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Research Article

Two-Scale Convergence of Stekloff Eigenvalue Problems in Perforated Domains

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By means of the two-scale convergence method, we investigate the asymptotic behavior of eigenvalues and eigenfunctions of Stekloff eigenvalue problems in perforated domains. We prove a concise and precise homogenization result including convergence of gradients of eigenfunctions which improves the understanding of the asymptotic behavior of eigenfunctions. It is also justified that the natural local problem is not an eigenvalue problem.

1. Introduction

We are interested in the spectral asymptotics (as $\varepsilon \to 0$) of the linear elliptic eigenvalue problem

$$\begin{gathered}
-\sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\left(a_{ij}\left(x, \frac{x}{\varepsilon}\right) \frac{\partial u_\varepsilon}{\partial x_j} \right) = 0, \quad \text{in } \Omega^\varepsilon, \\
\sum_{i,j=1}^{N} a_{ij}\left(x, \frac{x}{\varepsilon}\right) \frac{\partial u_\varepsilon}{\partial x_j} \nu_i = \lambda_\varepsilon u_\varepsilon, \quad \text{on } \partial T^\varepsilon, \\
u_\varepsilon = 0, \quad \text{on } \partial\Omega, \\
\varepsilon \int_{S^\varepsilon} |u_\varepsilon|^2 d\sigma_\varepsilon(x) = 1,
\end{gathered} \tag{1.1}$$

where Ω is a bounded open set in $\mathbb{R}^N_x$ (the numerical space of variables $x = (x_1, \ldots, x_N)$, with integer $N \geq 2$) with Lipschitz boundary $\partial\Omega$, $a_{ij} \in \mathcal{C}(\overline{\Omega}; L^\infty(\mathbb{R}^N_y))$ $(1 \leq i, j \leq N)$, with the symmetry condition $a_{ji} = \overline{a}_{ij}$, the periodicity hypothesis: for each $x \in \overline{\Omega}$ and for every $k \in \mathbb{Z}^N$ one has $a_{ij}(x, y+k) = a_{ij}(x, y)$ almost everywhere in $y \in \mathbb{R}^N_y$, and finally the ellipticity condition: there exists $\alpha > 0$ such that for any $x \in \overline{\Omega}$

$$\operatorname{Re} \sum_{i,j=1}^{N} a_{ij}(x, y)\xi_j \overline{\xi_i} \geq \alpha|\xi|^2 \tag{1.2}$$

for all $\xi \in \mathbb{C}^N$ and for almost all $y \in \mathbb{R}^N_y$, where $|\xi|^2 = |\xi_1|^2 + \cdots + |\xi_N|^2$.

The set Ω^ε ($\varepsilon > 0$) is a domain perforated as follows. Let $T \subset Y = (0,1)^N$ be a compact subset in $\mathbb{R}^N_y$ with smooth boundary $\partial T(\equiv S)$ and nonempty interior. For $\varepsilon > 0$, we define

$$\begin{gathered} t^\varepsilon = \left\{ k \in \mathbb{Z}^N : \varepsilon(k+T) \subset \Omega \right\}, \\ T^\varepsilon = \bigcup_{k \in t^\varepsilon} \varepsilon(k+T), \\ \Omega^\varepsilon = \Omega \setminus T^\varepsilon. \end{gathered} \tag{1.3}$$

In this setup, T is the reference hole, whereas $\varepsilon(k+T)$ is a hole of size ε and T^ε is the collection of the holes of the perforated domain Ω^ε. The family T^ε is made up with a finite number of holes since Ω is bounded. Finally, $\nu = (\nu_i)$ denotes the outer unit normal vector to $\partial T^\varepsilon(\equiv S^\varepsilon)$ with respect to Ω^ε.

The asymptotics of eigenvalue problems has been widely explored. Homogenization of eigenvalue problems in a fixed domain goes back to Kesavan [1, 2]. In perforated domains it was first considered by Rauch [3] and Rauch and Taylor [4], but the first homogenization results on this topic pertains to Vanninathan [5], where he considered eigenvalue problems for the laplace operator ($a_{ij} = \delta_{ij}$ (Kronecker symbol)) in perforated domains, and combined asymptotic expansion with Tartar's energy method to prove homogenization results. Concerning homogenization of eigenvalue problems in perforated domains, we also mention the work of Conca et al. [6], Douanla and Svanstedt [7], Kaizu [8], Ozawa and Roppongi [9], Roppongi [10], and Pastukhova [11] and the references therein. In this paper, we deal with the spectral asymptotics of Stekloff eigenvalue problems for an elliptic linear differential operator of order two in divergence form with variable coefficients depending on the macroscopic variable and one microscopic variable. We obtain a very accurate, precise, and concise homogenization result (Theorem 3.7) by using the two-scale convergence method [12–16] introduced by Nguetseng [15] and further developed by Allaire [12]. A convergence result for gradients of eigenfunctions is provided, which improves the understanding of the asymptotic behavior of eigenfunctions. We also justify that the natural local problem is not an eigenvalue problem.

Unless otherwise specified, vector spaces throughout are considered over the complex field, $\mathbb{C}$, and scalar functions are assumed to take complex values. Let us recall some basic notations. Let $Y = (0,1)^N$, and let $F(\mathbb{R}^N)$ be a given function space. We denote by $F_{\text{per}}(Y)$ the space of functions in $F_{\text{loc}}(\mathbb{R}^N)$ that are Y-periodic and by $F_\#(Y)$ the space of those functions $u \in F_{\text{per}}(Y)$ with $\int_Y u(y)dy = 0$. Finally, the letter E denotes throughout a family of strictly positive real numbers ($0 < \varepsilon \leq 1$) admitting 0 as accumulation point. The numerical space $\mathbb{R}^N$ and its open sets are provided with the Lebesgue measure denoted by $dx = dx_1 \cdots dx_N$.

The rest of the paper is organized as follows. In Section 2, we recall some results about the two-scale convergence method, and the homogenization process is consider in Section 3.

2. Two-Scale Convergence on Periodic Surfaces

We first recall the definition and the main compactness theorems of the two-scale convergence method. Let Ω be an open bounded set in $\mathbb{R}_x^N$ (integer $N \geq 2$) and $Y = (0,1)^N$, the unit cube.

