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Research Article

Quenching for a Reaction-Diffusion System with Coupled Inner Singular Absorption Terms

Shouming Zhou and Chunlai Mu

College of Mathematics and Statistics, Chongqing University, Chongqing 400044, China

Correspondence should be addressed to Shouming Zhou, zhoushouming76@163.com

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we devote to investigate the quenching phenomenon for a reaction-diffusion system with coupled singular absorption terms, $u_t = \Delta u - u^{-p_1}v^{-q_1}$, $v_t = \Delta v - u^{-p_2}v^{-q_2}$. The solutions of the system quenches in finite time for any initial data are obtained, and the blow-up of time derivatives at the quenching point is verified. Moreover, under appropriate hypotheses, the criteria to identify the simultaneous and nonsimultaneous quenching are found, and the four kinds of quenching rates for different nonlinear exponent regions are given. Finally, some numerical experiments are performed, which illustrate our results.

1. Introduction

This paper deals with the following nonlinear parabolic equations with null Neumann boundary conditions:

$$\begin{gathered} u_t = \Delta u - u^{-p_1}v^{-q_1}, \quad v_t = \Delta v - u^{-p_2}v^{-q_2}, \quad (x,t) \in \Omega \times (0,T), \\ \frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0, \quad (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \quad x \in \overline{\Omega}, \end{gathered} \tag{1.1}$$

where $p_i, q_i \geq 0$ for $i = 1, 2$, $\Omega \subset R^n$ is a bounded domain with smooth boundary, the initial data u_0 and v_0 are positive, smooth, and compatible with the boundary data.

Because of the singular nonlinearity inner absorption terms of (1.1), the so-called finite-time quenching may occur for the model. We say that the solution (u, v) of the problem

(1.1) quenches, if there exists a time $t = T < \infty$ (T denotes the quenching time, x denotes quenching point), such that

$$\lim_{t\to T^-} \inf \min\left\{\min_{\overline{\Omega}} u(x,t), \min_{\overline{\Omega}} v(x,t)\right\} = 0. \tag{1.2}$$

For a quenching solution (u, v) of (1.1), the inf norm of one of the components must tend to 0 as t tends to the quenching time T. The case when u quenches and v remains bounded from zero is called non-simultaneous quenching. We will call the case, when both components u and v quench at the same time, as simultaneous quenching. The purpose of this paper is to find a criteria to identify simultaneous and non-simultaneous quenching for (1.1) and then establish quenching rates for the different cases.

In order to motivate the main results for system (1.1), we recall some classical results for the related system. de Pablo et al., firstly distinguished non-simultaneous quenching from simultaneous one in [1]. They considered a heat system coupled via inner absorptions as follows:

$$\begin{gathered} u_t = u_{xx} - v^{-p}, \quad v_t = v_{xx} - u^{-q}, \quad (x,t) \in (0,1)\times(0,T), \\ u_x(0,t) = u_x(1,t) = v_x(0,t) = v_x(1,t) = 0, \quad t \in (0,T), \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \quad x \in [0,1]. \end{gathered} \tag{1.3}$$

Recently, Zheng and Wang deduced problem (1.3) to n-dimensional with positive Dirichlet boundary condition in [2]. Then, Zhou et al. have given a natural continuation for problem (1.3) beyond quenching time T for the case of non-simultaneous quenching in [3].

Replacing the coupled inner absorptions in (1.1) by the coupled boundary fluxes, one gets

$$\begin{gathered} u_t = u_{xx}, \quad v_t = v_{xx}, \quad (x,t) \in (0,1)\times(0,T), \\ u_x(0,t) = \left(u^{-p_1}v^{-q_1}\right)(0,t), \quad v_x(0,t) = \left(u^{-p_2}v^{-q_2}\right)(0,t), \quad t \in (0,T), \\ u_x(1,t) = 0, \quad v_x(1,t) = 0, \quad t \in (0,T), \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \quad x \in [0,1]. \end{gathered} \tag{1.4}$$

Recently, the simultaneous and non-simultaneous quenching for problem (1.4), and what is related to it, was studied by many authors (see [4–7] and references therein).

In order to investigate the problem (1.1), it is necessary to recall the blow-up problem of the following reaction-diffusion system:

$$\begin{gathered} u_t = \Delta u + u^{p_1}v^{q_1}, \quad v_t = \Delta v + u^{p_2}v^{q_2}, \quad (x,t) \in \Omega\times(0,T), \\ u = v = 0, \quad (x,t) \in \partial\Omega\times(0,T), \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \quad x \in \overline{\Omega}, \end{gathered} \tag{1.5}$$

with positive powers p_i, q_i $(i = 1,2)$ has been extensively studied by many authors for various problems such as global existence and finite time blow-up, Fujita exponents, non-simultaneous and simultaneous blow-up, and blow-up rates, (see [8–10] and references therein). However, unlike the blow-up problem, there are less papers consider the weakly coupled quenching problem like (1.1), differently from the generally considered, there are two additional singular factors, namely, $-v^{-p}$ and $-u^{-q}$ for the inner absorptions of u and v, respectively. In this paper, we will show real contributions of the two additional singular factors to the quenching behavior of solutions. Our main results are stated as follows.

