, \end{gathered} \tag{2.13}$$

and its extension is defined by

$$V_k(x) = \begin{cases} v_k(x), & x \in \overline{B}_k, \\ v_k(x) = \underline{u}(x), & x \in B_k^c. \end{cases} \tag{2.14}$$

Since $\underline{u}, \overline{u} \in W^{1,p}(\mathbb{R}^N, |x|^{-ap}) \cap L^\infty(\mathbb{R}^N)$, we have $V_k(x), U_k(x) \in W^{1,p}(\mathbb{R}^N, |x|^{-ap}) \cap L^\infty(\mathbb{R}^N)$. By Theorem 2.4 in [12], we have

$$\underline{u}(x) \le V_k(x) \le V_{k+1}(x) \le U_{k+1}(x) \le U_k(x) \le \overline{u}(x), \quad \text{a.e in } \mathbb{R}^N \tag{2.15}$$

for $k = 1, 2, \ldots$. In view of (2.15), the pointwise limits

$$V_0(x) = \lim_{k\to\infty} V_k(x), \qquad U_0(x) = \lim_{k\to\infty} U_k(x) \tag{2.16}$$

exist and $\underline{u}(x) \le V_0(x) \le U_0(x) \le \overline{u}(x)$ in $\mathbb{R}^N$.

Similar to the proof Theorem 1.1 in [10] and the proof of Theorem 7.5.1 in [23], it is not difficult to get from Theorem 2.1 that $U_0(x)$ is the maximal weak solution and $V_0(x)$ the minimal solution of (2.3), which satisfies (2.10) and $V_0, U_0 \in W^{1,p}(\mathbb{R}^N, |x|^{-ap}) \cap L^\infty(\mathbb{R}^N)$. This ends the proof of Lemma 2.3. $\square$

Our main results read as follows.

Theorem 2.4 (existence). *Let* $1 < p < N$, $0 \leq a < (N-p)/p$. *Assume the following.*

(A_1) *The nonnegative functions* $f(u), g(u)$ *are Lipschitz continuous and nondecreasing,* $f(0) = g(0) = 0$. *Additionally,* $\sup_{t\geq 0} t^{-m} f(t) < \infty$ *and* $\sup_{t\geq 0} t^{-n} g(t) < \infty$ *with* $0 < m < p-1 < n$.

(A_2) *The nonnegative functions* $h(x)$, $H(x)$ *are locally Hölder continuous. Let* $\overline{M}(r) = \max_{|x|=r}\{h(x), H(x)\}$. *If*

$$M_\infty = \int_0^\infty \left(s^{1-N+ap} \int_0^s t^{N-1} \overline{M}(t) dt \right)^{1/(p-1)} ds < \infty, \tag{2.17}$$

then there exists $\lambda_0 > 0$, *such that* $\lambda \in [0, \lambda_0)$, *and the problem* (1.1) *admits a weak solution* $u(x) \in W^{1,p}(\mathbb{R}^N, |x|^{-ap}) \cap L^\infty(\mathbb{R}^N)$.

Theorem 2.5 (nonexistence). *Let* $1 < p < N$, $0 \leq a < (N-p)/p$. *Assume that*

(A_3) $0 < m \leq p-1 < n$;

(A_4) *there exist* $\alpha_1, \alpha_2 \geq 0$ *such that*

$$q = \alpha_1 + \alpha_2 > p-1, \qquad \frac{\alpha_1}{n} + \frac{\alpha_2}{m} = 1; \tag{2.18}$$

(A_5) *the functions* $h(x), H(x) > 0$ *in* $\mathbb{R}^N$ *satisfy*

$$\limsup_{R\to\infty} (B_1(R))^{-\alpha_1(p-1)/nq} (b_1(R))^{-\alpha_2(p-1)/mq} R^{\sigma_1} < \infty, \tag{2.19}$$

where $\sigma_1 = N - p(1+a) - N(p-1)/q$ *and*

$$B_1(R) = \inf_{\Omega_R} H(x), \quad b_1(R) = \inf_{\Omega_R} h(x), \quad \Omega_R = \left\{ x \in \mathbb{R}^N \mid R \leq |x| \leq \sqrt{2}R \right\}, \quad R \geq 1. \tag{2.20}$$

Then the problem

$$\begin{gathered} -\operatorname{div}\left(|x|^{-ap} |\nabla u|^{p-2} \nabla u \right) = h(x) u^m + H(x) u^n, \quad x \in \mathbb{R}^N, \\ u(x) \geq 0, \quad x \in \mathbb{R}^N \end{gathered} \tag{2.21}$$

has no nontrivial solution $u(x) \in W^{1,p}(\mathbb{R}^N, |x|^{-ap})$.

