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Research Article

Nontrivial Solutions of the Asymmetric Beam System with Jumping Nonlinear Terms

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We investigate the existence of multiple nontrivial solutions (ξ,η) for perturbations $b_1[(u+2)^+-2]$and $b_2[(u+3)^+-3]$ of the beam system with Dirichlet boundary condition $L\xi=b_1[(\xi+3\eta+2)^+-2]$ in $(-\pi/2,\pi/2)\times\mathbb{R}$, $L\eta=b_2[(\xi+3\eta+3)^+-3]$ in $(-\pi/2,\pi/2)\times\mathbb{R}$, where $u^+=\max\{u,0\}$, and μ,ν are nonzero constants. Here L is the beam operator in $\mathbb{R}^2$, and the nonlinearity $(b_1[(u+2)^+-2]+b_2[(u+3)^+-3]$ crosses the eigenvalues of the beam operator.

1. Introduction

Let L be the beam operator in $\mathbb{R}^2$, $Lu=u_{tt}+u_{xxxx}$. In this paper, we investigate the existence of multiple nontrivial solutions (ξ,η) for perturbations $b_1[(\xi+3\eta+2)^+-2]$ and $b_2[(\xi+3\eta+3)^+-3]$ of the beam system with Dirichlet boundary condition

$$\begin{gathered}
L\xi=b_1\left[(\xi+3\eta+2)^+-2\right]\quad \text{in } \left(-\frac{\pi}{2},\frac{\pi}{2}\right)\times\mathbb{R},\\
L\eta=b_2\left[(\xi+3\eta+3)^+-3\right]\quad \text{in } \left(-\frac{\pi}{2},\frac{\pi}{2}\right)\times\mathbb{R},\\
\xi\left(\pm\frac{\pi}{2},t\right)=\xi_{xx}\left(\pm\frac{\pi}{2},t\right)=0,\\
\xi(x,t+\pi)=\xi(x,t)=\xi(-x,t),\\
\eta\left(\pm\frac{\pi}{2},t\right)=\eta_{xx}\left(\pm\frac{\pi}{2},t\right)=0,\\
\eta(x,t+\pi)=\eta(x,t)=\eta(-x,t),
\end{gathered}\tag{1.1}$$

where $u^+ = \max\{u, 0\}$ and the nonlinearity $(b_1[(u+2)^+ - 2] + b_2[(u+3)^+ - 3])$ crosses the eigenvalues of the beam operator. This system represents a bending beam supported by cables in the two directions.

In [1, 2], the authors investigated the multiplicity of solutions of a nonlinear suspension bridge equation in an interval $(-\pi/2, \pi/2)$

$$\begin{gathered} u_{tt} + u_{xxxx} + bu^+ = f(x) \quad \text{in } \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \times \mathbb{R}, \\ u\left(\pm\frac{\pi}{2}, t\right) = u_{xx}\left(\pm\frac{\pi}{2}, t\right) = 0, \\ u \text{ is } \pi-\text{periodic in } t \text{ and even in } x, \end{gathered} \tag{1.2}$$

where the nonlinearity $-(bu^+)$ crosses an eigenvalue. This equation represents a bending beam supported by cables under a load f. The constant b represents the restoring force if the cables stretch. The nonlinearity u^+ models the fact that cables resist expansion but do not resist compression.

In [2] Lazer and McKenna point out that the kind of nonlinearity $b[(u+1)^+ - 1]$,

$$\begin{gathered} u_{tt} + u_{xxxx} = \mathrm{b}\left[(u+1)^+ - 1\right] \quad \text{in } \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \times \mathbb{R}, \\ u\left(\pm\frac{\pi}{2}, t\right) = u_{xx}\left(\pm\frac{\pi}{2}, t\right) = 0, \\ u \text{ is } \pi-\text{periodic in } t \text{ and even in } x, \end{gathered} \tag{1.3}$$

can furnish a model to study travelling waves in suspension bridges. This is a one-dimensional beam equation that represents only the up and down travelling waves of the beam. But the beam has also the right and left travelling waves. Hence we can consider two-dimensional beam equation (1.1).

The nonlinear equation with jumping nonlinearity has been extensively studied by many authors. For the fourth order elliptic equation, Taratello [3] and Micheletti and Pistoia [4, 5] proved the existence of nontrivial solutions, by using degree theory and critical point theory, separately. For one-dimensional case, Lazer and McKenna [6] proved the existence of nontrivial solution by the global bifurcation method. For this jumping nonlinearity, we are interested in the multiple nontrivial solutions of the equation. Here we used variational reduction method to find the nontrivial solutions of problem (1.1).

In Section 2, we investigate some properties of the Hilbert space spanned by eigenfunctions of the beam operator. We show that only the trivial solution exists for problem (1.4) when $-3 < b_1,\ b_2 < 1$, and $-3 < b_1 + b_2 < 1$. In Section 3, we state the Mountain Pass Theorem. In Section 4, we investigate the existence of nontrivial solutions $u(x,t)$ for a perturbation $g(u) = b_1[(u+2)^+ - 2] + b_2[(u+3)^+ - 3]$ of the asymmetric beam equation

$$\begin{gathered} u_{tt} + u_{xxxx} = g(u) \quad \text{in } \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \times \mathbb{R}, \\ u\left(\pm\frac{\pi}{2}, t\right) = u_{xx}\left(\pm\frac{\pi}{2}, t\right) = 0, \\ u(x, t+\pi) = u(x,t) = u(-x,t), \end{gathered} \tag{1.4}$$

where $u^+ = \max\{u, 0\}$, and b_1, b_2 are constants. This equation satisfies Dirichlet boundary condition on the interval $(-\pi/2, \ \pi/2)$ and periodic condition on the variable t. We use the variational reduction method to apply mountain pass theorem in order to get the main result that for $-15 < b_1, \ b_2 < -3, -15 < b_1 + b_2 < -3$ (1.2) has at least three periodic solutions, two of which are nontrivial. In Section 5, we investigate the existence of multiple nontrivial solutions (ξ, η) for perturbations $b_1[(\xi + 3\eta + 2)^+ - 2]$ and $b_2[(\xi + 3\eta + 3)^+ - 3]$ of beam system (1.1). We also prove that for $-3 < b_1, \ 3b_2 < 1, -3 < b_1 + 3b_2 < 1$ (1.1) has only the trivial solution.