Theorem 1.1. *If* $p_1, p_2, q_1, q_2 \geq 0$ *and* $p_1 + p_2 + q_1 + q_2 \neq 0$, *then the solution of the system* (1.1) *quenches in finite time for every initial data.*

On the other hand, some authors understand quenching as blow-up of time derivatives while the solution itself remains bounded (see [11–13]). In present paper, we assume that the initial data satisfy

$$\Delta u_0 - u_0^{-p_1} v_0^{-q_1} < 0, \quad \Delta v_0 - u_0^{-p_2} v_0^{-q_2} < 0, \quad x \in \Omega. \tag{1.6}$$

Theorem 1.2. *Let* $\Omega = B_R = \{x \in \mathbb{R}^N : |x| < R\}$ *and the radial initial function satisfies* (1.6), *then* (u_t, v_t) *blows up in finite time.*

Next, we characterize the ranges of parameters to distinguish simultaneous and non-simultaneous quenching. In order to simplify our work, we deal with the radial solutions of (1.1) with $\Omega = B_R = \{x \in R^n : |x| < R\}$, and the radial increasing initial data satisfies (1.6). Thus we, see that $x = 0$ is the only quenching point (see [2, 14]). Without loss of generality, we only consider the non-simultaneous quenching with u remaining strictly positive, and our main results are stated as follows.

Theorem 1.3. *If* $p_2 \geq p_1 + 1$ *and* $q_1 \geq q_2 + 1$, *then any quenching in* (1.1) *must be simultaneous.*

Theorem 1.4. *If* $p_2 \geq p_1 + 1$ *and* $q_1 < q_2 + 1$, *then any quenching in* (1.1) *is non-simultaneous with* u *being strictly positive.*

Theorem 1.5. *If* $p_2 < p_1 + 1$ *and* $q_1 < q_2 + 1$, *then both simultaneous and non-simultaneous quenching may occur in* (1.1) *depending on the initial data.*

Remark 1.6. In particular, if we choose $p_1 = q_2 = 0$, $q_1, p_2 > 0$, then we obtain that the ranges of parameters to distinguish simultaneous and non-simultaneous quenching coincide with the problem (1.3) (see [1, 2]). Moreover, this criteria to identify the simultaneous and non-simultaneous quenching is the same with the problem (1.4) which coupled boundary fluxes (see [6]). This situation also happens for the blow-up problem (see [8, 10, 15]).

Next, we deal with quenching rates. To state our results more conveniently, we introduce the notation $f \sim g$ which means that there exist two finite positive constants c_1, c_2 such that $c_1 g \leq f \leq c_2 g$, and the two parameters α and β verifying

$$\begin{pmatrix} p_1 + 1 & q_1 \\ p_2 & q_2 + 1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \tag{1.7}$$

or equivalently,

$$\alpha = \frac{q_1 - q_2 - 1}{p_2 q_1 - (p_1+1)(q_2+1)}, \qquad \beta = \frac{p_2 - p_1 - 1}{p_2 q_1 - (p_1+1)(q_2+1)}. \tag{1.8}$$

In terms of parameters α and β, the quenching rates of problem (1.1) can be shown as follow.

Theorem 1.7. *If quenching is non-simultaneous and, for instance, v is the quenching variable, then $v(0,t) \sim (T-t)^{1/(q_2+1)}$ as $t \to T$.*

Theorem 1.8. *If quenching is simultaneous, then for t close to T, we have*

(i) $u(0,t) \sim (T-t)^{\alpha}$, $v(0,t) \sim (T-t)^{\beta}$ *for* $p_2 < p_1+1$, $q_1 < q_2+1$ *or* $p_2 > p_1+1$, $q_1 > q_2+1$;

(ii) $u(0,t), v(0,t) \sim (T-t)^{p_1+q_1+1}$ *for* $p_2 = p_1+1$ *and* $q_1 = q_2+1$;

(iii) $u(0,t) \sim (T-t)^{1/(p_1+1)} |\log(T-s)|^{-q_1/(q_2-q_1+1)(p_1+1)}$, $v(0,t) \sim |\log(T-t)|^{1/(q_2-q_1+1)}$ *for* $p_2 = p_1+1$ *and* $q_1 > q_2+1$.

The plan of this paper is organized as follows. In Section 2, we distinguish non-simultaneous quenching from simultaneous one. The four kinds of non-simultaneous and simultaneous quenching rates for different nonlinear exponent regions are given in Section 3. In the Section 4, we perform some numerical experiments which illustrate our results.

2. Simultaneous and Non-Simultaneous Quenching

Proof of Theorem 1.1. Assume that (u,v) is the classical solution of (1.1) with the maximal existence time T. The maximum principle implies $0 < u \le M := \|u_0(x)\|_{L^\infty}$ and $0 < v \le N := \|v_0(x)\|_{L^\infty}$ in $\Omega \times (0,T)$. Let $F(t) = \int_\Omega u(x,t)dx$, $G(t) = \int_\Omega v(x,t)dx$, $t \in [0,T)$. Hence, integrating (1.1) in space and using Green's formula, we have

$$F'(t) = \int_\Omega \Delta u - u^{-p_1} v^{-q_1} dx \le -M^{-p_1} N^{-q_1}, \qquad G'(t) \le -M^{-p_2} N^{-q_2}. \tag{2.1}$$

Consequently,

$$F(t) \le |\Omega| M - M^{-p_1} N^{-q_1} t, \qquad G(t) \le |\Omega| N - M^{-p_2} N^{-q_2} t. \tag{2.2}$$

Thus, the solution of the problem (1.1) quenches in finite time. The prove of Theorem 1.1 is complete. □

In order to prove Theorem 1.2, we need the following Lemma.

