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Research Article

A Beale-Kato-Madja Criterion for Magneto-Micropolar Fluid Equations with Partial Viscosity

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We study the incompressible magneto-micropolar fluid equations with partial viscosity in $\mathbb{R}^n (n = 2, 3)$. A blow-up criterion of smooth solutions is obtained. The result is analogous to the celebrated Beale-Kato-Majda type criterion for the inviscid Euler equations of incompressible fluids.

1. Introduction

The incompressible magneto-micropolar fluid equations in $\mathbb{R}^n$ $(n = 2, 3)$ take the following form:

$$\begin{gathered}
\partial_t u - (\mu + \chi)\Delta u + u \cdot \nabla u - b \cdot \nabla b + \nabla\left(p + \frac{1}{2}|b|^2\right) - \chi\nabla \times v = 0, \\
\partial_t v - \gamma\Delta v - \kappa\nabla \operatorname{div} v + 2\chi v + u \cdot \nabla v - \chi\nabla \times u = 0, \\
\partial_t b - \nu\Delta b + u \cdot \nabla b - b \cdot \nabla u = 0, \\
\nabla \cdot u = 0, \qquad \nabla \cdot b = 0,
\end{gathered} \tag{1.1}$$

where $u(t,x)$, $v(t,x)$, $b(t,x)$ and $p(t,x)$ denote the velocity of the fluid, the microrotational velocity, magnetic field, and hydrostatic pressure, respectively. μ is the kinematic viscosity, χ is the vortex viscosity, γ and κ are spin viscosities, and $1/\nu$ is the magnetic Reynold.

The incompressible magneto-micropolar fluid equation (1.1) has been studied extensively (see [1–7]). In [2], the authors have proven that a weak solution to (1.1) has fractional time derivatives of any order less than $1/2$ in the two-dimensional case. In the three-dimensional case, a uniqueness result similar to the one for Navier-Stokes equations is given and the same result concerning fractional derivatives is obtained, but only for a more regular weak solution. Rojas-Medar [4] established local existence and uniqueness of strong solutions by the Galerkin method. Rojas-Medar and Boldrini [5] also proved the existence of weak solutions by the Galerkin method, and in 2D case, also proved the uniqueness of the weak solutions. Ortega-Torres and Rojas-Medar [3] proved global existence of strong solutions for small initial data. A Beale-Kato-Majda type blow-up criterion for smooth solution (u, v, b) to (1.1) that relies on the vorticity of velocity $\nabla \times u$ only is obtained by Yuan [7]. For regularity results, refer to Yuan [6] and Gala [1].

If $b = 0$, (1.1) reduces to micropolar fluid equations. The micropolar fluid equations was first developed by Eringen [8]. It is a type of fluids which exhibits the microrotational effects and microrotational inertia, and can be viewed as a non-Newtonian fluid. Physically, micropolar fluid may represent fluids consisting of rigid, randomly oriented (or spherical particles) suspended in a viscous medium, where the deformation of fluid particles is ignored. It can describe many phenomena that appeared in a large number of complex fluids such as the suspensions, animal blood, and liquid crystals which cannot be characterized appropriately by the Navier-Stokes equations, and that it is important to the scientists working with the hydrodynamic-fluid problems and phenomena. For more background, we refer to [9] and references therein. The existences of weak and strong solutions for micropolar fluid equations were proved by Galdi and Rionero [10] and Yamaguchi [11], respectively. Regularity criteria of weak solutions to the micropolar fluid equations are investigated in [12]. In [13], the authors gave sufficient conditions on the kinematics pressure in order to obtain regularity and uniqueness of the weak solutions to the micropolar fluid equations. The convergence of weak solutions of the micropolar fluids in bounded domains of $\mathbb{R}^n$ was investigated (see [14]). When the viscosities tend to zero, in the limit, a fluid governed by an Euler-like system was found.