2. Preliminaries

Let L be the differential operator and $Lu = u_{tt} + u_{xxxx}$. Then the eigenvalue problem

$$\begin{gathered} Lu = \lambda u \quad \text{in } \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \times \mathbb{R}, \\ u\left(\pm\frac{\pi}{2}, t\right) = 0, \quad u(x, t + \pi) = u(x, t) = u(-x, t) \end{gathered} \tag{2.1}$$

has infinitely many eigenvalues $\lambda_{mn} = (2n + 1)^4 - 4m^2 (m, n = 0, 1, 2, \ldots)$ and corresponding normalized eigenfunctions $\phi_{mn}, \psi_{mn} (m, n \geq 0)$ given by

$$\begin{aligned} \phi_{0n} &= \frac{\sqrt{2}}{\pi} \cos(2n + 1)x \quad \text{for } n \geq 0, \\ \phi_{mn} &= \frac{2}{\pi} \cos 2mt \cdot \cos(2n + 1)x \quad \text{for } m > 0, n \geq 0, \\ \psi_{mn} &= \frac{2}{\pi} \sin 2mt \cdot \cos(2n + 1)x \quad \text{for } m > 0, n \geq 0. \end{aligned} \tag{2.2}$$

We note that all eigenvalues in the interval $(-19, \ 45)$ are given by

$$\lambda_{20} = -15 < \lambda_{10} = -3 < \lambda_{00} = 1 < \lambda_{41} = 17. \tag{2.3}$$

Let Ω be the square $(-\pi/2, \pi/2) \times (-\pi/2, \pi/2)$ and H_0 the Hilbert space defined by

$$H_0 = \left\{ u \in L^2(\Omega) \ : u \text{ is even in } x \right\}. \tag{2.4}$$

Then the set of functions $\{\phi_{mn}, \ \psi_{mn}\}$ is an orthonormal basis in H_0. Let us denote an element u in H_0 as

$$u = \sum (h_{mn}\phi_{mn} + k_{mn}\psi_{mn}), \tag{2.5}$$

and we define a subspace H of H_0 as

$$H = \left\{ u \in H_0 : \sum |\lambda_{mn}| \left(h_{mn}^2 + k_{mn}^2 \right) < \infty \right\}. \tag{2.6}$$

Then this is a complete normed space with a norm

$$\|u\|_H = \left[\sum |\lambda_{mn}|\left(h_{mn}^2 + k_{mn}^2\right)\right]^{1/2}. \tag{2.7}$$

Since $|\lambda_{mn}| \geq 1$ for all m, n, we have that

(i) $\|u\|_{\mathrm{H}} \geq \|u\|$, where $\|u\|$ denotes the L^2 norm of u;

(ii) $\|u\| = 0$ if and only if $\|u\|_H = 0$.

Define $L_\beta u = Lu + \beta u$. Then we have the following lemma (cf. [7]).

Lemma 2.1. *Let $\beta \in \mathbb{R}$, $\beta \neq -\lambda_{mn}$ $(m, n \geq 0)$. Then we have that*

$$L_\beta^{-1} \textit{ is a bounded linear operator from } H_0 \textit{ into } H. \tag{2.8}$$

Theorem 2.2. *Let $-3 < b_1,\ b_2 < 1$, and $-3 < b_1 + b_2 < 1$. Then the equation, with Dirichlet boundary condition,*

$$Lu = b_1\left[(u+2)^+ - 2\right] + b_2\left[(u+3)^+ - 3\right] \tag{2.9}$$

has only the trivial solution in H_0.

Proof. Since $\lambda_{10} = -3$ and $\lambda_{00} = 1$, let $\beta = -(1/2)(\lambda_{00} + \lambda_{10}) = -(1/2)(-3+1) = 1$. The equation is equivalent to

$$u = (L+\beta)^{-1}\left(b_1\left[(u+2)^+ - 2\right] + b_2\left[(u+3)^+ - 3\right] + \beta u\right). \tag{2.10}$$

By Lemma 2.1, $(L+\beta)^{-1}$ is a compact linear map from H_0 into H_0. Therefore, it is L^2 norm $(1/2)$. We note that

$$\begin{aligned} &\left\| b_1\left[(u_1+2)^+ - (u_2+2)^+\right] + b_2\left[(u_1+3)^+ - (u_2+3)^+\right] + \beta(u_1 - u_2)\right\| \\ &\quad \leq \max\left\{\left|b_1+\beta\right|, \left|b_2+\beta\right|, \left|b_1+b_2+\beta\right|, \left|\beta\right|\right\}\|u_1 - u_2\| < \frac{1}{2}(\lambda_{00} - \lambda_{10})\|u_1 - u_2\| \\ &\quad = 2\|u_1 - u_2\|. \end{aligned} \tag{2.11}$$

So the right-hand side of (2.10) defines a Lipschitz mapping of H_0 into H_0 with Lipschitz constant $\gamma < 1$. Therefore, by the contraction mapping principle, there exists a unique solution $u \in H_0$. Since $u \equiv 0$ is a solution of (2.10), $u \equiv 0$ is the unique solution. □

3. Mountain Pass Theorem

The mountain pass theorem concerns itself with proving the existence of critical points of functional $I \in C^1(E, \mathbb{R})$ which satisfy the Palais-Smale (PS) condition, which occurs repeatedly in critical point theory.

Definition 3.1. We say that I satisfies the Palais-Smale condition if any sequence $\{u_m\} \subset E$ for which $I(u_m)$ is bounded and $I'(u_m) \to 0$ as $m \to \infty$ possesses a convergent sequence.

The following deformation theorem is stated in [8].

Theorem 3.2. *Let E be a real Banach space and $I \in C^1(E, R)$. Suppose I satisfies Palais-Smale condition. Let N be a given neighborhood of the set K_c of the critical points of I at a given level c. Then there exists $\epsilon > 0$, as small as we want, and a deformation $\eta : [0,1] \times E \to E$ such that we denote by A_b the set $\{x \in E : I(x) \leq b\}$:*

(i) $\eta(0, x) = x$ *for all* $x \in E$,

(ii) $\eta(t, x) = x$ *for all* $x \in A_{c-2\epsilon} \cup (E \setminus A_{c+2c})$, *for all* $t \in [0,1]$,

(iii) $\eta(1, \cdot)(A_{c+\epsilon} \setminus N) \subset A_{c-\epsilon}$.

We state the Mountain Pass Theorem.