Lemma 2.1. *Assume that $\Omega = B_R = \{x \in \mathbb{R}^N : |x| < R\}$ and the radial nondecreasing initial data satisfy* (1.6), *then there exists a small $\delta > 0$ such that*

$$u_t < -\delta u^{-p_1} v^{-q_1}, \quad v_t < -\delta u^{-p_2} v^{-q_2}, \quad (x,t) \in \Omega \times (0,T). \tag{2.3}$$

Proof. Let $I = u_t + \delta u^{-p_1} v^{-q_1}$, $J = v_t + \delta u^{-p_2} v^{-q_2}$, $(x,t) \in \Omega \times (0,T)$. Thus,

$$\begin{aligned} I_t - \Delta I = {} & p_1 u^{-p_1-1} v^{-q_1} I + q_1 u^{-p_1} v^{-q_1-1} J - \delta p_1 (p_1+1) u^{-p_1-2} v^{-q_1} |\nabla u|^2 \\ & - \delta q_1 (q_1+1) u^{-p_1} v^{-q_1-2} |\nabla v|^2 - 2\delta p_1 q_1 u^{-p_1-1} v^{-q_1-1} \nabla u \cdot \nabla v. \end{aligned} \tag{2.4}$$

Since u and v are radial and nondecreasing in $|x|$, we have $\nabla u \cdot \nabla v = u_r v_r \geq 0$. A similar computation holds for J, and we obtain

$$\begin{aligned} I_t - \Delta I &\leq p_1 u^{-p_1-1} v^{-q_1} I + q_1 u^{-p_1} v^{-q_1-1} J, \quad (x,t) \in \Omega \times (0,T), \\ J_t - \Delta J &\leq p_2 u^{-p_2-1} v^{-q_2} I + q_2 u^{-p_2} v^{-q_2-1} J, \quad (x,t) \in \Omega \times (0,T), \end{aligned} \tag{2.5}$$

with boundary conditions

$$\frac{\partial I}{\partial n} = \frac{\partial J}{\partial n} = 0, \quad (x,t) \in \partial\Omega \times (0,T). \tag{2.6}$$

From (1.6), it is easy to deduce $u_t(x,0), v_t(x,0) \leq -\theta < 0$ in Ω (see [13, 14]). Choosing δ small enough, we have that the initial data verifying

$$\begin{aligned} I(x,0) &= u_t(x,0) + \delta u^{-p_1} v^{-q_1}(x,0) \leq 0, \quad x \in \partial\Omega, \\ J(x,0) &= v_t(x,0) + \delta u^{-p_2} v^{-q_2}(x,0) \leq 0, \quad x \in \partial\Omega. \end{aligned} \tag{2.7}$$

Hence, by the comparison result, we derive that

$$\begin{aligned} I(x,t) &= u_t(x,t) + \delta u^{-p_1} v^{-q_1}(x,t) \leq 0, \\ J(x,t) &= v_t(x,t) + \delta u^{-p_2} v^{-q_2}(x,t) \leq 0, \end{aligned} \quad (x,t) \in \Omega \times (0,T). \tag{2.8}$$

This proves Lemma 2.1. □

Proof of Theorem 1.2. This theorem is the direct result of Theorem 1.1 and Lemma 2.1. □

Next, we characterize the ranges of parameters to distinguish simultaneous and non-simultaneous quenching. By the hypothesis on the initial data, we obtain $\min_{x\in\overline{B}_R} u(x,t) = u(0,t)$, $\min_{x\in\overline{B}_R} v(x,t) = v(0,t)$ and $u_t(0,t) \geq -u^{-p_1} v^{-p_1}(0,t)$, $v_t(0,t) \geq -u^{-p_2} v^{-p_2}(0,t)$ for $t \in [0,T)$ (see [2, 14]). We collect the estimates of the time derivatives obtained before. Clearly, the only quenching point is $x = 0$ (see [2]), we only care for the original point,

$$-u^{-p_1} v^{-p_1}(0,t) \leq u_t(0,t) \leq -\delta u^{-p_1} v^{-p_1}(0,t), \quad t \in (0,T), \tag{2.9}$$

$$-u^{-p_2} v^{-p_2}(0,t) \leq v_t(0,t) \leq -\delta u^{-p_2} v^{-p_2}(0,t), \quad t \in (0,T). \tag{2.10}$$

Proof of Theorem 1.3. We argue by contradiction. Assume that there exists $a > 0$ such that $u \geq a$ on $\overline{B}_R \times [0,T)$ and v quenching at the time T. Through (2.10), we have

$v_t(0,t) \geq -u^{-p_2}v^{-p_2} \geq -a^{-p_2}v^{-p_2}$, integrating from t to T we get $v(t) \leq C(T-t)^{1/(q_2+1)}$. Together with (2.9) we have $u_t(0,t) \leq -C(T-t)^{-q_1/(q_2+1)}$. Integrating in $(0,T)$, we obtain

$$C\int_0^T (T-t)^{-q_1/(q_2+1)}dt \leq u(0,0) - u(0,T). \tag{2.11}$$

If $q_1/(q_2+1) \geq 1$, we have the left hand of the above inequality diverged. So, we get a contradiction. The proof of Theorem 1.3 is finished. □

Proof of Theorem 1.4. First, assume that $p_2 > p_1+1$ and $q_1 < q_2+1$. Combining (2.9) with (2.10), we get

$$\frac{1}{\delta}v^{q_2-q_1}v_t(0,t) \leq u^{p_1-p_2}u_t(0,t) \leq \delta v^{q_2-q_1}v_t(0,t). \tag{2.12}$$