Definition 2.1. A sequence $(u_\varepsilon)_{\varepsilon\in E} \subset L^2(\Omega)$ is said to two-scale converge in $L^2(\Omega)$ to some$u_0 \in L^2(\Omega \times Y)$ if, as $E \ni \varepsilon \to 0$,

$$\int_\Omega u_\varepsilon(x)\phi\left(x, \frac{x}{\varepsilon}\right)dx \longrightarrow \iint_{\Omega\times Y} u_0(x,y)\phi(x,y)\,dx\,dy \tag{2.1}$$

for all $\phi \in L^2(\Omega; \mathcal{C}_{\rm per}(Y))$.

Notation 1. We express this by writing $u_\varepsilon \xrightarrow{2s} u_0$ *in* $L^2(\Omega)$.

The following theorem is the backbone of the two-scale convergence method.

Theorem 2.2. *Let $(u_\varepsilon)_{\varepsilon\in E}$ be a bounded sequence in $L^2(\Omega)$. Then, a subsequence E' can be extracted from E such that, as $E' \ni \varepsilon \to 0$, the sequence $(u_\varepsilon)_{\varepsilon\in E'}$ two-scale converges in $L^2(\Omega)$ to some $u_0 \in L^2(\Omega \times Y)$.*

Here follows the cornerstone of two-scale convergence.

Theorem 2.3. *Let $(u_\varepsilon)_{\varepsilon\in E}$ be a bounded sequence in $H^1(\Omega)$. Then, a subsequence E' can be extracted from E such that, as $E' \ni \varepsilon \to 0$,*

$$\begin{gathered} u_\varepsilon \longrightarrow u_0, \quad \text{in } H^1(\Omega)\text{-weak}, \\ u_\varepsilon \longrightarrow u_0, \quad \text{in } L^2(\Omega), \\ \frac{\partial u_\varepsilon}{\partial x_j} \xrightarrow{2s} \frac{\partial u_0}{\partial x_j} + \frac{\partial u_1}{\partial y_j}, \quad \text{in } L^2(\Omega)\ (1 \leq j \leq N), \end{gathered} \tag{2.2}$$

where $u_0 \in H^1(\Omega)$ and $u_1 \in L^2(\Omega; H^1_\#(Y))$.

In the sequel, we denote by $d\sigma(y)$ $(y \in Y)$, $d\sigma_\varepsilon(x)$ $(x \in \Omega, \varepsilon \in E)$, the surface measures on S and S^ε, respectively. The surface measure of S is denoted by $|S|$. The space of squared integrable functions, with respect to the previous measures on S and S^ε are denoted by $L^2(S)$ and $L^2(S^\varepsilon)$, respectively. Since the volume of S^ε grows proportionally to $1/\varepsilon$ as $\varepsilon \to 0$, we endow $L^2(S^\varepsilon)$ with the scaled scalar product [17]

$$(u,v)_{L^2(S^\varepsilon)} = \varepsilon \int_{S^\varepsilon} u(x)v(x)d\sigma_\varepsilon(x) \quad \left(u, v \in L^2(S^\varepsilon)\right). \tag{2.3}$$

Definition 2.1 then generalizes as.

Definition 2.4. A sequence $(u_\varepsilon)_{\varepsilon\in E} \subset L^2(S^\varepsilon)$ is said to two-scale converge to some $u_0 \in L^2(\Omega \times S)$ if as follows. $E \ni \varepsilon \to 0$,

$$\varepsilon \int_{S^\varepsilon} u_\varepsilon(x)\phi\left(x, \frac{x}{\varepsilon}\right) d\sigma_\varepsilon(x) \longrightarrow \iint_{\Omega\times S} u_0(x,y)\phi(x,y)\,dx\,d\sigma(y) \tag{2.4}$$

for all $\phi \in \mathcal{C}(\overline{\Omega}; \mathcal{C}_{\text{per}}(Y))$.

The following result paves the way of the general version of Theorem 2.2.

Lemma 2.5. *Let $\phi \in \mathcal{C}(\overline{\Omega}; \mathcal{C}_{per}(Y))$. Then, we have*

$$\varepsilon \int_{S^\varepsilon} \left|\phi\left(x, \frac{x}{\varepsilon}\right)\right|^2 d\sigma_\varepsilon(x) \le C\|\phi\|_\infty^2 \tag{2.5}$$

for some constant C independent of ε and, as $E \ni \varepsilon \to 0$,

$$\varepsilon \int_{S^\varepsilon} \left|\phi\left(x, \frac{x}{\varepsilon}\right)\right|^2 d\sigma_\varepsilon(x) \longrightarrow \iint_{\Omega\times S} |\phi(x,y)|^2\,dx\,d\sigma(y). \tag{2.6}$$

Proof. The first part is left to the reader. Let $\varphi \in \mathcal{C}(\overline{\Omega})$ and $\psi \in \mathcal{C}_{\text{per}}(Y)$. We have

$$\varepsilon \int_{S^\varepsilon} \left|\varphi(x)\psi\left(\frac{x}{\varepsilon}\right)\right|^2 d\sigma_\varepsilon(x) = \varepsilon \sum_{k\in t^\varepsilon} \int_{\varepsilon(k+S)} \left|\varphi(x)\psi\left(\frac{x}{\varepsilon}\right)\right|^2 d\sigma_\varepsilon(x). \tag{2.7}$$

Using the second mean value theorem, for any $k \in t^\varepsilon$, we have

$$\int_{\varepsilon(k+S)} \left|\varphi(x)\psi\left(\frac{x}{\varepsilon}\right)\right|^2 d\sigma_\varepsilon(x) = |\varphi(x_k)|^2 \int_{\varepsilon(k+S)} \left|\psi\left(\frac{x}{\varepsilon}\right)\right|^2 d\sigma_\varepsilon(x) \tag{2.8}$$

for some $x_k \in \varepsilon(k+S) \subset \varepsilon(k+Y)$. Hence,

$$\begin{aligned}
\varepsilon \int_{S^\varepsilon} \left|\varphi(x)\psi\left(\frac{x}{\varepsilon}\right)\right|^2 d\sigma_\varepsilon(x) &= \varepsilon \sum_{k\in t^\varepsilon} \int_{\varepsilon(k+S)} \left|\varphi(x)\psi\left(\frac{x}{\varepsilon}\right)\right|^2 d\sigma_\varepsilon(x) \\
&= \varepsilon \sum_{k\in t^\varepsilon} \left|\varphi\left(x^k\right)\right|^2 \int_{\varepsilon(k+S)} \left|\psi\left(\frac{x}{\varepsilon}\right)\right|^2 d\sigma_\varepsilon(x) \\
&= \varepsilon \sum_{k\in t^\varepsilon} \left|\varphi\left(x^k\right)\right|^2 \varepsilon^{N-1} \int_{(k+S)} |\psi(y)|^2 d\sigma(y) \\
&= \left(\int_S |\psi(y)|^2 d\sigma(y)\right) \sum_{k\in t^\varepsilon} \varepsilon^N \left|\varphi\left(x^k\right)\right|^2.
\end{aligned} \tag{2.9}$$