Theorem 3.3. *Let E be a real Banach space and $I \in C^1(E, \mathbb{R})$ satisfy (PS) condition. Suppose that*

(I_1) *there are constants $\rho, \alpha > 0$ such that $I|_{\partial B_\rho(0)} \geq I(0) + \alpha$, and*

(I_2) *there is an $e \in E \setminus \overline{B}_\rho$ such that $I(e) \leq I(0)$.*

Then I possesses a critical value $c \geq \alpha$. Moreover, c can be characterized as

$$c = \inf_{g \in \Gamma} \max_{u \in g([0,1])} I(u), \tag{3.1}$$

where

$$\Gamma = \{g \in C([0,1], E) \mid g(0) = 0, g(1) = e\}. \tag{3.2}$$

4. Critical Point Theory and Multiple Nontrivial Solutions

We investigate the existence of multiple solutions of (1.1) when $-7 < b_1,\ b_2 < -3 - 7 < b_1 + b_2 < -3$. We define a functional on H by

$$J(u) = \int_\Omega \left[\frac{1}{2}\left(-|u_t|^2 + |u_{xx}|^2\right) - \frac{b_1}{2}\left|(u+2)^+\right|^2 + 2b_1 u - \frac{b_2}{2}\left|(u+3)^+\right|^2 + 3b_2 u\right] dx\, dt. \tag{4.1}$$

Then the functional J is well defined in H and the solutions of (1.4) coincide with the critical points of $J(u)$. Now we investigate the property of functional J.

Lemma 4.1 (cf. [7]). *$J(u)$ is continuous and Frechet differentiable at each $u \in H$ with*

$$DJ(u)v = \int_\Omega \left(Lu - b_1(u+2)^+ + 2b_1 - b_2(u+3)^+ + 3b_2\right) v\, dx\, dt, \qquad v \in H. \tag{4.2}$$

We will use a variational reduction method to apply the mountain pass theorem.

Let V = closure of span$\{\phi_{10}, \psi_{10}\}$ be the two-dimensional subspace of H. Both of them have the same eigenvalue λ_{10}. Then $\|v\|_H = \sqrt{3}\|v\|$ for $v \in V$. Let W be the orthogonal complement of V in H. Let $P : H \to V$ denote H onto V and $I - P : H \to W$ denote H onto W. Then every element $u \in H$ is expressed by

$$u = v + \mathrm{w}, \tag{4.3}$$

where $v = Pu$, $w = (I - P)u$.

Lemma 4.2. *Let $-15 < b_1,\ b_2 < -3$, and $-15 < b_1 + b_2 < -3$. Let $v \in V$ be given. Then we have that there exists a unique solution $z \in W$ of equation*

$$Lz + (I - P)\left[-b_1(v + z + 2)^+ + 2b_1 - b_2(v + z + 3)^+ + 3b_2\right] = 0 \quad \text{in } W. \tag{4.4}$$

Let $z = \theta(v)$. Then θ satisfies a uniform Lipschitz continuous on v with respect to the L^2 norm (also the norm $\|\cdot\|_H$).

Proof. Choose $\beta = 7$ and let $g(\xi) = b_1(\xi + 2)^+ + b_2(\xi + 3)^+ + \beta\xi$. Then (4.4) can be written as

$$z = (L + \beta)^{-1}(I - P)\left[g(v + z) - (b_1 + b_2)\right]. \tag{4.5}$$

Since $(L+\beta)^{-1}(I-P)$ is a self-adjoint, compact, linear map from $(I-P)H$ into itself, the eigenvalues of $(L + \beta)^{-1}(I - P)$ in W are $(\lambda_{mn} + \beta)^{-1}$, where $\lambda_{mn} > 1$ or $\lambda_{mn} \le -15$. Therefore, $\|(L + \beta)^{-1}(I - P)\|$ is $1/4$. Since

$$\left|g(\xi_1) - g(\xi_2)\right| \le \max\left\{\left|b_1 + \beta\right|, \left|b_2 + \beta\right|, \left|b_1 + b_2 + \beta\right|, \left|\beta\right|\right\}|\xi_1 - \xi_2| < 8|\xi_1 - \xi_2|, \tag{4.6}$$

the right-hand side of (4.5) defines a Lipschitz mapping because for fixed $v \in V(I - P)H_0$ maps into itself. By the contraction mapping principle, there exists a unique $z \in (I - P)H_0$ (also $z \in (I - P)H$) for fixed $v \in V$. Since $(L + \beta)^{-1}$ is bounded from H to W there exists a unique solution $z \in W$ of (4.4) for given $v \in V$.

Let

$$\gamma = \frac{\max\left\{\left|b_1 + \beta\right|, \left|b_2 + \beta\right|, \left|b_1 + b_2 + \beta\right|, \left|\beta\right|\right\}}{8}. \tag{4.7}$$

Then $0 < \gamma < 1$. If $z_1 = \theta(v_1)$ and $z_2 = \theta(v_2)$ for any $v_1, v_2 \in V$, then

$$\|z_1 - z_2\| \le \left\|(L + \beta)^{-1}(I - P)\right\|\left\|\left(g(v_1 + z_1) - g(v_2 + z_2)\right)\right\| < \gamma(\|v_1 - v_2\| + \|z_1 - z_2\|). \tag{4.8}$$

Hence

$$\|z_1 - z_2\| \le \frac{\gamma}{1 - \gamma}\|v_1 - v_2\|. \tag{4.9}$$

Since $\|(L+\beta)^{-1}(I-P)\|_H \leq 1/8\|u\|$,

$$\begin{aligned}\|z_1 - z_2\|_H &= \left\| (L+\beta)^{-1}(I-P)\big(g(v_1+z_1) - g(v_2+z_2)\big) \right\|_H \\ &\leq (\|z_1 - z_2\| + \|v_1 - v_2\|) \\ &\leq \frac{1}{1-\gamma}\|v_1 - v_2\|_H.\end{aligned} \tag{4.10}$$

Therefore, θ is continuous on V with respect to norm $\|\cdot\|$ (also, to $\|\cdot\|_H$). □

Lemma 4.3. *If $\widetilde{J} : V \to \mathbb{R}$ is defined by $\widetilde{J}(v) = J(v+\theta(v))$, then $\widetilde{J}$ is a continuous Frechet derivative $D\widetilde{J}$ with respect to V and*

$$D\widetilde{J}(v)s = DJ(v+\theta(v))(s) \quad \forall\ s \in V. \tag{4.11}$$

If v_0 is a critical point of $\widetilde{J}$, then $v_0 + \theta(v_0)$ is a solution of (1.4) *and conversely every solution of* (1.4) *is of this form.*