Since $q_1 - q_2 < 1 < p_2 - p_1$, integrating the first inequality in the (2.12) from 0 to t, we have

$$v^{q_2-q_1+1}(0,t) \leq C_1 - C_2u^{p_1-p_2+1}(0,t), \tag{2.13}$$

where $C_1,\ C_2$ are positive constants, the above inequality requires that u remains positive up to the quenching time. The case $q_1 - q_2 < 1 \leq p_2 - p_1$ can be treated in an analogous way. The proof of Theorem 1.4 is complete. □

Proof of Theorem 1.5. If $p_1 = p_2 = p_3 = p_4 = 1$ and the initial data $u_0(x) = v_0(x)$ on $\overline{B}_R$, thus, it is easy to see that for problem (1.1) simultaneous quenching occurs.

On the other hand, we want to choose v_0 small in order that the quenching time T (through Theorem 1.1, we get $T \leq \min(|\Omega|M^{p_1+1}N^{q_1}, |\Omega|\ M^{p_2}N^{q_2+1})$) be so small that u does not have time to vanish.

Let $u_0 > 0$ be fixed. From $u_t, v_t \leq -\theta$ in $\Omega \times (0,T)$, we obtain

$$u(0,t) \geq \theta(T-t), \quad v(0,t) \geq \theta(T-t), \quad t \in (0,T). \tag{2.14}$$

Together with the estimate (2.12), we get

$$v_t(0,t) \geq -u^{-p_2}v^{-p_2}(0,t) \geq -\theta^{-p_2}(T-t)^{-p_2}v^{-p_2}(0,t). \tag{2.15}$$

Integrating in $[0,t]$, we obtain

$$\begin{aligned}\frac{1}{q_2+1}v^{q_2+1}(0,t) &\geq \frac{1}{q_2+1}v^{q_2+1}(0,0) - \theta^{-p_2}\int_0^t (T-s)^{-p_2}ds\\ &\geq \frac{1}{q_2+1}v^{q_2+1}(0,0) - \theta^{-p_2}\int_0^T (T-s)^{-p_2}ds\\ &\geq \frac{1}{q_2+1}v^{q_2+1}(0,0) - \frac{\theta^{-p_2}}{1-p_2}T^{1-p_2}.\end{aligned} \tag{2.16}$$

It is easy to see that the last term of the above inequality is strictly positive, if T is small enough and $p_2 < 1$, therefore, we prove that, under the condition $p_2 < p_1 + 1$ and $q_1 < q_2 + 1$, for the solution of (1.1) non-simultaneous quenching may occur. The proof of Theorem 1.5 is complete. □

3. Quenching Rates

In this section, we deal with the all possible quenching rates in model (1.1).

Proof of Theorem 1.7. Under the condition of Theorem 1.7, it holds that $a \le u(0,t) \le M$. By (2.10), we have

$$-a^{-p_2}v^{-p_2}(0,t) \le v_t(0,t) \le -\delta M^{-p_2}v^{-p_2}(0,t), \quad t \in (0,T). \tag{3.1}$$

Thus,

$$v(0,t) \sim (T-t)^{1/(q_2+1)} \quad \text{as } t \longrightarrow T. \tag{3.2}$$

The proof of Theorem 1.7 is complete. □

Proof of Theorem 1.8. (i) Assume that the quenching of problem (1.1) is simultaneous with $p_2 > p_1 + 1,\ q_1 > q_2 + 1$, integrating (2.12) yields

$$c_1\left(v^{q_2-q_1+1}(0,t) - v_0^{q_2-q_1+1}(0)\right) \le c_2\left(u^{p_1-p_2+1}(0,t) - u_0^{p_1-p_2+1}(0)\right), \tag{3.3}$$

where $c_1 = 1/\delta(q_2 - q_1 + 1),\ c_2 = 1/(p_1 - p_2 + 1)$. Since we assume that u, v quench at T, we have $v^{q_2-q_1+1}(0,t) \to \infty, u^{p_1-p_2+1} \to \infty$ as $t \to T$.

On the other hand, from $q_2 - q_1 + 1 < 0$ and $p_1 - p_2 + 1 < 0$, we get, a positive constant C_1 such that

$$v^{q_2-q_1+1}(0,t) \ge C_1 u^{p_1-p_2+1}(0,t), \quad \text{as } t \longrightarrow T. \tag{3.4}$$

Similarly, we can show that there exists a positive constant C_2 such that

$$u^{p_1-p_2+1}(0,t) \ge C_2 v^{q_2-q_1+1}(0,t), \quad \text{as } t \longrightarrow T. \tag{3.5}$$

Consequently,

$$u^{p_1-p_2+1}(0,t) \sim v^{q_2-q_1+1}(0,t), \quad \text{as } t \longrightarrow T. \tag{3.6}$$

Recalling the estimates (2.9) and (2.10), we obtain

$$u_t(0,t) \sim -u^{-q_1(p_1-p_2+1)/(q_2-q_1+1)-p_1}, \qquad v_t(0,t) \sim -v^{-p_2(q_2-q_1+1)/(p_1-p_2+1)-q_2}. \tag{3.7}$$

Integrating from t to T, we get

$$u(0,t) \sim (T-t)^{\alpha}, \quad v(0,t) \sim (T-t)^{\beta}, \quad \text{as } t \longrightarrow T. \tag{3.8}$$