Remark 2.6. If assumption (2.19) holds, then

$$A_\infty = \int_0^\infty \left(s^{1-N+ap} \int_0^s t^{N-1} (B_1(t))^{\lambda_1} (b_1(t))^{\lambda_2} dt \right)^{1/(p-1)} ds = \infty, \tag{2.22}$$

with $\lambda_1 = \alpha_1/n$, $\lambda_2 = \alpha_2/m$.

In fact, for this case, there exist $t_0 \geq 1$ and $C_0 > 0$ such that

$$(B_1(t))^{\lambda_1} (b_1(t))^{\lambda_2} \geq C_0 t^{\sigma_1 q/(p-1)} = C_0 t^{-N+(q/(p-1))(N-p(1+a))} \tag{2.23}$$

for $t \geq t_0$. Therefore,

$$\begin{aligned} A_\infty &\geq \int_{t_0}^\infty \left(s^{1-N+ap} \int_0^s t^{N-1} (B_1(t))^{\lambda_1} (b_1(t))^{\lambda_2} dt \right)^{1/(p-1)} ds \\ &\geq C_1 \int_{t_0}^\infty s^{(1-N+ap+q(N-p(1+a))/(p-1))/(p-1)} ds = \infty. \end{aligned} \tag{2.24}$$

So, condition (2.19) implies (2.22).

3. Proof of Existence

Before proofing the existence, we present some preliminary lemmas which will be useful in what follows.

Lemma 3.1. *Suppose that $\rho(x) \geq 0$, $\not\equiv 0$ is local Hölder continuous and satisfies*

$$\rho_\infty = \int_0^\infty \left(s^{1-N+ap} \int_0^s t^{N-1} \overline{\rho}(t) dt \right)^{1/(p-1)} ds < \infty. \tag{3.1}$$

Then the problem

$$\begin{gathered} -\operatorname{div}\left(|x|^{-ap} |\nabla u|^{p-2} \nabla u \right) = \rho(x), \quad x \in \mathbb{R}^N, \\ u(x) > 0, \quad x \in \mathbb{R}^N, \quad \lim_{|x| \to \infty} u(x) = 0 \end{gathered} \tag{3.2}$$

has a weak solution $u(x) \in W^{1,p}(\mathbb{R}^N, |x|^{-ap}) \cap L^\infty(\mathbb{R}^N)$, where $\overline{\rho}(t) = \max_{|x|=t} \rho(x)$.

Proof. Let $\underline{\rho}(t) = \min_{|x|=t}\rho(x)$. Then $\underline{\rho}(t) \le \overline{\rho}(t)$. Denote

$$V(|x|) = V(r) = \int_r^\infty \left(s^{1-N+ap} \int_0^s t^{N-1}\underline{\rho}(t)dt \right)^{1/(p-1)} ds, \quad r = |x| \ge 0,$$
$$U(|x|) = U(r) = \int_r^\infty \left(s^{1-N+ap} \int_0^s t^{N-1}\overline{\rho}(t)dt \right)^{1/(p-1)} ds, \quad r = |x| \ge 0. \tag{3.3}$$

Obviously, $\lim_{|x|\to\infty}V(|x|) = \lim_{|x|\to\infty}U(|x|) = 0$ and $V(|x|) \le U(|x|)$. It is easy to verify that

$$-\operatorname{div}\left(|x|^{-ap}|\nabla V|^{p-2}\nabla V\right) = \underline{\rho}(x), \quad x \in \mathbb{R}^N,$$
$$-\operatorname{div}\left(|x|^{-ap}|\nabla U|^{p-2}\nabla U\right) = \overline{\rho}(x), \quad x \in \mathbb{R}^N. \tag{3.4}$$