But, as $E \ni \varepsilon \to 0$,

$$\sum_{k \in t^{\varepsilon}} \varepsilon^{N} \left| \varphi\left(x^{k}\right) \right|^{2} \longrightarrow \int_{\Omega} |\varphi(x)|^{2} dx, \tag{2.10}$$

and the proof is completed due to the density of $\mathcal{C}(\overline{\Omega}) \otimes \mathcal{C}_{\text{per}}(Y)$ in $\mathcal{C}(\overline{\Omega}; \mathcal{C}_{\text{per}}(Y))$. □

Remark 2.6. Even if often used (see, e.g., [13, 17]), this is the first time Lemma 2.5 is rigorously proved. It is worth noticing that because of a trace issue one cannot replace therein the space $\mathcal{C}(\overline{\Omega}; \mathcal{C}_{\text{per}}(Y))$ by $L^{2}(\Omega; \mathcal{C}_{\text{per}}(Y))$.

Theorem 2.2 generalizes as follows.

Theorem 2.7. *Let $(u_{\varepsilon})_{\varepsilon \in E}$ be a sequence in $L^{2}(S^{\varepsilon})$ such that*

$$\varepsilon \int_{S^{\varepsilon}} |u_{\varepsilon}(x)|^{2} d\sigma_{\varepsilon}(x) \leq C, \tag{2.11}$$

where C is a positive constant independent of ε. There exists a subsequence E' of E such that $(u_{\varepsilon})_{\varepsilon \in E'}$ two-scale converges to some $u_{0} \in L^{2}(\Omega; L^{2}(S))$ in the sense of Definition 2.4.

Proof. Put $F_{\varepsilon}(\phi) = \varepsilon \int_{S^{\varepsilon}} u_{\varepsilon}(x) \phi(x, (x/\varepsilon)) d\sigma_{\varepsilon}(x)$ for $\phi \in \mathcal{C}(\overline{\Omega}; \mathcal{C}_{\text{per}}(Y))$. We have

$$\left| F_{\varepsilon}(\phi) \right| \leq C \left(\varepsilon \int_{S^{\varepsilon}} \left| \phi\left(x, \frac{x}{\varepsilon}\right) \right|^{2} d\sigma_{\varepsilon}(x) \right)^{1/2} \leq C \|\phi\|_{\infty}, \tag{2.12}$$

which allows us to view F_{ε} as a continuous linear form on $\mathcal{C}(\overline{\Omega}; \mathcal{C}_{\text{per}}(Y))$. Hence, there exists a bounded sequence of measures $(\mu_{\varepsilon})_{\varepsilon \in E}$ such that $F_{\varepsilon}(\phi) = \langle \mu_{\varepsilon}, \phi \rangle$. Due to the separability of $\mathcal{C}(\overline{\Omega}; \mathcal{C}_{\text{per}}(Y))$ there exists a subsequence E' of E such that in the weak * topology of the dual of $\mathcal{C}(\overline{\Omega}; \mathcal{C}_{\text{per}}(Y))$ we have $\mu_{\varepsilon} \to \mu_{0}$ as $E' \ni \varepsilon \to 0$. A limit passage ($E' \ni \varepsilon \to 0$) in (2.12) yields

$$\left| \langle \mu_{0}, \phi \rangle \right| \leq C \left(\iint_{\Omega \times S} |\phi(x, y)|^{2} dx \, d\sigma(y) \right)^{1/2}. \tag{2.13}$$

But μ_{0} is a continuous form on $L^{2}(\Omega; L^{2}(S))$ by density of $\mathcal{C}(\overline{\Omega}; \mathcal{C}_{\text{per}}(Y))$ in the later space, and there exists $u_{0} \in L^{2}(\Omega; L^{2}(S))$ such that

$$\langle \mu_{0}, \phi \rangle = \iint_{\Omega \times S} u_{0}(x, y) \phi(x, y) dx \, d\sigma(y) \tag{2.14}$$

for all $\phi \in \mathcal{C}(\overline{\Omega}; \mathcal{C}_{\text{per}}(Y))$, which completes the proof. □

In the case when $(u_{\varepsilon})_{\varepsilon \in E}$ is the sequence of traces on S^{ε} of functions in $H^{1}(\Omega)$, a link can be established between its usual and surface two-scale limits. The following proposition whose proof's outlines can be found in [13] clarifies this.

Proposition 2.8. *Let $(u_\varepsilon)_{\varepsilon\in E} \subset H^1(\Omega)$ be such that*

$$\|u_\varepsilon\|_{L^2(\Omega)} + \varepsilon\|Du_\varepsilon\|_{L^2(\Omega)^N} \leq C, \tag{2.15}$$

where C is a positive constant independent of ε and D denotes the usual gradient. The sequence of traces of $(u_\varepsilon)_{\varepsilon\in E}$ on S^ε satisfies

$$\varepsilon\int_{S^\varepsilon} |u_\varepsilon(x)|^2 d\sigma_\varepsilon(x) \leq C \quad (\varepsilon \in E), \tag{2.16}$$

and up to a subsequence E' of E, it two-scale converges in the sense of Definition 2.4 to some $u_0 \in L^2(\Omega; L^2(S))$ which is nothing but the trace on S of the usual two-scale limit, a function in $L^2(\Omega; H^1_\#(Y))$. More precisely, as $E' \ni \varepsilon \to 0$,

$$\begin{gathered}
\varepsilon\int_{S^\varepsilon} u_\varepsilon(x)\phi\left(x,\frac{x}{\varepsilon}\right)d\sigma_\varepsilon(x) \longrightarrow \iint_{\Omega\times S} u_0(x,y)\phi(x,y)\,dx\,d\sigma(y),\\
\int_\Omega u_\varepsilon(x)\phi\left(x,\frac{x}{\varepsilon}\right)dx\,dy \longrightarrow \iint_{\Omega\times Y} u_0(x,y)\phi(x,y)\,dx\,dy
\end{gathered} \tag{2.17}$$

for all $\phi \in \mathcal{C}(\overline{\Omega}; \mathcal{C}_{per}(Y))$.