If $p_2 < p_1 + 1$ and $q_1 < q_2 + 1$, we deduce the quenching rate by a bootstrap argument. First, by (2.9), we get $u_t(0,t) \le -\delta u^{-p_1} N^{-q_1}$, it follows that $u \ge c(T-t)^{1/(p_1+1)}$. Employing (2.10), we get $v_t(0,t) \ge -u^{-p_2} v^{-q_2} \ge -c(T-t)^{-p_2/(p_1+1)} v^{-q_2}$, that is, $v(0,t) \le c(T-t)^{(p_1-p_2+1)/(q_2+1)(p_1+1)}$. Repeating this procedure, we obtain $u(0,t) \ge c(T-t)^{\alpha_n}$, $v(0,t) \ge c(T-t)^{\beta_n}$, where $\alpha_n,\ \beta_n$ satisfy

$$\begin{gathered}(p_1+1)\alpha_{n+1} = 1 - q_1\beta_n, \quad (q_2+1)\beta_{n+1} = 1 - p_2\alpha_n, \\ \alpha_0 = \frac{1}{p_1+1}, \qquad \beta_0 = \frac{p_1 - p_2 + 1}{(q_2+1)(p_1+1)}.\end{gathered} \tag{3.9}$$

One can check that $\alpha_n \to \alpha,\ \beta_n \to \beta$ ($\alpha,\ \beta$ define by (1.8)), and the all positive constants c are bounded. Therefore, passing to the limit, we get $u(0,t) \ge c(T-t)^{\alpha},\ v(0,t) \ge c(T-t)^{\beta}$. The reverse inequalities can be obtained in the same way.

(ii) If $p_2 = p_1 + 1$ and $q_1 = q_2 + 1$, we have $p_1 + q_1 = p_2 + q_2$. It is easy to see that $u(0,t) \sim v(0,t)$ as $t \to T$, from (2.9) and (2.10), we obtain

$$u(0,t), v(0,t) \sim (T-t)^{p_1+q_1+1}, \quad \text{as } t \longrightarrow T. \tag{3.10}$$

(iii) If $p_2 = p_1 + 1$ and $q_1 > q_2 + 1$, from (2.9), we get

$$u(0,t) \sim \exp\left(-cv^{q_2-q_1+1}(0,t)\right). \tag{3.11}$$

Recalling the estimate (2.10), we get

$$v_t(0,t) \sim -\exp\left(p_2 c v^{q_2-q_1+1}\right) v^{-q_2}, \tag{3.12}$$

that is,

$$\int_{v(0,t)}^{0} \exp\left(-p_2 c y^{q_2-q_1+1}\right) y^{q_2}\, dy \sim -(T-t). \tag{3.13}$$

Let $p_2 c y^{q_2-q_1+1}(s) = w(s)$, we have

$$\int_{p_2 c v^{q_2-q_1+1}(0,t)}^{\infty} c w^{q_1/(q_2-q_1+1)} e^{-w}\, dw \sim (T-t). \tag{3.14}$$

It is known that the incomplete Gamma function $\Gamma(a,z) = \int_z^\infty w^{a-1}e^{-w}dw$ satisfies $\Gamma(a,z) \sim z^{a-1}e^{-z}$ for $z \to \infty$. With $a-1 = q_1/(q_2-q_1+1)$, we obtain

$$(T-t) \sim v^{q_2} \exp\left(-cp_2 v^{q_2-q_1+1}\right), \tag{3.15}$$

and hence,

$$v(0,t) \sim \left|\log(T-t)\right|^{1/(q_2-q_1+1)}. \tag{3.16}$$

Next, we deduce the behaviour for u. Combining with (2.9) and (3.16), we have

$$u^{p_1}u_t(0,t) \sim -\left|\log(T-t)\right|^{-q_1/(q_2-q_1+1)}. \tag{3.17}$$

Integrating from t to T,

$$u^{p_1+1}(0,t) \sim \int_t^T \left|\log(T-s)\right|^{-q_1/(q_2-q_1+1)}ds. \tag{3.18}$$

Setting $\log(T-s) = -z$, we get

$$u^{p_1+1}(0,t) \sim \int_{-\log(T-t)}^{\infty} z^{-q_1/(q_2-q_1+1)}e^{-z}ds. \tag{3.19}$$

For the incomplete Gamma function $\Gamma(a, -\log(T-t))$ with $a-1 = -q_1/(q_2-q_1+1)$, we obtain

$$u(0,t) \sim (T-t)^{1/(p_1+1)}\left|\log(T-s)\right|^{-q_1/(q_2-q_1+1)(p_1+1)}. \tag{3.20}$$

The proof of Theorem 1.8 is complete. □

4. Numerical Experiments

In this section, we perform some numerical experiments, which illustrate our results. Now we introduce the numerical scheme for the space discretization, we discretize applying linear finite elements with mass lumping in a uniform mesh for the space variable and keeping t continuous, it is well known that this discretization in space coincides with the classic central finite difference second-order scheme, (see [16]), Mass lumping is widely used in parabolic problems with blow-up and quenching, (see, e.g., [17, 18]).

Let us consider the uniform partition of size h of the interval $[-L, L]$, ($x_i = ih$, $h = L/N$, $i = 1, \ldots, N$), and its associated standard piecewise linear finite element space V_h.