This shows that $V(|x|)$ (resp., $U(|x|)$) is a lower (resp., upper) solution of (3.2). Then by Lemma 2.3, there exists a weak solution $u(x)$ for problem (3.2) satisfying $u(x) \in W^{1,p}(\mathbb{R}^N, |x|^{-ap}) \cap L^\infty(\mathbb{R}^N)$, and

$$V(|x|) \le u(x) \le U(|x|), \quad x \in \mathbb{R}^N. \tag{3.5}$$

□

Lemma 3.2. *Let $N \ge 3$. If*

(1)

$$\int_1^\infty \left(t^{1+ap}\overline{\rho}(t)\right)^{1/(p-1)} dt < \infty, \quad \textit{if } 1 < p \le 2,\ N > 2 + ap, \tag{3.6}$$

(2)

$$\int_0^\infty t^{(N(p-2)+1+ap)/(p-1)}\overline{\rho}(t)dt < \infty, \quad \textit{if } p > 2,\ N > p(1+a), \tag{3.7}$$

one has $\rho_\infty < \infty$.

Proof. (1) Since $1 < p \le 2$, $1/(p-1) \ge 1$. By the Hölder inequality, we obtain

$$\begin{aligned}
\rho_\infty &= \int_0^\infty \left(s^{1-N+ap}\int_0^s t^{N-1}\overline{\rho}(t)dt\right)^{1/(p-1)} ds \\
&\le \int_0^\infty s^{(1-N+ap)/(p-1)}\left[\left(\int_0^s \left(t^{N-1}\overline{\rho}(t)\right)^{1/(p-1)} dt\right)^{p-1}\left(\int_0^s dt\right)^{2-p}\right]^{1/(p-1)} ds \\
&= \int_0^\infty s^{(3-N+ap-p)/(p-1)}\int_0^s \left(t^{N-1}\overline{\rho}(t)\right)^{1/(p-1)} dt\, ds \\
&= \int_0^\infty t^{(N-1)/(p-1)}\left(\overline{\rho}(t)\right)^{1/(p-1)}\int_t^\infty s^{(3-N+ap-p)/(p-1)} ds\, dt \\
&= \frac{p-1}{N-2-ap}\int_0^\infty \left(t^{1+ap}\overline{\rho}(t)\right)^{1/(p-1)} dt < \infty.
\end{aligned} \tag{3.8}$$

(2) If $p > 2$ and $N > p(a+1)$, we take $p' = N - (N - p(a+1))/(p-1)$ and then $p' \in (p(1+a), N)$.

Note that

$$\begin{aligned}
&\int_0^1 \left(s^{1-N+ap}\int_0^s t^{N-1}\overline{\rho}(t)dt\right)^{1/(p-1)} ds \le \left(\int_0^1 \overline{\rho}(t)dt\right)^{1/(p-1)} < \infty, \\
&\int_1^\infty \left(s^{1-N+ap}\int_0^s t^{N-1}\overline{\rho}(t)dt\right)^{1/(p-1)} ds \le \int_1^\infty s^{(ap+1-p')/(p-1)}\left(\int_0^s t^{p'-1}\overline{\rho}(t)dt\right)^{1/(p-1)} ds \\
&\qquad \le \frac{(p-1)^2}{(p-2)(N-p(a+1))}\left(\int_0^\infty t^{(N(p-2)+1+ap)/(p-1)}\overline{\rho}(t)dt\right)^{1/(p-1)} \\
&\qquad < \infty.
\end{aligned} \tag{3.9}$$

This implies $\rho_\infty < \infty$ and ends the proof of Lemma 3.2. □

Corollary 3.3. *If $\overline{\rho}(t) = \max_{|x|=t}\rho(x)$ satisfies the conditions in Lemma 3.2, then the problem* (3.2) *admits a solution $u(x) \in W^{1,p}(\mathbb{R}^N, |x|^{-ap}) \cap L^\infty(\mathbb{R}^N)$.*

Lemma 3.4. *Suppose that $f(t) \ge 0$ is nondecreasing and $\sup_{t\ge 0} t^{-m} f(t) < \infty$ with $m < p-1$. Additionally, let the function $h(x) \ge 0$ be locally Hölder continuous and satisfy*

$$h_\infty = \int_0^\infty \left(s^{1-N+ap}\int_0^s t^{N-1}\overline{h}(t)dt\right)^{1/(p-1)} ds < \infty, \tag{3.10}$$

where $\overline{h}(t) = \max_{|x|=t} h(x)$. *Then the problem*

$$-\operatorname{div}\left(|x|^{-ap}|\nabla u|^{p-2}\nabla u\right) = h(x)f(u), \quad x \in \mathbb{R}^N,$$
$$u(x) > 0, \quad x \in \mathbb{R}^N, \quad \lim_{|x|\to\infty} u(x) = 0 \tag{3.11}$$

has a weak solution $u(x) \in W^{1,p}(\mathbb{R}^N, |x|^{-ap}) \cap L^\infty(\mathbb{R}^N)$.