3. Homogenization Procedure

We make use of the notations introduced earlier in Section 1. Before we proceed we need a few details.

3.1. Preliminaries

We introduce the characteristic function χ_G of

$$G = \mathbb{R}^N_y \setminus \Theta \tag{3.1}$$

with

$$\Theta = \bigcup_{k\in\mathbb{Z}^N} (k+T). \tag{3.2}$$

It follows from the closeness of T that Θ is closed in $\mathbb{R}^N_y$ so that G is an open subset of $\mathbb{R}^N_y$. Next, let $\varepsilon \in E$ be arbitrarily fixed, and define

$$V_\varepsilon = \left\{u \in H^1(\Omega^\varepsilon) : u = 0 \text{ on } \partial\Omega\right\}. \tag{3.3}$$

We equip V_ε with the $H^1(\Omega^\varepsilon)$-norm which makes it a Hilbert space. We recall the following classical result [18].

Proposition 3.1. *For each $\varepsilon \in E$ there exists an operator P_ε of V_ε into $H_0^1(\Omega)$ with the following properties:*

(i) *P_ε sends continuously and linearly V_ε into $H_0^1(\Omega)$;*

(ii) *$(P_\varepsilon v)|_{\Omega^\varepsilon} = v$ for all $v \in V_\varepsilon$;*

(iii) *$\|D(P_\varepsilon v)\|_{L^2(\Omega)^N} \leq c\|Dv\|_{L^2(\Omega^\varepsilon)^N}$ for all $v \in V_\varepsilon$, where c is a constant independent of ε and D denotes the usual gradient operator.*

It is also a well-known fact that, under the hypotheses mentioned earlier in the introduction, the spectral problem (1.1) has an increasing sequence of eigenvalues $\{\lambda_\varepsilon^k\}_{k=1}^\infty$,

$$\begin{gathered} 0 < \lambda_\varepsilon^1 \leq \lambda_\varepsilon^2 \leq \lambda_\varepsilon^3 \leq \cdots \leq \lambda_\varepsilon^n, \\ \lambda_\varepsilon^n \longrightarrow +\infty, \quad \text{as } n \longrightarrow +\infty. \end{gathered} \tag{3.4}$$

It is to be noted that if the coefficients a_{ij}^ε are real valued then the first eigenvalue λ_1^ε is isolated. Moreover, to each eigenvalue, λ_ε^k is attached to an eigenvector $u_\varepsilon^k \in V_\varepsilon$ and $\{u_\varepsilon^k\}_{k=1}^\infty$ is an orthonormal basis in $L^2(S^\varepsilon)$. In the sequel, the couple $(\lambda_\varepsilon^k, u_\varepsilon^k)$ will be referred to as eigencouple without further ado.

We finally recall the Courant-Fisher minimax principle which gives a useful (as will be seen later) characterization of the eigenvalues to problem (1.1). To this end, we introduce the Rayleigh quotient defined, for each $v \in V_\varepsilon \setminus \{0\}$, by

$$R^\varepsilon(v) = \frac{\int_{\Omega^\varepsilon} (A^\varepsilon Dv, Dv)\,dx}{\int_{S^\varepsilon} |v|^2 d\sigma_\varepsilon(x)}, \tag{3.5}$$

where A^ε is the N^2-square matrix $(a_{ij}^\varepsilon)_{1\leq i,j\leq N}$ and D denotes the usual gradient. Denoting by E^k ($k \geq 0$) the collection of all subspaces of dimension k of V_ε, the minimax principle is stated as follows: for any $k \geq 1$, the k'th eigenvalue to (1.1) is given by

$$\lambda_\varepsilon^k = \min_{W\in E^k}\left(\max_{v\in W\setminus\{0\}} R^\varepsilon(v)\right) = \max_{W\in E^{k-1}}\left(\min_{v\in W^\perp\setminus\{0\}} R^\varepsilon(v)\right). \tag{3.6}$$

In particular, the first eigenvalue satisfies

$$\lambda_\varepsilon^1 = \min_{v\in V_\varepsilon\setminus\{0\}} R^\varepsilon(v), \tag{3.7}$$

and every minimum in (3.6) is an eigenvector associated with λ_ε^1.

Now, let $Q^\varepsilon = \Omega \setminus (\varepsilon\Theta)$. This is an open set in $\mathbb{R}^N$, and $\Omega^\varepsilon \setminus Q^\varepsilon$ is the intersection of Ω with the collection of the holes crossing the boundary $\partial\Omega$. We have the following result which implies, as will be seen later, that the holes crossing the boundary $\partial\Omega$ are of no effects as regards the homogenization process since they are in arbitrary narrow stripe along the boundary.

Lemma 3.2 (see [19]). *Let $K \subset \Omega$ be a compact set independent of ε. There is some $\varepsilon_0 > 0$ such that $\Omega^\varepsilon \setminus Q^\varepsilon \subset \Omega \setminus K$ for any $0 < \varepsilon \leq \varepsilon_0$.*

Next, we introduce the space

$$\mathbb{F}_0^1 = H_0^1(\Omega) \times L^2\Big(\Omega; H_\#^1(Y)\Big). \tag{3.8}$$

Endowed with the following norm

$$\|\mathbf{v}\|_{\mathbb{F}_0^1} = \left\| D_x v_0 + D_y v_1 \right\|_{L^2(\Omega\times Y)} \quad \Big(\mathbf{v} = (v_0, v_1) \in \mathbb{F}_0^1\Big), \tag{3.9}$$

$\mathbb{F}_0^1$ *is a Hilbert space admitting* $F_0^\infty = \mathfrak{D}(\Omega) \times [\mathfrak{D}(\Omega) \otimes \mathcal{C}_\#^\infty(Y)]$ *as a dense subspace. This being so, for* $\mathbf{u}, \mathbf{v} \in \mathbb{F}_0^1 \times \mathbb{F}_0^1$, *let*

$$a_\Omega(\mathbf{u}, \mathbf{v}) = \sum_{i,j=1}^{N} \iint_{\Omega\times Y^*} a_{ij}(x,y) \left(\frac{\partial u_0}{\partial x_j} + \frac{\partial u_1}{\partial y_j} \right) \left(\overline{\frac{\partial v_0}{\partial x_j}} + \overline{\frac{\partial v_1}{\partial y_j}} \right) dx\, dy. \tag{3.10}$$