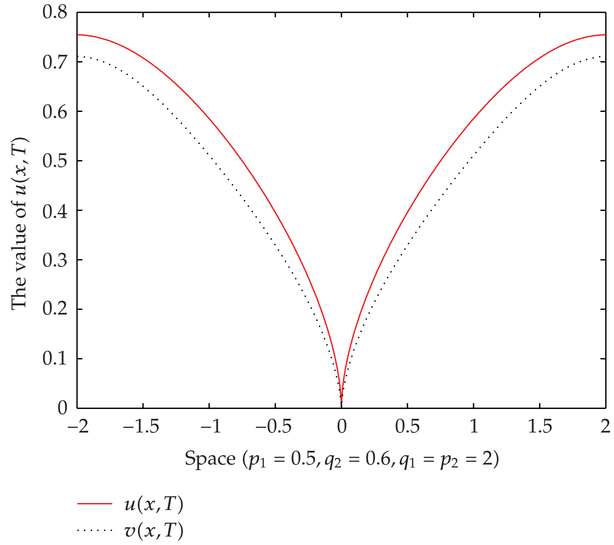

Figure 1: The value of the solution at the quenching time $T = 0.132431$.

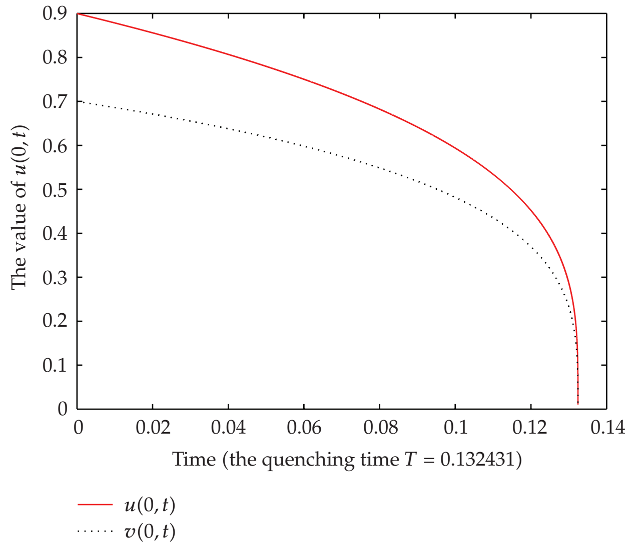

Figure 2: Evolution at the point $x_0 = 0$ of the solution ($p_1 = 0.5,\ q_2 = 0.6,\ q_1 = p_2 = 2$).

The semidiscrete approximation $(u_h(t), v_h(t)) \in V_h$ obtained by the finite element method with mass lumping is defined as

$$\begin{aligned} \int_{-L}^{L} ((u_h)_t w)^I dx &= \int_{-L}^{L} (u_h)_x w_x dx - \int_{-L}^{L} \left((v_h)^{-p} w\right)^I dx, \quad \forall w \in V_h,\ \forall t \in (0,T), \\ \int_{-L}^{L} ((v_h)_t w)^I dx &= \int_{-L}^{L} (v_h)_x w_x dx - \int_{-L}^{L} \left((u_h)^{-q} w\right)^I dx, \quad \forall w \in V_h,\ \forall t \in (0,T), \end{aligned} \tag{4.1}$$

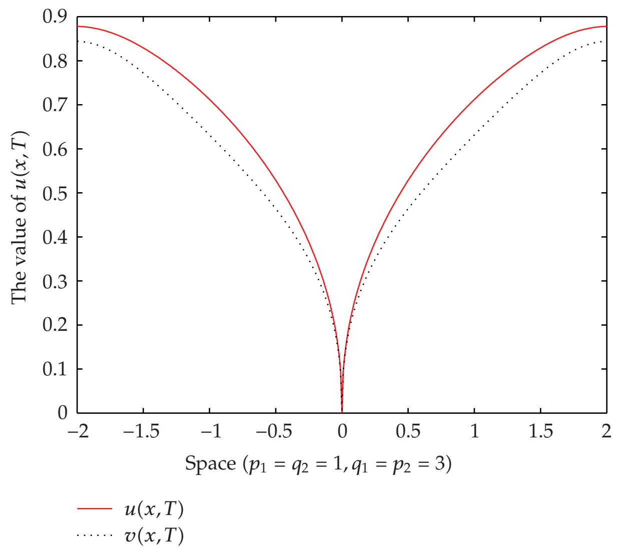

Figure 3: The value of the solution at the quenching time $T = 0.0660034$.

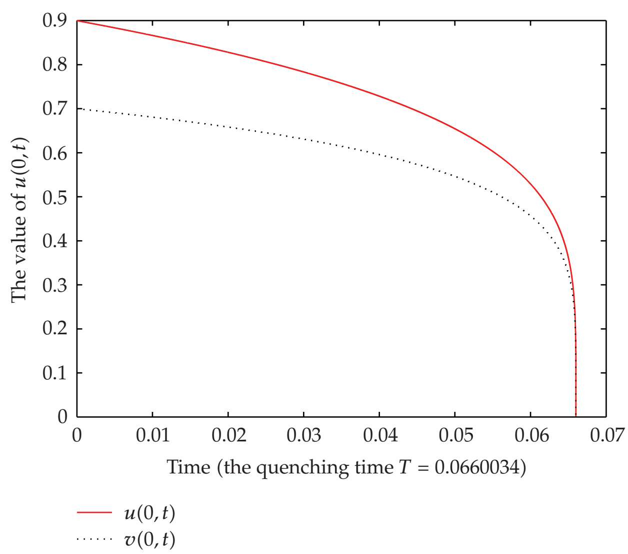

Figure 4: Evolution at the point $x_0 = 0$ of the solution $(p_1 = q_2 = 1,\ q_1 = p_2 = 3)$.

where the superindex I denotes the Lagrange interpolation.