Proof. We first consider the problem

$$-\operatorname{div}\left(|x|^{-ap}|\nabla u|^{p-2}\nabla u\right) = h(x), \quad x \in \mathbb{R}^N,$$
$$u(x) > 0, \quad x \in \mathbb{R}^N, \quad \lim_{|x|\to\infty} u(x) = 0. \tag{3.12}$$

By Lemma 3.1, there is a solution $w_h(x)$ for (3.12) satisfying $w_h(x) \in W^{1,p}(\mathbb{R}^N, |x|^{-ap}) \cap L^\infty(\mathbb{R}^N)$. In order to get the existence of solution for (3.11), we chose a pair of upper-lower solution of the equation in (3.11) by means of $w_h(x)$.

Let $t > 0$. It is easy to verify that $u_h = tw_h$ is an upper solution of

$$-\operatorname{div}\left(|x|^{-ap}|\nabla u|^{p-2}\nabla u\right) = h(x)f(u), \quad x \in \mathbb{R}^N \tag{3.13}$$

if and only if

$$-\operatorname{div}\left(|x|^{-ap}|\nabla u_h|^{p-2}\nabla u_h\right) \geq h(x)f(u_h), \quad x \in \mathbb{R}^N \tag{3.14}$$

or

$$t^{p-1} \geq f(tw_h), \quad x \in \mathbb{R}^N. \tag{3.15}$$

By the assumption on $f(u)$, we know that there exists $c_0 > 0$, such that $f(t) \leq c_0 t^m$. So, $c_0 t^m \|w_h\|_\infty^m \geq f(t\|w_h\|_\infty) \geq f(tw_h)$. Then we take $t_0 = (c_0\|w_h\|_\infty^m)^{1/(p-1-m)}$ so that $u_h = tw_h$ $(t > t_0)$ is an upper solution of (3.13).

We now construct a lower solution of (3.13). Consider the boundary value problem

$$-\operatorname{div}\left(|x|^{-ap}|\nabla v|^{p-2}\nabla v\right) = h(x)f(v), \quad x \in B_k,$$
$$v > 0, \quad x \in B_k, \quad v = 0, \quad x \in \partial B_k \tag{3.16}$$

for $k = 1, 2, \ldots$.

By Theorem 3.1 in [12], there exists a solution $v_k \in W^{1,p}(B_k, |x|^{-ap}) \cap L^\infty(B_k)$ for (3.16). We define an extension by $v_k(x) = 0$ for $|x| \geq k$. Then, by Theorem 2.4 in [12] and Díaz-Saá's inequality in [24], we get

$$v_1(x) \leq v_2(x) \leq \cdots \leq v_k(x) \leq v_{k+1}(x) \leq \cdots \leq u_h(x), \quad x \in B_k. \tag{3.17}$$

Setting $v(x) = \lim_{k\to\infty} v_k(x)$ and performing some standard computations, we see that $v \in W^{1,p}(\mathbb{R}^N, |x|^{-ap}) \cap L^\infty(\mathbb{R}^N)$,

$$\begin{gathered} -\operatorname{div}\left(|x|^{-ap}|\nabla v|^{p-2}\nabla v\right) = h(x)f(v), \quad x \in \mathbb{R}^N, \\ v(x) > 0, \quad x \in \mathbb{R}^N, \quad \lim_{|x|\to\infty} v(x) = 0, \end{gathered} \tag{3.18}$$

and $v(x) \le u_h(x)$ in $\mathbb{R}^N$. Then, our result follows from Lemma 2.3. □

We now give the proof of Theorem 2.4.