This defines a hermitian, continuous sesquilinear form on $\mathbb{F}_0^1 \times \mathbb{F}_0^1$. *We will need the following results.*

Lemma 3.3. *Fix* $\Phi = (\psi_0, \psi_1) \in F_0^\infty$, *and define* $\Phi_\varepsilon : \Omega \to \mathbb{C}(\varepsilon > 0)$ *by*

$$\Phi_\varepsilon(x) = \psi_0(x) + \varepsilon \psi_1\left(x, \frac{x}{\varepsilon}\right) \quad (x \in \Omega). \tag{3.11}$$

If $(u_\varepsilon)_{\varepsilon\in E} \subset H_0^1(\Omega)$ *is such that*

$$\frac{\partial u_\varepsilon}{\partial x_i} \xrightarrow{2s} \frac{\partial u_0}{\partial x_i} + \frac{\partial u_1}{\partial y_i}, \quad \text{in } L^2(\Omega) \quad (1 \le i \le N) \tag{3.12}$$

as $E \ni \varepsilon \to 0$, *where* $\mathbf{u} = (u_0, u_1) \in \mathbb{F}_0^1$, *then*

$$a^\varepsilon(u_\varepsilon, \Phi_\varepsilon) \longrightarrow a_\Omega(\mathbf{u}, \Phi) \tag{3.13}$$

as $E \ni \varepsilon \to 0$, *where*

$$a^\varepsilon(u_\varepsilon, \Phi_\varepsilon) = \sum_{i,j=1}^{N} \int_{\Omega^\varepsilon} a_{ij}^\varepsilon \frac{\partial u_\varepsilon}{\partial x_j} \frac{\overline{\partial \Phi_\varepsilon}}{\partial x_i} dx. \tag{3.14}$$

Proof. For $\varepsilon > 0$, $\Phi_\varepsilon \in \mathfrak{D}(\Omega)$ and all the functions $\Phi_\varepsilon(\varepsilon > 0)$ have their supports contained in a fixed compact set $K \subset \Omega$. Thanks to Lemma 3.3, there is some $\varepsilon_0 > 0$ such that

$$\Phi_\varepsilon = 0, \quad \text{in } \Omega^\varepsilon \setminus Q^\varepsilon \ (E \ni \varepsilon \le \varepsilon_0). \tag{3.15}$$

Using the decomposition $\Omega^\varepsilon = Q^\varepsilon \cup (\Omega^\varepsilon \setminus Q^\varepsilon)$ and the equality $Q^\varepsilon = \Omega \cap \varepsilon G$, we get for $E \ni \varepsilon \le \varepsilon_0$

$$\begin{aligned}
a^\varepsilon(u_\varepsilon, \Phi_\varepsilon) &= \sum_{i,j=1}^N \int_{\Omega^\varepsilon} a_{ij}\left(x, \frac{x}{\varepsilon}\right) \frac{\partial u_\varepsilon}{\partial x_j} \overline{\frac{\partial \Phi_\varepsilon}{\partial x_i}} dx \\
&= \sum_{i,j=1}^N \int_{Q^\varepsilon} a_{ij}\left(x, \frac{x}{\varepsilon}\right) \frac{\partial u_\varepsilon}{\partial x_j} \overline{\frac{\partial \Phi_\varepsilon}{\partial x_i}} dx \\
&= \sum_{i,j=1}^N \int_{\Omega\cap\varepsilon G} a_{ij}\left(x, \frac{x}{\varepsilon}\right) \frac{\partial u_\varepsilon}{\partial x_j} \overline{\frac{\partial \Phi_\varepsilon}{\partial x_i}} dx \\
&= \sum_{i,j=1}^N \int_{\Omega} a_{ij}\left(x, \frac{x}{\varepsilon}\right) \chi_{\varepsilon G}(x) \frac{\partial u_\varepsilon}{\partial x_j} \overline{\frac{\partial \Phi_\varepsilon}{\partial x_i}} dx \\
&= \sum_{i,j=1}^N \int_{\Omega} a_{ij}\left(x, \frac{x}{\varepsilon}\right) \chi_{G}\left(\frac{x}{\varepsilon}\right) \frac{\partial u_\varepsilon}{\partial x_j} \overline{\frac{\partial \Phi_\varepsilon}{\partial x_i}} dx.
\end{aligned} \tag{3.16}$$

Bear in mind that, as $E \ni \varepsilon \to 0$, we have (see, e.g., [19, Lemma 2.4])

$$\sum_{i,j=1}^N \frac{\partial u_\varepsilon}{\partial x_j} \overline{\frac{\partial \Phi_\varepsilon}{\partial x_i}} \xrightarrow{2s} \sum_{i,j=1}^N \left(\frac{\partial u_0}{\partial x_j} + \frac{\partial u_1}{\partial y_j}\right)\left(\overline{\frac{\partial \psi_0}{\partial x_j}} + \overline{\frac{\partial \psi_1}{\partial y_j}}\right), \quad \text{in } L^2(\Omega). \tag{3.17}$$

We also recall that $a_{ij}(x,y)\chi_G(y) \in \mathcal{C}(\overline{\Omega}; L^2_{\text{per}}(Y))(1 \le i,j \le N)$ and that Property (2.1) in Definition 2.1 still holds for f in $\mathcal{C}(\overline{\Omega}; L^2_{\text{per}}(Y))$ instead of $L^2(\Omega; \mathcal{C}_{\text{per}}(Y))$ whenever the two-scale convergence therein is ensured (see, e.g., [14, Theorem 15]). Thus, as $E \ni \varepsilon \to 0$,

$$\begin{aligned}
a^\varepsilon(u_\varepsilon, \Phi_\varepsilon) &= \sum_{i,j=1}^N \int_{\Omega} a_{ij}\left(x, \frac{x}{\varepsilon}\right) \chi_{G}\left(\frac{x}{\varepsilon}\right) \frac{\partial u_\varepsilon}{\partial x_j} \overline{\frac{\partial \Phi_\varepsilon}{\partial x_i}} dx \\
&\longrightarrow \sum_{i,j=1}^N \iint_{\Omega\times Y} a_{ij}(x,y)\chi_G(y) \left(\frac{\partial u_0}{\partial x_j} + \frac{\partial u_1}{\partial y_j}\right)\left(\overline{\frac{\partial \psi_0}{\partial x_j}} + \overline{\frac{\partial \psi_1}{\partial y_j}}\right) dx\, dy \\
&= \sum_{i,j=1}^N \iint_{\Omega\times Y^*} a_{ij}(x,y) \left(\frac{\partial u_0}{\partial x_j} + \frac{\partial u_1}{\partial y_j}\right)\left(\overline{\frac{\partial \psi_0}{\partial x_j}} + \overline{\frac{\partial \psi_1}{\partial y_j}}\right) dx\, dy \\
&= a_\Omega(\mathbf{u}, \Phi),
\end{aligned} \tag{3.18}$$