We denote with $(U(t), V(t)) = ((u_1, \ldots, u_N), (v_1, \ldots, v_N))$ the values of the numerical approximation at the nodes $x_i = ih$ and the time t. Thus,

$$(u_h(x,t), v_h(x,t)) = \left(\sum_{k=1}^{N} u_k(t)\varphi(x), \sum_{k=1}^{N} v_k(t)\varphi(x) \right), \tag{4.2}$$

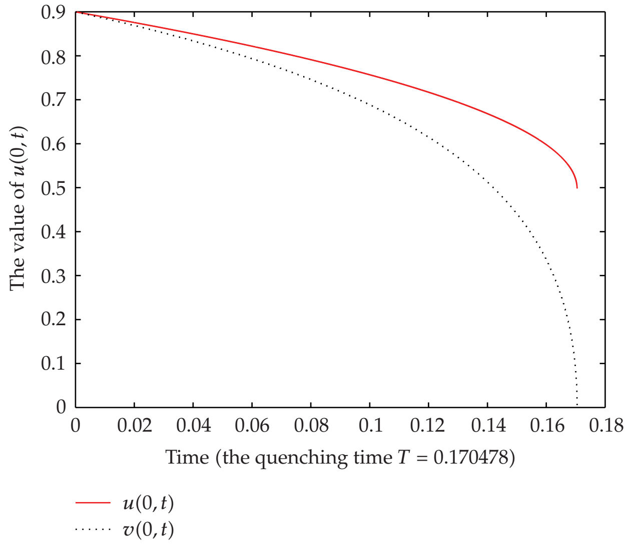

Figure 5: Evolution at the point $x_0 = 0$ of the solution ($p_1 = q_1 = q_2 = 1,\ p_2 = 3$).

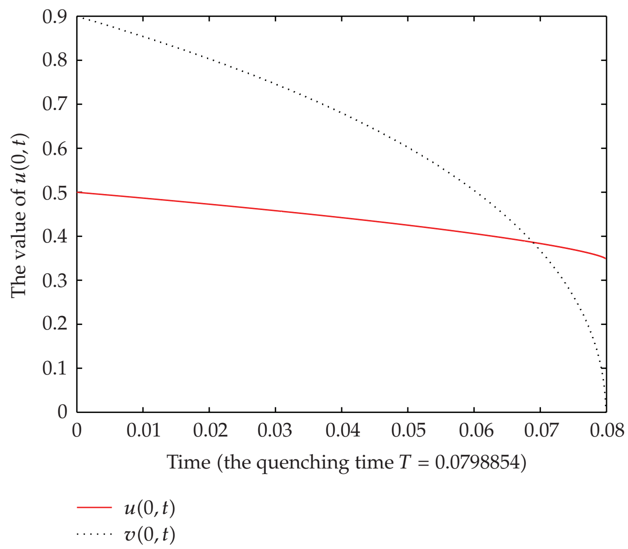

Figure 6: Evolution at the point $x_0 = 0$ of the solution ($p_1 = 0.6,\ q_1 = 0.5,\ p_2 = 2,\ q_2 = 1$).

where $\{\varphi_k\}$ is the standard base of V_h. Then $(U(t), v(t))$ satisfies the following ODE system:

$$\begin{aligned} MU'(t) &= -AU(t) - MU^{-p_1}V^{-q_1}(t), \\ MV'(t) &= -AV(t) - MU^{-p_2}V^{-q_2}(t), \\ (U(0), V(0)) &= \left(\phi^I, \varphi^I\right), \end{aligned} \tag{4.3}$$

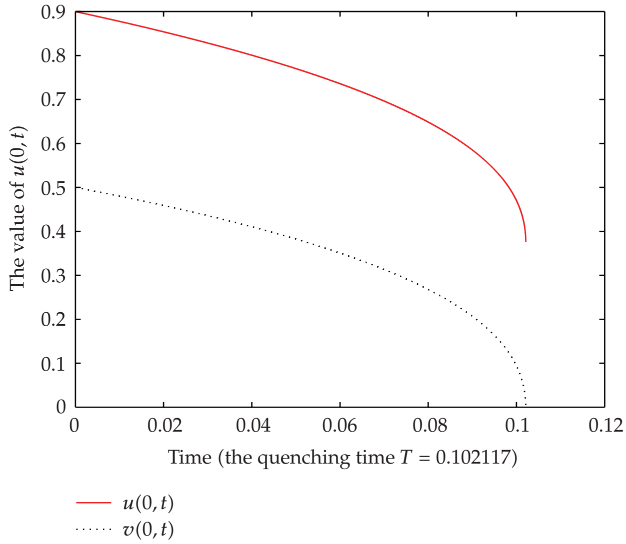

Figure 7: Evolution at the point $x_0 = 0$ of the solution $(p_1 = q_1 = p_2 = q_2 = 1)$.