Proof. Let $v \in V$ and set $z = \theta(v)$. If $w \in W$, then from (4.4)

$$\begin{aligned}\int_\Omega &\big(-\theta(v)_t w_t + \theta(v)_x w_x - b_1(v+\theta(v)+2)^+ w + 2b_1 w \\ &-b_2(v+\theta(v)+3)^+ w + 3b_2 w\big)\,dt\,dx = 0.\end{aligned} \tag{4.12}$$

Since $\int_\Omega v_t w_t = 0$ and $\int_\Omega v_x w_x = 0$,

$$DJ(v+\theta(v))(w) = 0 \quad \forall\ w \in W. \tag{4.13}$$

Let W_1, W_2 be the two subspaces of H defined as follows:

$$\begin{aligned} W_1 &= \text{closure of } \operatorname{span}\{\phi_{mn}, \psi_{mn} \mid \lambda_{mn} \leq -15\}, \\ W_2 &= \text{closure of } \operatorname{span}\{\phi_{mn}, \psi_{mn} \mid \lambda_{mn} \geq 1\}.\end{aligned} \tag{4.14}$$

Given $v \in V$ and considering the function h: $W_1 \times W_2 \to$ defined by

$$h(w_1, w_2) = J(v + w_1 + w_2), \tag{4.15}$$

the function h has continuous partial Fréchet derivatives $D_1 h$ and $D_2 h$ with respect to its first and second variables given by

$$\begin{aligned} D_1 h(w_1, w_2)(y_1) &= DJ(v+w_1+w_2)(y_1) \quad \text{for } y_1 \in W_1, \\ D_2 h(w_1, w_2)(y_2) &= DJ(v+w_1+w_2)(y_2) \quad \text{for } y_2 \in W_2.\end{aligned} \tag{4.16}$$

Therefore, let $\theta(v) = \theta_1(v) + \theta_2(v)$ with $\theta_1(v) \in W_1$ and $\theta_2(v) \in W_2$. Then by Lemma 4.2

$$\begin{aligned} D_1 h(\theta_1(v), \theta_2(v))(y_1) &= 0, \quad \text{for } y_1 \in W_1 \\ D_2 h(\theta_1(v), \theta_2(v))(y_2) &= 0, \quad \text{for } y_2 \in W_2. \end{aligned} \tag{4.17}$$

If $w_2, y_2 \in W_2$ and $w_1 \in W_1$, then

$$\begin{aligned} &\left[Dh(w_1, w_2) - Dh(w_1, y_2)\right](w_2 - y_2) \\ &\quad = (DJ(v + w_1 + w_2) - DJ(v + w_1 + y_2))(w_2 - y_2) \\ &\quad = \int_\Omega -\left|(w_2 - y_2)_t\right|^2 + \left|(w_2 - y_2)_{xx}^2\right| - b_1\left[(v + w_1 + w_2 + 2)^+\right. \\ &\qquad - (v + w_1 + y_2 + 2)^+ - b_2(v + w_1 + w_2 + 3)^+ \\ &\qquad \left. -(v + w_1 + y_2 + 3)^+\right](w_2 - y_2)\, dt\, dx. \end{aligned} \tag{4.18}$$

Since $(s^+ - t^+)\ (s - t) \geq 0$ for any $s, t \in \mathbb{R}$ and $-7 < b_1, b_2, b_1 + b_2 < -3$, it is easy to know that

$$\begin{aligned} &\int_\Omega -b_1\left[(v + w_1 + w_2 + 2)^+ - (v + w_1 + y_2 + 2)^+\right](w_2 - y_2) \\ &\quad - b_2\left[(v + w_1 + w_2 + 3)^+ - (v + w_1 + y_2 + 3)^+\right](w_2 - y_2)\, dx\, dt \geq 0. \end{aligned} \tag{4.19}$$

And

$$\int_\Omega \left[-\left|(w_2 - y_2)_t\right|^2 + (w_2 - y_2)_{xx}^2\right] dt\, dx = \|w_2 - y_2\|_H^2. \tag{4.20}$$

It follows that

$$(Dh(w_1, w_2) - Dh(w_1, y_2))(w_2 - y_2) \geq \|w_2 - y_2\|_H^2. \tag{4.21}$$

Therefore, h is strictly convex with respect to the second variable.

Similarly, using the fact that $-b_j(s^+ - t^+)(s - t) \leq -b_j(s - t)^2$ for any $s, t \in \mathbb{R}$, if w_1 and y_1 are in W_1 and $w_2 \in W_2$, then

$$\begin{aligned} &(D_1 h(w_1, w_2) - D_1 h(y_1, w_2))(w_1 - y_1) \\ &\quad \leq -\|w_1 - y_1\|_H^2 - b_1\|w_1 - y_1\|^2 - b_2\|w_1 - y_1\|^2 \\ &\quad \leq \left(-1 - \frac{b_1 + b_2}{7}\right)\|w_1 - y_1\|_H^2, \end{aligned} \tag{4.22}$$

where $-15 < b_1 + b_2 < -3$. Therefore, h is strictly concave with respect to the first variable. From (4.17), it follows that

$$J(v+\theta_1(v)+\theta_2(v)) \le J(v+\theta_1(v)+y_2) \quad \text{for any } y_2 \in W_2,$$
$$J(v+\theta_1(v)+\theta_2(v)) \ge J(v+y_1+\theta_2(v)) \quad \text{for any } y_1 \in W_1, \tag{4.23}$$

with equality if and only if $y_1 = \theta_1(v), y_2 = \theta_2(v)$.

Since h is strictly concave (convex) with respect to its first (second) variable, [9, Theorem 2.3] implies that $\widetilde{J}$ is C^1 with respect to v and

$$D\widetilde{J}(v)(s) = DJ(v+\theta(v))(s), \quad \text{any } s \in V. \tag{4.24}$$

Suppose that there exists $v_0 \in V$ such that $D\widetilde{J}(v_0) = 0$. From (4.24), it follows that $DJ(v_0+\theta(v_0))(v) = 0$ for all $v \in V$. Then by Lemma 4.2, it follows that $DJ(v_0+\theta(v_0))v = 0$ for any $v \in H$. Therefore, $u = v_0 + \theta(v_0)$ is a solution of (1.4).