Proof of Theorem 2.4. Let $u_M \in W^{1,p}(\mathbb{R}^N, |x|^{-ap}) \cap L^\infty(\mathbb{R}^N)$ be a solution of the problem

$$\begin{gathered} -\operatorname{div}\left(|x|^{-ap}|\nabla u|^{p-2}\nabla u\right) = M(x), \quad x \in \mathbb{R}^N, \\ u > 0, \quad x \in \mathbb{R}^N, \quad \lim_{|x|\to\infty} u(x) = 0, \end{gathered} \tag{3.19}$$

where $M(x) = \max\{h(x), H(x)\}$. We see that $w = tu_M$ $(t > 0)$ is an upper solution of the equation

$$-\operatorname{div}\left(|x|^{-ap}|\nabla u|^{p-2}\nabla u\right) = M(x)\big(f(u) + \lambda g(u)\big), \quad x \in \mathbb{R}^N \tag{3.20}$$

if and only if

$$-\operatorname{div}\left(|x|^{-ap}|\nabla w|^{p-2}\nabla w\right) \ge M(x)\big(f(w) + \lambda g(w)\big), \quad x \in \mathbb{R}^N \tag{3.21}$$

or

$$t^{p-1} \ge f(tu_M) + \lambda g(tu_M), \quad x \in \mathbb{R}^N. \tag{3.22}$$

Since

$$\sup_{t\ge0} t^{-m} f(t) < \infty, \qquad \sup_{t\ge0} t^{-n} g(t) < \infty, \tag{3.23}$$

we have a constant $c_0 > 0$, such that

$$f(t) \le c_0 t^m, \quad g(t) \le c_0 t^n, \quad \forall t \ge 0. \tag{3.24}$$

Denote

$$\phi(t) = \frac{t^{p-1} - c_0 t^m \|u_M\|_\infty^m}{c_0 t^n \|u_M\|_\infty^n}. \tag{3.25}$$

Since $m < p-1 < n$, we have $\lim_{t\to 0^+}\phi(t) = -\infty$, $\lim_{t\to\infty}\phi(t) = 0$ and there exist $t_0 > 0$, such that $\phi'(t) > 0$ for $0 \le t < t_0$ and $\phi'(t) < 0$ for $t > t_0$. Then $\phi(t_0) = \max_{t>0}\phi(t)$. A simple computation shows that

$$t_0 = \left(\frac{c_0(n-m)}{n-p+1}\|u_M\|_\infty^m\right)^{1/(p-1-m)}. \tag{3.26}$$

Thus

$$\lambda_0 = \phi(t_0) = c_0^{(m-n)/(p-1-m)}\|u_M\|_\infty^{(m-n)(p-1)/(p-1-m)}\left[\frac{p-1-m}{n-p+1}\right]\left[\frac{n-m}{n-p+1}\right]^{(m-n)/(p-1-m)} > 0. \tag{3.27}$$

Hence, for any $0 < \lambda < \lambda_0$, there exists a unique $t_\lambda > 0$, such that $\lambda = \phi(t_\lambda)$. That is

$$t_\lambda^{p-1} = c_0 t_\lambda^m\|u_M\|_\infty^m + c_0\lambda t_\lambda^n\|u_M\|_\infty^n \ge f(t_\lambda u_M) + \lambda g(t_\lambda u_M). \tag{3.28}$$

Now defining $w = t_\lambda u_M$, we get

$$\begin{aligned} -\operatorname{div}\left(|x|^{-ap}|\nabla w|^{p-2}\nabla w\right) &= t_\lambda^{p-1}\left(-\operatorname{div}\left(|x|^{-ap}|\nabla u_M|^{p-2}\nabla u_M\right)\right) \\ &= M(x)t_\lambda^{p-1} \\ &= M(x)\left(t_\lambda^m\|u_M\|_\infty^m + \lambda t_\lambda^n\|u_M\|_\infty^n\right) \\ &\ge M(x)\left(f(w) + \lambda g(w)\right). \end{aligned} \tag{3.29}$$

This shows that w is an upper solution of (3.20). Noting that

$$M(x)\left(f(w) + \lambda g(w)\right) \ge h(x)f(w) + \lambda H(x)g(w), \tag{3.30}$$

we know that w is an upper solution of (1.1). Let v be a solution of (3.11). Obviously, v is a lower solution of (1.1). We now show that $v(x) \le w(x)$ in $\mathbb{R}^N$.