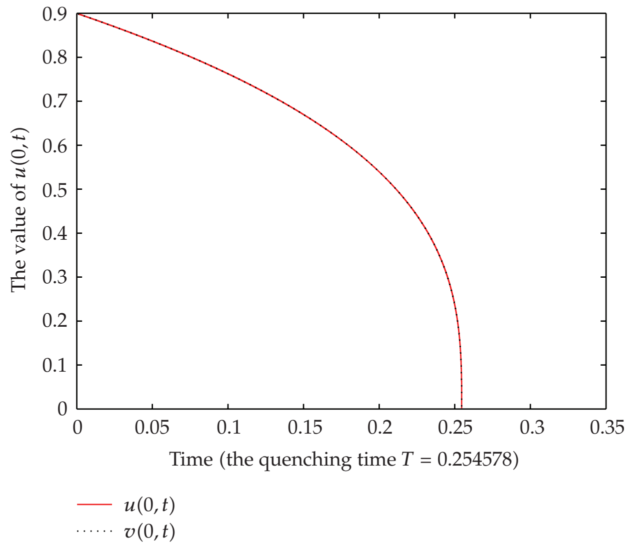

Figure 8: Evolution at the point $x_0 = 0$ of the solution $(p_1 = q_1 = p_2 = q_2 = 1)$.

where M is the mass matrix obtained with lumping, A is the stiffness matrix, and (ϕ^I, φ^I) is the Lagrange interpolation of the initial datum $(\phi(x), \varphi(x))$.

We take $\Omega = [-2, 2]$ and $-2 = x_1 < \cdots < x_N = 2$, $0 = t_1 < \cdots < t_M = T$. Writing the system (4.3) explicitly, we get the following ODE system:

$$\delta_t u(x_1,t_j) = \frac{2}{h^2}\left(u(x_2,t_j) - u(x_1,t_j)\right) - u^{-p_1}v^{-q_1}(x_1,t_j), \quad 1 \le j \le M,$$

$$\delta_t v(x_1,t_j) = \frac{2}{h^2}\left(v(x_2,t_j) - v(x_1,t_j)\right) - u^{-p_2}v^{-q_2}(x_1,t_j), \quad 1 \le j \le M,$$

$$\delta_t u(x_i,t_j) = \frac{\left(u(x_{i+1},t_j) - 2u(x_i,t_j) + u(x_{i-1},t_j)\right)}{h^2} - u^{-p_1}v^{-q_1}(x_i,t_j),$$
$$1 \le i \le N-1, \quad 1 \le j \le M,$$

$$\delta_t v(x_i,t_j) = \frac{\left(v(x_{i+1},t_j) - 2v(x_i,t_j) + v(x_{i-1},t_j)\right)}{h^2} - u^{-p_2}v^{-q_2}(x_i,t_j), \tag{4.4}$$
$$1 \le i \le N-1, \quad 1 \le j \le M,$$

$$\delta_t u(x_N,t_j) = \frac{2}{h^2}\left(u(x_{N-1},t_j) - u(x_N,t_j)\right) - u^{-p_1}v^{-q_1}(x_N,t_j), \quad 1 \le j \le M,$$

$$\delta_t v(x_N,t_j) = \frac{2}{h^2}\left(v(x_{N-1},t_j) - v(x_N,t_j)\right) - u^{-p_2}v^{-q_2}(x_N,t_j), \quad 1 \le j \le M,$$

$$u(x_i,t_1) = u_0(ih), v(x_i,t_1) = v_0(ih), \quad 1 \le i \le N,$$

where $\delta_t v(x_i,t_j) = u(x_i,t_{j+1})/\Delta t_j$ and $h = 0.01$. In order to show the evolution in time of a numerical solution, we chose $\Delta t_j = \lambda U_j$, $U_j = \min_{1\le i\le N} u(x_i,t_j)$, and $0 < \lambda < 1$ which will be choose later.

First, we consider the case $p_1 = 0.5$, $q_2 = 0.6$, $p_2 = q_1 = 2$, and the initial data $u_0 = 1-(1/10)\sin((\pi/4)(s+2)), v_0 = 1-(3/10)\sin((\pi/4)(s+2))$, We observe that the solutions of (1.1) quenching only at the origin, if the symmetric initial data with a unique minimum at $x = 0$ (see Figure 1), and the quenching is simultaneous (see Figure 2); If we take $p_2 = q_1 = 3$, $p_1 = q_2 = 1$, and the same initial data (see Figures 3 and 4), then we obtain the results which accords with Theorem 1.3.

Next, we take $p_1 = q_1 = q_2 = 1$, $p_2 = 3$ with the same initial data $u_0(x) = \phi(s) = 1-(1/10)\sin((\pi/4)(s+2))$. In this case the quenching in (1.1) is non-simultaneous with u being strictly positive (see Figure 5); If we choose $p_1 = 0.6$, $q_1 = 0.5$, $p_2 = 2$, and $q_2 = 1$ with the initial data $u_0 = 1-(1/10)\sin((\pi/4)(s+2))$, $v_0 = 1-(1/2)\sin((\pi/4)(s+2))$ (see Figure 6), then we can see that our results coincide with Theorem 1.4.

Finally, we choose $p_1 = q_1 = p_2 = q_2 = 1$ In Figure 7, we take the initial data $u_0 = 1-(1/10)\sin((\pi/4)(s+2))$, $v_0 = 1-(1/2)\sin((\pi/4)(s+2))$, and in Figure 8 we take the different initial data both equal to $1-(1/10)\sin((\pi/4)(s+2))$, we can see that both non-simultaneous quenching and simultaneous quenching may occur in (1.1), depending on the initial data.

Acknowledgments

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