Conversely, if u is a solution of (1.4) and $v_0 = Pu$, then $D\widetilde{J}(v_0)v = 0$ for any $v \in H$. □

Lemma 4.4. *Let $-15 < b_1,\ b_2 < -3$, and $-15 < b_1 + b_2 = b < -3$. Then there exists a small open neighborhood B of 0 in V such that $v = 0$ is a strict local minimum of $\widetilde{J}$.*

Proof. For $-15 < b_1,\ b_2 < -3$, and $-15 < b_1 + b_2 = b < -3$, problem (1.4) has a trivial solution $u_0 = 0$. Thus we have $0 = u_0 = v + \theta(v)$. Since the subspace W is orthogonal complement of subspace V, we get $v = 0$ and $\theta(v) = 0$. Furthermore, $\theta(0)$ is the unique solution of (4.4) in W for $v = 0$. The trivial solution u_0 is of the form $u_0 = 0 + \theta(0)$ and $I + \theta$, where I is an identity map on V, θ is continuous, it follows that there exists a small open neighborhood B of 0 in V such that if $v \in B$ then $v+\theta(v)+2 > 0$, $v+\theta(v)+3 > 0$. By Lemma 4.2, $\theta(0) = 0$ is the solution of (4.5) for any $v \in B$. Therefore, if $v \in B$, then for $z = \theta(v)$ we have $z = 0$. Thus

$$\begin{aligned}
\widetilde{J}(v) = J(v+z) &= \int_\Omega \Big[\frac{1}{2}\Big(-|(v+z)_t|^2 + |(v+z)_{xx}|^2\Big) - \frac{b_1}{2}\big|(v+z+2)^+\big|^2 \\
&\quad +2b_1(v+z) - \frac{b_2}{2}\big|(v+z+3)^+\big|^2 + 3b_2(v+z)\Big]\,dt\,dx \\
&= \int_\Omega \Big[\frac{1}{2}\Big(-|v_t|^2 + |v_{xx}|^2\Big) - \frac{b_1}{2}(v+1)^2 + b_1 v - \frac{b_2}{2}(v+2)^2 + 2b_2 v\Big]\,dt\,dx \\
&= \int_\Omega \Big[\frac{1}{2}\Big(-|v_t|^2 + |v_{xx}|^2\Big) - \frac{b_1}{2}v^2 - 2b_1 - \frac{b_2}{2}v^2 - \frac{9}{2}b_2\Big]\,dt\,dx.
\end{aligned} \tag{4.25}$$

If $v \in V$, then $Lv = -3v$. Therefore, in B,

$$\widetilde{J}(v) = \widetilde{J}(v) - \widetilde{J}(0) = \int_\Omega \Big[\frac{1}{2}\Big(-|v_t|^2 + |v_x|^2\Big) - \frac{b}{2}v^2\Big]\,dt\,dx = \frac{1}{2}(-3-b)\int_\Omega v^2 dt\,dx \ge 0, \tag{4.26}$$

where $-15 < b = b_1 + b_2 < -3$. It follows that $v = 0$ is a strict local point of minimum of $\widetilde{J}$. □

Proposition 4.5. *If $-15 < b < 1$, then the equation $Lu - bu^+ = 0$ admits only the trivial solution $u = 0$ in H_0.*

Proof. $H_1 = \text{span}\{\cos x \cos 2mt, m \geq 0\}$ is invariant under L and under the map $u \mapsto bu^+$. So the spectrum σ_1 of L restricted to H_1 contains $\lambda_{10} = -3$ in $(-15, 1)$. The spectrum σ_2 of L restricted to $H_2 = H_1^\perp$ contains $\lambda_{10} = -3$ in $(-15, 1)$. From the symmetry theorem in [10], any solution $y(t)\cos x$ of this equation satisfies $y'' + y - by^+ = 0$. This nontrivial periodic solution is periodic with periodic $\pi + (\pi/\sqrt{-b+1}) \neq \pi$. This shows that there is no nontrivial solution of $Lv - bv^+ = 0$. □

Lemma 4.6. *Let $b = b_1 + b_2$ and $-15 < b_1,\ b_2,\ b < -3$. Then the functional $\widetilde{J}$, defined on V, satisfies the Palais-Smale condition.*

Proof. Let $\{v_n\} \subset V$ be a Palais-Smale sequence that is $\widetilde{J}(v_n)$ is bounded and $D\widetilde{J}(v_n) \to 0$ in V. Since V is two-dimensional, it is enough to prove that $\{v_n\}$ is bounded in V.

Let u_n be the solution of (1.4) with $u_n = v_n + \theta(v_n)$ where $v_n \in V$. So

$$Lu_n - b_1(u_n + 2)^+ + 2b_1 - b_2(u_n + 3)^+ + 3b_2 = DJ(u_n) \quad \text{in } H. \tag{4.27}$$

By contradiction, we suppose that $\|v_n\| \to +\infty$, also $\|u_n\| \to +\infty$. Dividing by $\|u_n\|$ and taking $w_n = \dfrac{u_n}{\|u_n\|}$, we get

$$Lw_n - b_1\left(w_n + \frac{2}{\|u_n\|}\right)^+ + \frac{2b_1}{\|u_n\|} - b_2\left(w_n + \frac{3}{\|u_n\|}\right)^+ + \frac{3b_2}{\|u_n\|} = \frac{(DJ(u_n))}{\|u_n\|} \longrightarrow 0. \tag{4.28}$$

Since $\|w_n\| = 1$, we get $w_n \to w_0$ weakly in H_0. Since L^{-1} is a compact operator, passing to a subsequence, we get $w_n \to w_0$ strongly in H_0. Taking the limit of both sides of (4.28), it follows that

$$Lw_0 - bw_0^+ = 0, \tag{4.29}$$

with $\|w_0\| \neq 0$. This contradicts to the fact that for $-15 < b < -3$ the following equation

$$Lu - bu^+ = 0 \quad \text{in } H_0 \tag{4.30}$$

has only the trivial solution by Proposition 4.5. Hence $\{v_n\}$ is bounded in V. □

We now define the functional on H, for $-15 < b < -3$,

$$J^*(u) = \int_\Omega \left[-\frac{1}{2}\left(-|u_t|^2 + |u_x|^2\right) - \frac{b}{2}|u^+|^2 dx\, dt.\right. \tag{4.31}$$

The critical points of $J^*(u)$ coincide with solutions of the equation

$$Lu - bu^+ = 0 \quad \text{in } H_0. \tag{4.32}$$

The above equation ($-15 < b < -3$) has only the trivial solution and hence $J^*(u)$ has only one critical point $u = 0$.

Given $v \in V$, let $\theta^*(v) = \theta(v) \in W$ be the unique solution of the equation

$$Lz + (I - P)\left[-b_1(v + z + 2)^+ + 2b_1 - b_2(v + z + 3)^+ + 3b_2\right] = 0 \quad \text{in } W, \tag{4.33}$$

where $-15 < b_1,\ b_2,\ b_1 + b_2 = b < -3$. Let us define the reduced functional $\widetilde{J}^*(v)$ on V by $J(v + \theta^*(v))$. We note that we can obtain the same results as Lemmas 4.1 and 4.2 when we replace $\theta(v)$ and $\widetilde{J}(v)$ by $\theta^*(v)$ and $\widetilde{J}^*(v)$. We also note that, for $-15 < b < -3$, $\widetilde{J}^*(v)$ has only the critical point $v = 0$.