Since $\phi'(t) < 0$ for $t > t_0$ and $\phi(t) \to 0$ as $t \to +\infty$, then for any $\lambda \in (0, \lambda_0)$, there exist $t_\lambda > 0$, such that $\lambda = \phi(t_\lambda)$. Without loss of generality, let $t_\lambda > t_0$.

From the proof of Lemma 3.4 and the definition of $u_M(x)$, we have $u_h(x) = tw_h(x) \le tu_M(x)$ for $t > t_0$. Further, by (3.17), we get $v_k(x) \le t_\lambda u_M(x) = w(x)$. Letting $k \to +\infty$, we obtain $v(x) \le w(x)$ in R^N.

By Lemma 2.3, there exists a solution $u \in W^{1,p}(\mathbb{R}^N, |x|^{-ap}) \cap L^\infty(\mathbb{R}^N)$ for the problem (1.1). We then complete the proof of Theorem 2.4. □

Remark 3.5. The nonlinear term $F(x,u) = h(x)f(u) + \lambda H(x)g(u)$ can be regarded as a perturbation of the nonlinear term $h(x)f(u)$.

4. Proof of Nonexistence

In order to prove the nonexistence of nontrivial solution of the problem (2.21), we use the test function method, which has been used in [25] and references therein. Some modification has been made in our proof. The proof is based on argument by contradiction which involves a priori estimate for a nonnegative solution of (2.21) by carefully choosing the special test function and scaling argument.

Proof of Theorem 2.5. Let $\phi_0(s) \in C_0^1[0,\infty)$ be defined by

$$\phi_0(s) = \begin{cases} 1, & 0 \le s < 1, \\ (l-k)^{-1}\left(l(2-s)^k - k(2-s)^l\right), & 1 \le s \le 2, \\ 0, & s > 2, \end{cases} \tag{4.1}$$

and put $\phi(x) = \phi_0(R^{-2}|x|^2)$, by which the parameters $l > k > 2$ will be determined later. It is not difficult to verify that $0 \le \phi_0(s) \le 1$ and $|\phi_0'(s)| \le \beta_0 \phi_0^{1-1/k}(s)$, where $\beta_0 = k(l/(l-k))^{1/k}$.

Suppose that $u(x)$ is a solution to problem (2.21). Without loss of generality, we can assume that $u(x) > 0$ in $\mathbb{R}^N$ (otherwise, we consider $u_\epsilon = u + \epsilon$ and let $\epsilon \downarrow 0$). Let $\alpha < 0$ be a parameter (α will also be chosen below).

By the Young inequality, we get

$$h(x)u^m + H(x)u^n \ge H^{\alpha_1/n}(x)h^{\alpha_2/m}(x)u^q \equiv H^{\lambda_1}(x)h^{\lambda_2}(x)u^q, \tag{4.2}$$

where α_1, α_2, and q satisfy (2.18) and $\lambda_1 = \alpha_1/n$, $\lambda_2 = \alpha_2/m$.

Multiplying the equation in (2.21) by $u^\alpha \phi$ and integrating by parts, we obtain

$$\int_{\mathbb{R}^N} H^{\lambda_1}h^{\lambda_2}u^{q+\alpha}\phi dx \le \alpha \int_{\mathbb{R}^N} |x|^{-ap}u^{\alpha-1}|\nabla u|^p \phi dx + \int_{\mathbb{R}^N} |x|^{-ap}u^{\alpha}|\nabla u|^{p-1}\left|\nabla\phi\right| dx. \tag{4.3}$$

Then applying the Young inequality with parameter $\varepsilon > 0$, we have

$$\int_{\mathbb{R}^N} H^{\lambda_1}h^{\lambda_2}u^{q+\alpha}\phi dx + \beta_\varepsilon \int_{\mathbb{R}^N} |x|^{-ap}|\nabla u|^p u^{\alpha-1}\phi dx \le C_\varepsilon \int_{\mathbb{R}^N} |x|^{-ap}u^{p+\alpha-1}\left|\nabla\phi\right|^p \phi^{1-p} dx, \tag{4.4}$$

where $\beta_\varepsilon = |\alpha| - \varepsilon > 0$.