which completes the proof. ◻

We now construct and point out the main properties of the so-called homogenized coefficients. Let $1 \leq j \leq N$, and fix $x \in \overline{\Omega}$. Put

$$a(x;u,v) = \sum_{i,j=1}^{N} \int_{Y^*} a_{ij}(x,y) \frac{\partial u}{\partial y_j} \overline{\frac{\partial v}{\partial y_i}} dy,$$
$$l_j(x,v) = \sum_{k=1}^{N} \int_{Y^*} a_{kj}(x,y) \overline{\frac{\partial v}{\partial y_k}} dy \tag{3.19}$$

for $u,v \in H^1_\#(Y)$. Equipped with the seminorm

$$N(u) = \|D_y u\|_{L^2(Y^*)^N} \quad \left(u \in H^1_\#(Y)\right), \tag{3.20}$$

$H^1_\#(Y)$ is a pre-Hilbert space that is nonseparate and noncomplete. Let $H^1_\#(Y^*)$ be its separated completion with respect to the seminorm $N(\cdot)$ and $\mathbf{i}$ the canonical mapping of $H^1_\#(Y)$ into $H^1_\#(Y^*)$. We recall that

(i) $H^1_\#(Y^*)$ is a Hilbert space;

(ii) $\mathbf{i}$ is linear;

(iii) $\mathbf{i}(H^1_\#(Y))$ is dense in $H^1_\#(Y^*)$;

(iv) $\|\mathbf{i}(u)\|_{H^1_\#(Y^*)} = N(u)$ for every u in $H^1_\#(Y)$;

(v) if F is a Banach space and l a continuous linear mapping of $H^1_\#(Y)$ into F, then there exists a unique continuous linear mapping $L : H^1_\#(Y^*) \to F$ such that $l = L \circ \mathbf{i}$.

Proposition 3.4. *Let $j = 1, \ldots, N$, and fix x in $\overline{\Omega}$. The noncoercive local variational problem*

$$u \in H^1_\#(Y), \quad a(x;u,v) = l_j(x,v), \quad \forall v \in H^1_\#(Y) \tag{3.21}$$

admits at least one solution. Moreover, if $\chi^j(x)$ and $\theta^j(x)$ are two solutions,

$$D_y \chi^j(x) = D_y \theta^j(x) \quad a.e.\ in\ Y^*. \tag{3.22}$$

Proof. Proceeding as in the proof of [19, Lemma 2.5], we can prove that there exists a unique hermitian, coercive, continuous sesquilinear form $A(x;\cdot,\cdot)$ on $H^1_\#(Y^*) \times H^1_\#(Y^*)$ such that $A(x;\mathbf{i}(u),\mathbf{i}(v)) = a(x;u,v)$ for all $u,v \in H^1_\#(Y)$. Based on (v) above, we consider the antilinear form $\mathbf{l}_j(x,\cdot)$ on $H^1_\#(Y^*)$ such that $\mathbf{l}_j(x,\mathbf{i}(u)) = l_j(x,u)$ for any $u \in H^1_\#(Y)$. Then, $\chi^j(x) \in H^1_\#(Y)$ satisfies (3.21) if and only if $\mathbf{i}(\chi^j(x))$ satisfies

$$\mathbf{i}\left(\chi^j(x)\right) \in H^1_\#(Y^*), \quad A\left(x;\mathbf{i}\left(\chi^j(x)\right),V\right) = \mathbf{l}_j(x,V), \quad \forall V \in H^1_\#(Y^*). \tag{3.23}$$

But $\mathbf{i}(\chi^j(x))$ is uniquely determined by (3.23) (see, e.g., [20, page 216]). We deduce that (3.21) admits at least one solution, and if $\chi^j(x)$ and $\theta^j(x)$ are two solutions, then $\mathbf{i}(\chi^j(x)) = \mathbf{i}(\theta^j(x))$, which means that $\chi^j(x)$ and $\theta^j(x)$ have the same neighborhoods in $H^1_\#(Y)$ or equivalently $N(\chi^j(x) - \theta^j(x)) = 0$. Hence, (3.22). □

Corollary 3.5. *Let $1 \leq i, j \leq N$, and x fixed in $\overline{\Omega}$. Let $\chi^j(x) \in H^1_\#(Y)$ be a solution to* (3.21). *The following homogenized coefficients*

$$q_{ij}(x) = \int_{Y^*} a_{ij}(x,y)dy - \sum_{l=1}^{N} \int_{Y^*} a_{il}(x,y) \frac{\partial \chi^j}{\partial y_l}(x,y)dy \tag{3.24}$$

are well defined in the sense that they do not depend on the solution to (3.21).

Lemma 3.6. *The following assertions are true:*

(i) $q_{ij} \in \mathcal{C}(\overline{\Omega})$,

(ii) $q_{ji} = \overline{q}_{ij}$,

(iii) *there exists a constant $\alpha_0 > 0$ such that*

$$\operatorname{Re} \sum_{i,j=1}^{N} q_{ij}(x)\xi_j\overline{\xi}_i \geq \alpha_0 |\xi|^2 \tag{3.25}$$

for all $x \in \overline{\Omega}$ and all $\xi \in \mathbb{C}^N$.

Proof. See for example, [21]. □

We are now in a position to state the main result of this paper.