Lemma 4.7. *Let $-15 < b_1,\ b_2 < -3$, $b = b_1 + b_2$, and $-15 < b < -3$. Then we have $\widetilde{J}^*(v) < 0$ for all $v \in V$ with $v \neq 0$.*

The proof of the lemma can be found in [1].

Lemma 4.8. *Let $-15 < b_1, b_2 < -3$, $b = b_1 + b_2$, and $-15 < b < -3$. Then we have*

$$\lim_{\|v\| \to \infty} \widetilde{J}(v) \longrightarrow -\infty \tag{4.34}$$

for all $v \in V$ (certainly for also the norm $\|\cdot\|_H$).

Proof. Suppose that it is not true that

$$\lim_{\|v\| \to \infty} \widetilde{J}(v) \longrightarrow -\infty. \tag{4.35}$$

Then there exists a sequence (v_n) in V and a constant C such that

$$\begin{aligned} &\lim_{n \to \infty} \|v_n\| \longrightarrow \infty, \\ &\widetilde{J}(v_n) = \int_\Omega \bigg(\frac{1}{2} L(v_n + \theta(v_n)) \cdot (v_n + \theta(v_n)) - \frac{b_1}{2} \left|(v_n + \theta(v_n) + 2)^+\right|^2 + 2b_1(v_n + \theta(v_n)) \\ &\qquad - \frac{b_2}{2} \left|(v_n + \theta(v_n) + 3)^+\right|^2 + 3b_1(v_n + \theta(v_n))\,dt\,dx \geq C \bigg). \end{aligned} \tag{4.36}$$

Given $v_n \in V$, let $w_n = \theta(v_n)$ be the unique solution of the equation

$$Lw + (I - P)\left[-b_1(v_n + w + 2)^+ + 2b_1 - b_2(v_n + w + 3)^+ + 3b_2\right] = 0 \quad \text{in } W. \tag{4.37}$$

Let $z_n = v_n + w_n$, $v_n^* = v_n/\|v_n\|$, and $z_n^* = z_n/\|v_n\|$. Then $z_n^* = v_n^* + w_n^*$. By dividing $\|v_n\|$, we have

$$\begin{aligned} w_n^* = {} & L^{-1}(I-P)\left(b_1\left(\frac{v_n+w_n+2}{\|v_n\|}\right)^+ - \frac{2b_1}{\|v_n\|}\right) \\ & + L^{-1}(I-P)\left(b_2\left(\frac{v_n+w_n+3}{\|v_n\|}\right)^+ - \frac{3b_2}{\|v_n\|}\right) \quad \text{in } W. \end{aligned} \tag{4.38}$$

By Lemma 4.2, $w_n = \theta(v_n)$ is Lipschitz continuous on V. So the sequence $\{w_n + v_n/\|v_n\|\}$ is bounded in H. Since $\lim_{n\to\infty}(1/\|v_n\|) = 0$ and $\lim_{n\to\infty}(b_j/\|v_n\|) = 0$ $(j = 1,\ 2)$, it follows that $b_1(v_n + w_n + 2/\|v_n\|)^+ - 2b_1/\|v_n\|$ and $b_2((v_n + w_n + 3)/\|v_n\|)^+ - 3b_2/\|v_n\|$ are bounded in H. Since L^{-1} is a compact operator, there is a subsequence of w_n^* converging to some w^* in W, denoted by itself. Since V is a two-dimensional space, assume that sequence (v_n^*) converges to $v^* \in V$ with $\|v^*\| = 1$. Therefore, we can get that the sequence (z_n^*) converges to an element z^* in H.

On the other hand, since $\widetilde{J}(v_n) \geq C$, dividing this inequality by $\|v_n\|^2$, we get

$$\begin{aligned} \int_\Omega & \frac{1}{2}L(z_n^*)\cdot z_n^* - \frac{b_1}{2}\left(\left(z_n^* + \frac{2}{\|v_n\|}\right)^+\right)^2 + 2b_1\frac{z_n^*}{\|v_n\|} \\ & - \frac{b_2}{2}\left(\left(z_n^* + \frac{3}{\|v_n\|}\right)^+\right)^2 + 3b_3\frac{z_n^*}{\|v_n\|}\,dt\,dx \geq \frac{C}{\|v_n\|^2}. \end{aligned} \tag{4.39}$$

By Lemma 4.2, it follows that for any $y \in W$

$$\int_\Omega \left[-(z_n)_t y_t + (z_n)_x y_x - b_1(z_n+2)^+ y + 2b_1 y - b_2(z_n+3)^+ y + 3b_2 y\right] dt\,dx = 0. \tag{4.40}$$

If we set $y = w_n$ in (4.40) and divide by $\|v_n\|^2$, then we obtain

$$\int_\Omega \left[-\left|(w_n^*)_t\right|^2 + \left|(w_n^*)_{xx}\right|^2 - b_1(z_n^*)^+ w_n^* + \frac{2b_1}{\|v_n\|}w_n^* - b_2(z_n^*)^+ w_n^* + \frac{3b_2}{\|v_n\|}w_n^*\right] dt\,dx = 0. \tag{4.41}$$

Let $y \in W$ be arbitrary. Dividing (4.40) by $\|v_n\|$ and letting $n \to \infty$, we obtain

$$\int_\Omega \left[-(z^*)_t y_t + (z^*)_{xx} y_{xx} - b(z^*)^+ y\right] dt\,dx = 0, \tag{4.42}$$

where $b = b_1 + b_2$. Then (4.42) can be written in the form $D\widetilde{J}^*(v^* + w^*)(y) = 0$ for all $y \in W$. Put $w^* = \theta(v^*)$. Letting $n \to \infty$ in (4.41), we obtain

$$\begin{aligned}
\lim_{n\to\infty} \int_\Omega &\left(-\left|(w_n^*)_t\right|^2 + \left|(w_n^*)_{xx}\right|^2\right) dt\,dx \\
&= \lim_{n\to\infty} \int_\Omega b(z_n^*)^+ w_n^* - \frac{b}{\|v_n\|} w_n^* dt\,dx \\
&= \int_\Omega b(z^*)^+ w^* dtdz \\
&= \int_\Omega (-(z^*)_t (w^*)_t + (z^*)_x (w^*)_{xx}) dt\,dx \\
&= \int_\Omega \left(-|(w^*)_t|^2 + |(w^*)_{xx}|^2\right) dt\,dx,
\end{aligned} \tag{4.43}$$

where we have used (4.42). Hence

$$\lim_{n\to\infty} \int_\Omega \left[-\left|(z_n^*)_t\right|^2 + \left|(z_n^*)_{xx}\right|^2\right] dt\,dx = \int_\Omega \left[-|(z^*)_t|^2 + |(z^*)_{xx}|^2\right] dt\,dx. \tag{4.44}$$