Similarly, let us multiply the equation in (2.21) by ϕ and integrate by parts:

$$\begin{aligned} \int_{\mathbb{R}^N} H^{\lambda_1}h^{\lambda_2}u^q\phi dx &\le \int_{\Omega_R} |x|^{-ap}|\nabla u|^{p-1}\left|\nabla\phi\right| dx \\ &\le \left(\int_{\Omega_R} |x|^{-ap}|\nabla u|^p u^{\alpha-1}\phi dx\right)^{(p-1)/p} \left(\int_{\Omega_R} |x|^{-ap}\left|\nabla\phi\right|^p \phi^{1-p} u^{(1-\alpha)(p-1)} dx\right)^{1/p}. \end{aligned} \tag{4.5}$$

By (4.4),

$$\int_{\mathbb{R}^N} |x|^{-ap}|\nabla u|^p u^{\alpha-1}\phi dx \le C\int_{\Omega_R} |x|^{-ap}\left|\nabla\phi\right|^p \phi^{1-p}u^{p+\alpha-1}dx. \tag{4.6}$$

Now, we apply the Hölder inequality to the integral on the right-hand side of (4.6):

$$\begin{aligned}&\int_{\Omega_R} |x|^{-ap}\left|\nabla\phi\right|^p\phi^{1-p}u^{p+\alpha-1}dx\\ &\quad\le\left(\int_{\Omega_R} H^{\lambda_1}h^{\lambda_2}u^q\phi dx\right)^{1/\lambda}\left(\int_{\Omega_R}|x|^{-ap\lambda'}\left|\nabla\phi\right|^{p\lambda'}\phi^{1-p\lambda'}\left(H^{\lambda_1}h^{\lambda_2}\right)^{1-\lambda'}dx\right)^{1/\lambda'}\end{aligned} \tag{4.7}$$

with $\lambda = q/(p+\alpha-1) > 1$, $\lambda' = q/(q-p-\alpha+1)$ and $\Omega_R = \{x \in \mathbb{R}^N \mid R \le |x| \le \sqrt{2}R\}$.

Since $q > p-1$, we chose $\alpha < 0$ so small that $q > (p-1)(1-\alpha)$. Then, we have

$$\begin{aligned}&\int_{\Omega_R} |x|^{-ap}\left|\nabla\phi\right|^p\phi^{1-p}u^{(1-\alpha)(p-1)}dx\\ &\quad\le\left(\int_{\Omega_R} H^{\lambda_1}h^{\lambda_2}u^q\phi dx\right)^{1/\mu}\left(\int_{\Omega_R}|x|^{-ap\mu'}|\nabla\phi|^{p\mu'}\phi^{1-p\mu'}\left(H^{\lambda_1}h^{\lambda_2}\right)^{1-\mu'}dx\right)^{1/\mu'}\end{aligned} \tag{4.8}$$

with $\mu = q/(1-\alpha)(p-1) > 1$, $\mu' = q/(q-(1-\alpha)(p-1))$.

Since $\phi(x) = \phi_0(R^{-2}|x|^2)$, $|\nabla\phi(x)| \le C_0R^{-1}\phi_0^{1-1/k}(|\xi|) = C_0R^{-1}\phi_0^{1-1/k}$ with $x = R\xi$. Then we get

$$\begin{aligned}&\int_{\Omega_R} |x|^{-ap\lambda'}\left|\nabla\phi\right|^{p\lambda'}\phi^{1-p\lambda'}\left(H^{\lambda_1}h^{\lambda_2}\right)^{1-\lambda'}dx\\ &\quad\le CR^{N-(1+a)p\lambda'}(B_1(R))^{\lambda_1(1-\lambda')}(b_1(R))^{\lambda_2(1-\lambda')}\int_{\Omega_1}\phi_0^{(1-1/k)p\lambda'}(|\xi|)\phi_0^{1-p\lambda'}(|\xi|)d\xi,\\ &\int_{\Omega_R} |x|^{-ap\mu'}\left|\nabla_x\phi\right|^{p\mu'}\phi^{1-p\mu'}H^{1-\mu'}dx\\ &\quad\le CR^{N-(1+a)p\mu'}(B_1(R))^{\lambda_1(1-\mu')}(b_1(R))^{\lambda_2(1-\mu')}\int_{\Omega_1}\phi_0^{(1-1/k)p\mu'}(|\xi|)\phi_0^{1-p\mu'}(|\xi|)d\xi,\end{aligned} \tag{4.9}$$

where $B_1(R) = \inf_{\Omega_R} H(x)$ and $b_1(R) = \inf_{\Omega_R} h(x)$.