3.2. *Homogenization Result*

Theorem 3.7. *For each $k \geq 1$ and each $\varepsilon \in E$, let $(\lambda^k_\varepsilon, u^k_\varepsilon)$ be the k'th eigencouple to* (1.1). *Then, there exists a subsequence E' of E such that*

$$\frac{1}{\varepsilon}\lambda^k_\varepsilon \longrightarrow \lambda^k_0, \quad \text{in } \mathbb{C} \text{ as } E \ni \varepsilon \longrightarrow 0, \tag{3.26}$$

$$P_\varepsilon u^k_\varepsilon \longrightarrow u^k_0, \quad \text{in } H^1_0(\Omega)\text{-weak as } E' \ni \varepsilon \longrightarrow 0, \tag{3.27}$$

$$P_\varepsilon u^k_\varepsilon \longrightarrow u^k_0, \quad \text{in } L^2(\Omega) \text{ as } E' \ni \varepsilon \longrightarrow 0, \tag{3.28}$$

$$\frac{\partial P_\varepsilon u^k_\varepsilon}{\partial x_j} \xrightarrow{2s} \frac{\partial u^k_0}{\partial x_j} + \frac{\partial u^k_1}{\partial y_j}, \quad \text{in } L^2(\Omega), \text{ as } E' \ni \varepsilon \longrightarrow 0 \quad (1 \leq j \leq N), \tag{3.29}$$

where $(\lambda^k_0, u^k_0) \in \mathbb{C} \times H^1_0(\Omega)$ is the k'th eigencouple to the spectral problem

$$-\sum_{i,j=1}^{N}\frac{\partial}{\partial x_i}\left(\frac{1}{|S|}q_{ij}(x)\frac{\partial u_0}{\partial x_j}\right)=\lambda_0 u_0, \quad in\ \Omega,$$
$$u_0=0, \quad on\ \partial\Omega, \tag{3.30}$$
$$\int_\Omega |u_0|^2 dx=\frac{1}{|S|},$$

where $u_1^k \in L^2(\Omega; H^1_\#(Y))$. *Moreover, for almost every* $x\in\Omega$, *the following hold true:*

(i) $u_1^k(x)$ *is a solution to the noncoercive variational problem*

$$u_1^k(x)\in H^1_\#(Y),$$
$$a\left(x; u_1^k(x), v\right)=-\sum_{i,j=1}^{N}\frac{\partial u_0^k}{\partial x_j}\int_{Y^*}a_{ij}(x,y)\frac{\overline{\partial v}}{\partial y_i}dy, \tag{3.31}$$
$$\forall v\in H^1_\#(Y).$$

(ii) *We have*

$$\mathbf{i}\left(u_1^k(x)\right)=\sum_{j=1}^{N}\frac{\partial u_0^k}{\partial x_j}(x)\mathbf{i}\left(\chi^j(x)\right), \tag{3.32}$$

where χ^j *is any function in* $H^1_\#(Y)$ *defined by the cell problem* (3.21).

Proof. Let us first recall that, according to the properties of the coefficients q_{ij} (Lemma 3.6), the spectral problem (3.30) admits a sequence of eigencouples with similar properties to those of problem (1.1). However, this is also proved by our homogenization process.

Now, fix $k\ge 1$. There exists a constant $0<c_1<\infty$ independent of ε such that

$$0<\lambda_\varepsilon^k\le c_1\mu_\varepsilon^k, \tag{3.33}$$

where

$$\mu_\varepsilon^k=\min_{W\in E^k}\left(\max_{v\in W\setminus\{0\}}\frac{\int_{\Omega^\varepsilon}|Dv|^2dx}{\int_{S^\varepsilon}|u_\varepsilon|^2 d\sigma_\varepsilon(x)}\right), \tag{3.34}$$

E^k still being the collection of subspaces of dimension k of V_ε. But it is proved in [5, Proposition 12.1] that $0<\mu_\varepsilon^k<c_2\varepsilon$, c_2 being a constant independent of ε. Hence the sequence $((1/\varepsilon)\lambda_\varepsilon^k)_{\varepsilon\in E}$ is bounded in $\mathbb{C}$.

Clearly, for fixed $E\ni\varepsilon>0$, u_ε^k lies in V_ε and

$$\sum_{i,j=1}^{N}\int_{\Omega^\varepsilon}a_{ij}^\varepsilon\frac{\partial u_\varepsilon^k}{\partial x_j}\frac{\overline{\partial v}}{\partial x_i}dx=\left(\frac{1}{\varepsilon}\lambda_\varepsilon^k\right)\varepsilon\int_{S^\varepsilon}u_\varepsilon^k\overline{v}\,d\sigma_\varepsilon(x) \tag{3.35}$$

for any $v \in V_\varepsilon$. Bear in mind that $\varepsilon \int_{S^\varepsilon} |u_\varepsilon^k|^2 d\sigma_\varepsilon(x) = 1$, and chose $v = u_\varepsilon^k$ in (3.35). The boundedness of the sequence $((1/\varepsilon)\lambda_\varepsilon^k)_{\varepsilon\in E}$ and the ellipticity assumption (1.2) implies at once by means of Proposition 3.1 that the sequence $(P_\varepsilon u_\varepsilon^k)_{\varepsilon\in E}$ is bounded in $H_0^1(\Omega)$. Theorem 2.3 and Proposition 2.8 apply simultaneously and give us $\mathbf{u}^k = (u_0^k, u_1^k) \in \mathbb{F}_0^1$ such that for some $\lambda_0^k \in \mathbb{C}$ and some subsequence $E' \subset E$ we have (3.26)–(3.29), where (3.28) is a direct consequence of (3.27) by the Rellich-Kondrachov theorem. For fixed $\varepsilon \in E'$, let Φ_ε be as in Lemma 3.3. Multiplying both sides of the first equality in (1.1) by Φ_ε and integrating over Ω leads us to the variational ε-problem

$$\sum_{i,j=1}^{N} \int_{\Omega^\varepsilon} a_{ij}^\varepsilon \frac{\partial P_\varepsilon u_\varepsilon^k}{\partial x_j} \frac{\overline{\partial \Phi_\varepsilon}}{\partial x_i} dx = \left(\frac{1}{\varepsilon}\lambda_\varepsilon^k\right) \varepsilon \int_{S^\varepsilon} \left(P_\varepsilon u_\varepsilon^k\right) \overline{\Phi}_\varepsilon d\sigma_\varepsilon(x). \tag{3.36}$$

Sending $\varepsilon \in E'$ to 0, keeping (3.26)–(3.29) and Lemma 3.3 in mind, we obtain

$$\sum_{i,j=1}^{N} \iint_{\Omega\times Y^*} a_{ij} \left(\frac{\partial u_0}{\partial x_j} + \frac{\partial u_1}{\partial y_j}\right) \left(\frac{\overline{\partial \psi_0}}{\partial x_j} + \frac{\overline{\partial \psi_1}}{\partial y_j}\right) dx\, dy = \lambda_0^k \iint_{\Omega\times S} u_0^k \overline{\psi_0} dx\, d\sigma(y). \tag{3.37}$$