Letting $n \to \infty$ in (4.39), we obtain

$$\widetilde{J}^*(v^*) = \int_\Omega \left[\frac{1}{2}\left(-|(z^*)_t|^2 + |(z^*)_{xx}|^2\right) + \frac{b}{2}\left|(z^*)^+\right|^2\right] dt\,dx \geq 0. \tag{4.45}$$

Since $\|v^*\| = 1$, this contradicts to the fact that $\widetilde{J}^*(v) < 0$ for all $v \neq 0$. This proves that $\lim_{\|v\|\to\infty} \widetilde{J}(v) \to -\infty$. □

Now we state the main result in this paper.

Theorem 4.9. *Let $-15 < b_1,\ b_2 < -3$, $b = b_1 + b_2$, and $-15 < b < -3$. Then there exist at least three solutions of the equation*

$$\begin{gathered}
u_{tt} + u_{xxxx} = b_1\left[(u+2)^+ - 2\right] + b_2\left[(u+3)^+ - 3\right] \quad \text{in } \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \times \mathbb{R}, \\
u\left(\pm\frac{\pi}{2}, t\right) = 0, \qquad u(x, t+\pi) = u(x, t),
\end{gathered} \tag{4.46}$$

two of which are nontrivial solutions.

Proof. We remark that $u = 0$ is the trivial solution of problem (1.4). Then $v = 0$ is a critical point of functional $\widetilde{J}$. Next we want to find others critical points of $\widetilde{J}$ which are corresponding to the solutions of problem (1.4).

By Lemma 4.4, there exists a small open neighborhood B of 0 in V such that $v = 0$ is a strict local point of minimum of $\widetilde{J}$. Since $\lim_{\|v\|_H \to \infty} \widetilde{J}(v) \to -\infty$ from Lemma 4.8 and V is a two-dimensional space, there exists a critical point $v_0 \in V$ of $\widetilde{J}$ such that

$$\widetilde{J}(v_0) = \max_{v \in V} \widetilde{J}(v). \tag{4.47}$$

Let B_{v_0} be an open neighborhood of v_0 in V such that $B \cap B_{v_0} = \emptyset$. Since $\lim_{\|v\|_H \to \infty} \widetilde{J}(v) \to -\infty$, we can choose $v_1 \in V \setminus (B \cup B_{v_0})$ such that $\widetilde{J}(v_1) < \widetilde{J}(0)$. Since $\widetilde{J}$ satisfies the Palais-Smale condition, by the Mountain Pass Theorem (Theorem 3.3), there is a critical value

$$c = \inf_{\gamma \in \Gamma} \sup_{\gamma} \widetilde{J}(v), \tag{4.48}$$

where $\Gamma = \{\gamma \in C([0,1], E) \mid \gamma(0) = 0, \gamma(1) = v_0\}$.

If $\widetilde{J}(v_0) \neq c$, then there exists a critical point v of $\widetilde{J}$ at level c such that $v \neq v_0$, 0 (since $c \neq \widetilde{J}(v_0)$ and $c > \widetilde{J}(0)$). Therefore, in case $\widetilde{J}(v_0) \neq c$, the functional $\widetilde{J}(v)$ has also at least 3 critical points $0, v_0, v$.

If $\widetilde{J}(v_0) = c$, then define

$$c' = \inf_{\gamma \in \Gamma'} \sup_{\gamma} \widetilde{J}(v), \tag{4.49}$$

where $\Gamma' = \{\gamma \in \Gamma : \gamma \cap B_{v_0} = \emptyset\}$. Hence,

$$c = \inf_{\gamma \in \Gamma} \sup_{\gamma} \widetilde{J}(v) \leq \inf_{\gamma \in \Gamma'} \sup_{\gamma} \widetilde{J}(v) \leq \max_{v \in V} \widetilde{J}(v) = c. \tag{4.50}$$

That is $c = c'$. By contradiction, assume $K_c = \{v \in V \mid \widetilde{J}(v) = c \text{ and } D\widetilde{J}(v) = 0\} = \{v_0\}$. Use the functional $\widetilde{J}$ for the deformation theorem (Theorem 4.9) and taking $\epsilon < (1/2)(c - \widetilde{J}(0))$. We choose $\gamma \in \Gamma'$ such that $\sup_{\gamma} \widetilde{J} \leq c$. From the deformation theorem (Theorem 3.2), $\eta(1, \cdot) \circ \gamma \in \Gamma$ and

$$c = \inf_{\gamma \in \Gamma} \sup_{\gamma} \widetilde{J}(v) \leq \sup_{\eta(1,\cdot)\circ\gamma} \widetilde{J}(v) \leq c - \epsilon, \tag{4.51}$$

which is a contradiction. Therefore, there exists a critical point v of $\widetilde{J}$ at level c such that $v \neq v_0$, 0, which means that (1.4) has at least three critical points. Since $\|v\|_H$ and $\|v_0\|_H \neq 0$, these two critical points coincide with two nontrivial period solutions of problem (1.4). □

5. Nontrivial Solutions for the Beam System

In this section, we investigate the existence of multiple nontrivial solutions (ξ, η) for perturbations $b_1[(\xi + 3\eta + 2)^+ - 2]$ and $b_2[(\xi + 3\eta + 3)^+ - 3]$ of the beam system with Dirichlet boundary condition

$$
\begin{aligned}
L\xi &= b_1\left[(\xi + 3\eta + 2)^+ - 2\right] \quad \text{in } \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \times \mathbb{R}, \\
L\eta &= b_2\left[(\xi + 3\eta + 3)^+ - 3\right] \quad \text{in } \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \times \mathbb{R}, \\
\xi\left(\pm\frac{\pi}{2}, t\right) &= \xi_{xx}\left(\pm\frac{\pi}{2}, t\right) = 0, \\
\xi(x, t + \pi) &= \xi(x, t) = \xi(-x, t), \\
\eta\left(\pm\frac{\pi}{2}, t\right) &= \eta_{xx}\left(\pm\frac{\pi}{2}, t\right) = 0, \\
\eta(x, t + \pi) &= \eta_{xx}(x, t) = \eta(-x, t),
\end{aligned}
\tag{5.1}
$$

where $u^+ = \max\{u, 0\}$ and the nonlinearity $(b_1[(u + 2)^+ - 2] + 3b_2[(u + 3)^+ - 3]$ crosses the eigenvalues of the beam operator.