Let $k > max\{p\lambda', p\mu'\}$. Then,

$$\int_{\Omega_1}\phi_0^{(1-1/k)p\lambda'}(|\xi|)\phi_0^{1-p\lambda'}(|\xi|)d\xi \le \int_{\Omega_1}\phi_0(|\xi|)d\xi \le |\Omega_1|. \tag{4.10}$$

Similarly,

$$\int_{\Omega_1} \phi_0^{(1-1/k)p\mu'}(|\xi|)\phi_0^{1-p\mu'}(|\xi|)d\xi \le |\Omega_1|. \tag{4.11}$$

Then it follows from (4.5)–(4.11) that

$$\left(\int_{\mathbb{R}^N} H^{\lambda_1}h^{\lambda_2}u^q\phi dx\right)^{1-s} \le CR^{\sigma_1}(B_1(R))^{\sigma_2}(b_1(R))^{\sigma_3} \tag{4.12}$$

with $s = (p-1)/p\lambda + 1/p\mu = (p-1)/q < 1$ and

$$\begin{aligned}
\sigma_1 &= \frac{p-1}{p\lambda'}(N-(1+a)p\lambda') + \frac{1}{p\mu'}(N-(1+a)p\mu') = N - p(1+a) - \frac{N(p-1)}{q},\\
\sigma_2 &= \frac{\lambda_1(p-1)}{p\lambda'}(1-\lambda') + \frac{\lambda_1}{p\mu'}(1-\mu') = -\frac{\lambda_1(p-1)}{q},\\
\sigma_3 &= \frac{\lambda_2(p-1)}{p\lambda'}(1-\lambda') + \frac{\lambda_2}{p\mu'}(1-\mu') = -\frac{\lambda_2(p-1)}{q}.
\end{aligned} \tag{4.13}$$

If $\limsup_{R\to\infty} R^{\sigma_1}(B_1(R))^{\sigma_2}(b_1(R))^{\sigma_3} = 0$, it follows from (4.12) that

$$\int_{\mathbb{R}^N} H^{\lambda_1}h^{\lambda_2}u^q dx = 0. \tag{4.14}$$

This implies that $u(x) = 0$, a.e. in $\mathbb{R}^N$. That is, u is a trivial solution for (2.21).

If $\limsup_{R\to\infty} R^{\sigma_1}(B_1(R))^{\sigma_2}(b_1(R))^{\sigma_3} = C_1 < \infty$, then (4.12) gives that

$$\begin{gathered}
\int_{\mathbb{R}^N} H^{\lambda_1}h^{\lambda_2}u^q dx < \infty,\\
\lim_{R\to\infty}\int_{\Omega_R} H^{\lambda_1}h^{\lambda_2}u^q dx = 0.
\end{gathered} \tag{4.15}$$

By (4.5), we derive

$$\begin{aligned}
\int_{B_R} H^{\lambda_1}h^{\lambda_2}u^q dx &\le \int_{B_{2R}} H^{\lambda_1}h^{\lambda_2}u^q\phi dx\\
&\le \left(\int_{\Omega_R}|x|^{-ap}|\nabla u|^p u^{\alpha-1}\phi dx\right)^{(p-1)/p}\left(\int_{\Omega_R}|x|^{-ap}|\nabla\phi|^p\phi^{1-p}u^{(1-\alpha)(p-1)}dx\right)^{1/p}.
\end{aligned} \tag{4.16}$$

Reasoning as in the first part of the proof, we infer that

$$\begin{aligned}\int_{B_R} H^{\lambda_1} h^{\lambda_2} u^q dx &\le C R^{\sigma_1} (B_1(R))^{\sigma_2} (b_1(R))^{\sigma_3} \left(\int_{\Omega_R} H^{\lambda_1} h^{\lambda_2} u^q \phi dx \right)^{(p-1)/q} \\ &\le C C_1 \left(\int_{\Omega_R} H^{\lambda_1} h^{\lambda_2} u^q \phi dx \right)^{(p-1)/q}.\end{aligned} \tag{4.17}$$

Letting $R \to \infty$ in (4.17), we obtain (4.14). Thus, $u = 0$, a.e. in $\mathbb{R}^N$. Then the proof of Theorem 2.5 is completed. □

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