The right-hand side follows by means of Proposition 2.8 as explained:

$$\begin{aligned} \varepsilon \int_{S^\varepsilon} \left(P_\varepsilon u_\varepsilon^k\right) \overline{\Phi}_\varepsilon d\sigma_\varepsilon(x) &= \varepsilon \int_{S^\varepsilon} \left(P_\varepsilon u_\varepsilon^k\right) \overline{\psi}_0 d\sigma_\varepsilon(x) + \varepsilon \left(\varepsilon \int_{S^\varepsilon} \left(P_\varepsilon u_\varepsilon^k\right) \overline{\psi}_1 \left(x, \frac{x}{\varepsilon}\right) d\sigma_\varepsilon(x)\right) \\ &\longrightarrow \iint_{\Omega\times S} u_0^k \overline{\psi_0} dx\, d\sigma(y) + 0, \quad \text{as } E' \ni \varepsilon \longrightarrow 0. \end{aligned} \tag{3.38}$$

Therefore, $(\lambda_0^k, \mathbf{u}^k) \in \mathbb{C} \times \mathbb{F}_0^1$ solves the following *global homogenized spectral problem*:

$$\begin{aligned} &\text{find } (\lambda, \mathbf{u}) \in \mathbb{C} \times \mathbb{F}_0^1 \text{ such that} \\ &\sum_{i,j=1}^{N} \iint_{\Omega\times Y^*} a_{ij} \left(\frac{\partial u_0}{\partial x_j} + \frac{\partial u_1}{\partial y_j}\right) \left(\frac{\overline{\partial \psi_0}}{\partial x_i} + \frac{\overline{\partial \psi_1}}{\partial y_i}\right) dx\, dy = \lambda |S| \int_\Omega u_0 \overline{\psi}_0 dx, \quad \forall \Phi \in \mathbb{F}_0^1. \end{aligned} \tag{3.39}$$

To prove (i), choose $\Phi = (\psi_0, \psi_1)$ in (3.39) such that $\psi_0 = 0$ and $\psi_1 = \varphi \otimes v_1$, where $\varphi \in \mathfrak{D}(\Omega)$ and $v_1 \in H_\#^1(Y)$ to get

$$\int_\Omega \varphi(x) \left[\sum_{i,j=1}^{N} \int_{Y^*} a_{ij} \left(\frac{\partial u_0^k}{\partial x_j} + \frac{\partial u_1^k}{\partial y_j}\right) \frac{\overline{\partial v_1}}{\partial y_i} dy\right] dx = 0. \tag{3.40}$$

Hence, by the arbitrariness of φ, we have a.e. in Ω

$$\sum_{i,j=1}^{N} \int_{Y^*} a_{ij} \left(\frac{\partial u_0^k}{\partial x_j} + \frac{\partial u_1^k}{\partial y_j}\right) \frac{\overline{\partial v_1}}{\partial y_i} dy = 0 \tag{3.41}$$

for any v_1 in $H_\#^1(Y)$, which is nothing but (3.31).

Regarding (ii), pick any $\chi^j(x)$ solution to the cell problem (3.21), and put $z(x) = \sum_{j=1}^{N}(\partial u_0^k/\partial x_j)(x)\chi^j(x)$.

By multiplying both sides of (3.21) by $-(\partial u_0^k/\partial x_j)(x)$ and then summing over $1 \leq j \leq N$, we see that $z(x)$ satisfies (3.31). Hence, $\mathbf{i}(z(x)) = \mathbf{i}(u^k(x))$ by uniqueness of the solution to the coercive variational problem in $H^1_\#(Y^*)$ corresponding to the noncoercive variational problem (3.31) (see the proof of Proposition 3.4). Thus, (3.32) follows since $\mathbf{i}$ is linear.

Now, by considering $\Phi = (\psi_0, \psi_1)$ in (3.39) such that $\psi_1 = 0$ and $\psi_0 \in \mathfrak{D}(\Omega)$, we get

$$\sum_{i,j=1}^{N}\iint_{\Omega\times Y^*} a_{ij}\left(\frac{\partial u_0^k}{\partial x_j} + \frac{\partial u_1^k}{\partial y_j}\right)\frac{\overline{\partial\psi_0}}{\partial x_i}\,dx\,dy = |S|\lambda_0^k\int_\Omega u_0^k\overline{\psi}_0 dx. \tag{3.42}$$

As (3.32) is equivalent (see the proof of Proposition 3.4) to

$$D_y u_1^k(x) = \sum_{j=1}^{N}\frac{\partial u_0^k}{\partial x_j}(x)D_y\chi^j(x), \quad \text{a.e. in } Y^*, \tag{3.43}$$

we arrive at

$$\sum_{i,j=1}^{N}\int_\Omega\left[\int_{Y^*} a_{ij}dy - \sum_{l=1}^{N}\int_{Y^*} a_{il}\frac{\partial\chi^j}{\partial y_l}dy\right]\frac{\partial u_0^k}{\partial x_j}\frac{\overline{\partial\psi_0}}{\partial x_i}\,dx = |S|\lambda_0^k\int_\Omega u_0^k\overline{\psi}_0 dx, \tag{3.44}$$

that is, (see (3.24))

$$\sum_{i,j=1}^{N}\int_\Omega \frac{1}{|S|}q_{ij}(x)\frac{\partial u_0^k}{\partial x_j}\frac{\overline{\partial\psi_0}}{\partial x_i}dx = \lambda_0^k\int_\Omega u_0^k\overline{\psi}_0 dx. \tag{3.45}$$

Thanks to the arbitrariness of ψ_0 and the weak derivative formula, we conclude that (λ_0^k, u_0^k) is the k'th eigencouple to (3.30) and the whole sequence $((1/\varepsilon)\lambda_\varepsilon^k)_{\varepsilon\in E}$ converges.

Finally, by using (3.28) and a similar line of reasoning as in the proof of Lemma 2.5, we arrive at

$$\lim_{E'\ni\varepsilon\to 0}\varepsilon\int_{S^\varepsilon}\left|P^\varepsilon u_\varepsilon^k\right|\left|P^\varepsilon u_\varepsilon^l\right|d\sigma_\varepsilon(x) = |S|\int_\Omega\left|u_0^k\right|\left|u_0^l\right|dx. \tag{3.46}$$

The normalization condition in (3.30) follows thereby, and moreover $\{u_0^k\}_{k=1}^\infty$ is an orthogonal basis in $L^2(\Omega)$. □

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