Theorem 5.1. *Let $-15 < b_1,\ 3b_2 < -3$, $b = b_1 + 3b_2$, and $-15 < b < -3$. Then beam system* (5.1) *has at least three solutions (ξ, η), two of which are nontrivial solutions.*

Proof. From problem (5.1), we get the equation

$$
\begin{aligned}
L(\xi + 3\eta) &= g(\xi + 3\eta + 2) \quad \text{in } \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \times \mathbb{R}, \\
\xi\left(\pm\frac{\pi}{2}, t\right) &= \xi_{xx}\left(\pm\frac{\pi}{2}, t\right) = 0, \\
\xi(x, t + \pi) &= \xi(x, t) = \xi(-x, t), \\
\eta\left(\pm\frac{\pi}{2}, t\right) &= \eta_{xx}\left(\pm\frac{\pi}{2}, t\right) = 0, \\
\eta(x, t + \pi) &= \eta_{xx}(x, t) = \eta(-x, t),
\end{aligned}
\tag{5.2}
$$

where the nonlinearity $g(u) = b_1[(u + 2)^+ - 2] + 3b_2[(u + 3)^+ - 3]$.

Let $w = \xi + 3\eta$. Then the above equation is equivalent to

$$
\begin{aligned}
Lw &= b_1\left[(w + 2)^+ - 2\right] + 3b_2\left[(w + 3)^+ - 3\right] \quad \text{in } \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \times \mathbb{R}, \\
w\left(\pm\frac{\pi}{2}, t\right) &= w_{xx}\left(\pm\frac{\pi}{2}, t\right) = 0, \\
w(x, t + \pi) &= w(x, t) = w(-x, t).
\end{aligned}
\tag{5.3}
$$

Since $-15 < b_1,\ 3b_2 < -3$, $b = b_1 + 3b_2$, and $-15 < b < -3$, the above equation has at least three solutions, two of which are nontrivial solutions, say w_1, w_2. Hence we get the solutions (ξ, η) of problem (5.1) from the following systems:

$$
\begin{gathered}
L\xi = b_1\left[w_i + 2)^+ - 2\right] \quad \text{in } \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \times \mathbb{R}, \\
L\eta = b_2\left[(w_i + 3)^+ - 3\right] \quad \text{in } \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \times \mathbb{R}, \\
\xi\left(\pm\frac{\pi}{2}, t\right) = \xi_{xx}\left(\pm\frac{\pi}{2}, t\right) = 0, \\
\xi(x, t + \pi) = \xi(x, t) = \xi(-x, t), \\
\eta\left(\pm\frac{\pi}{2}, t\right) = \eta_{xx}\left(\pm\frac{\pi}{2}, t\right) = 0, \\
\eta(x, t + \pi) = \eta_{xx}(x, t) = \eta(-x, t),
\end{gathered}
\tag{5.4}
$$

where $i = 0, 1, 2$ and $w_0 = 0$. When $i = 0$, from the above equation, we get the trivial solution $(\xi, \eta) = (0, 0)$. When $i = 1, 2$, from the above equation, we get the nontrivial solutions (ξ_1, η_1), (ξ_2, η_2).Therefore, system(5.1) has at least three solutions (ξ, η), two of which are nontrivial solutions. □

Theorem 5.2. *Let* $-3 < b_1,\ 3b_2 < 1$, *and* $-3 < b_1 + 3b_2 < 1$. *Then system* (5.1) *has only the trivial solution* $(\xi, \eta) = (0, 0)$.

Proof. From problem (5.1), we get the equation

$$
\begin{gathered}
L(\xi + 3\eta) = g(\xi + 3\eta + 2) \quad \text{in } \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \times \mathbb{R}, \\
\xi\left(\pm\frac{\pi}{2}, t\right) = \xi_{xx}\left(\pm\frac{\pi}{2}, t\right) = 0, \\
\xi(x, t + \pi) = \xi(x, t) = \xi(-x, t), \\
\eta\left(\pm\frac{\pi}{2}, t\right) = \eta_{xx}\left(\pm\frac{\pi}{2}, t\right) = 0, \\
\eta(x, t + \pi) = \eta_{xx}(x, t) = \eta(-x, t),
\end{gathered}
\tag{5.5}
$$

where the nonlinearity $g(u) = b_1[(u + 2)^+ - 2] + 3b_2[(u + 3)^+ - 3]$.

Let $w = \xi + 3\eta$. Then the above equation is equivalent to

$$
\begin{gathered}
Lw = b_1\left[(w + 2)^+ - 2\right] + 3b_2\left[(w + 3)^+ - 3\right] \quad \text{in} \quad \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \times \mathbb{R}, \\
w\left(\pm\frac{\pi}{2}, t\right) = w_{xx}\left(\pm\frac{\pi}{2}, t\right) = 0, \\
w(x, t + \pi) = w(x, t) = w(-x, t).
\end{gathered}
\tag{5.6}
$$

Since $-3 < b_1,\ 3b_2 < 1$, and $-3 < b_1 + 3b_2 < 1$, by Theorem 2.2, the above equation has the trivial solution. Hence we have the trivial solution $(\xi, \eta = (0,0)$ of problem (5.1) from the following system:

$$\begin{gathered}
L\xi = b_1\left[0+2)^+ - 2\right] \quad \text{in } \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \times \mathbb{R}, \\
L\eta = b_2\left[(0+3)^+ - 3\right] \quad \text{in } \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \times \mathbb{R}, \\
\xi\left(\pm\frac{\pi}{2}, t\right) = \xi_{xx}\left(\pm\frac{\pi}{2}, t\right) = 0, \\
\xi(x, t+\pi) = \xi(x,t) = \xi(-x,t), \\
\eta\left(\pm\frac{\pi}{2}, t\right) = \eta_{xx}\left(\pm\frac{\pi}{2}, t\right) = 0, \\
\eta(x, t+\pi) = \eta_{xx}(x,t) = \eta(-x,t).
\end{gathered} \tag{5.7}$$

From (5.9), we get the trivial solution $(\xi, \eta) = (0,0)$. □

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