If both $v = 0$ and $\chi = 0$, then (1.1) reduces to be the magneto-hydrodynamic (MHD) equations. There are numerous important progresses on the fundamental issue of the regularity for the weak solution to MHD systems (see [15–23]). Zhou [18] established Serrin-type regularity criteria in term of the velocity only. Logarithmically improved regularity criteria for MHD equations were established in [16, 23]. Regularity criteria for the 3D MHD equations in term of the pressure were obtained [19]. Zhou and Gala [20] obtained regularity criteria of solutions in term of u and $\nabla \times u$ in the multiplier spaces. A new regularity criterion for weak solutions to the viscous MHD equations in terms of the vorticity field in Morrey-Campanato spaces was established (see [21]). In [22], a regularity criterion $\nabla b \in L^1(0, T; \mathrm{BMO}(\mathbb{R}^2))$ for the 2D MHD system with zero magnetic diffusivity was obtained.

Regularity criteria for the generalized viscous MHD equations were also obtained in [24]. Logarithmically improved regularity criteria for two related models to MHD equations were established in [25] and [26], respectively. Lei and Zhou [27] studied the magneto-hydrodynamic equations with $v = 0$ and $\mu = \chi = 0$. Caflisch et al. [28] and Zhang and Liu [29] obtained blow-up criteria of smooth solutions to 3-D ideal MHD equations, respectively. Cannone et al. [30] showed a losing estimate for the ideal MHD equations and applied it to establish an improved blow-up criterion of smooth solutions to ideal MHD equations.

In this paper, we consider the magneto-micropolar fluid equations (1.1) with partial viscosity, that is, $\mu = \chi = 0$. Without loss of generality, we take $\gamma = \kappa = \nu = 1$. The corresponding magneto-micropolar fluid equations thus reads

$$\begin{aligned}
&\partial_t u + u \cdot \nabla u - b \cdot \nabla b + \nabla\left(p + \frac{1}{2}|b|^2\right) = 0,\\
&\partial_t v - \Delta v - \nabla \operatorname{div} v + u \cdot \nabla v = 0,\\
&\partial_t b - \Delta b + u \cdot \nabla b - b \cdot \nabla u = 0,\\
&\nabla \cdot u = 0, \qquad \nabla \cdot b = 0.
\end{aligned} \tag{1.2}$$

In the absence of global well-posedness, the development of blow-up/non blow-up theory is of major importance for both theoretical and practical purposes. For incompressible Euler and Navier-Stokes equations, the well-known Beale-Kato-Majda's criterion [31] says that any solution u is smooth up to time T under the assumption that $\int_0^T \|\nabla \times u(t)\|_{L^\infty} dt < \infty$. Beale-Kato-Majdas criterion is slightly improved by Kozono and Taniuchi [32] under the assumption $\int_0^T \|\nabla \times u(t)\|_{\mathrm{BMO}} dt < \infty$. In this paper, we obtain a Beale-Kato-Majda type blow-up criterion of smooth solutions to the magneto-micropolar fluid equations (1.2).

Now we state our results as follows.

Theorem 1.1. *Let $u_0, v_0, b_0 \in H^m(\mathbb{R}^n)$ $(n = 2, 3)$, $m \geq 3$ with $\nabla \cdot u_0 = 0$, $\nabla \cdot b_0 = 0$. Assume that (u, v, b) is a smooth solution to* (1.2) *with initial data $u(0, x) = u_0(x)$, $v(0, x) = v_0(x)$, $b(0, x) = b_0(x)$ for $0 \leq t < T$. If u satisfies*

$$\int_0^T \frac{\|\nabla \times u(t)\|_{BMO}}{\sqrt{\ln(e + \|\nabla \times u(t)\|_{BMO})}} dt < \infty, \tag{1.3}$$

then the solution (u, v, b) can be extended beyond $t = T$.

We have the following corollary immediately.

Corollary 1.2. *Let $u_0, v_0, b_0 \in H^m(\mathbb{R}^n)$ $(n = 2, 3)$, $m \geq 3$ with $\nabla \cdot u_0 = 0$, $\nabla \cdot b_0 = 0$. Assume that (u, v, b) is a smooth solution to* (1.2) *with initial data $u(0, x) = u_0(x)$, $v(0, x) = v_0(x)$, $b(0, x) = b_0(x)$ for $0 \leq t < T$. Suppose that T is the maximal existence time, then*

$$\int_0^T \frac{\|\nabla \times u(t)\|_{BMO}}{\sqrt{\ln(e + \|\nabla \times u(t)\|_{BMO})}} dt = \infty. \tag{1.4}$$

The paper is organized as follows. We first state some preliminaries on functional settings and some important inequalities in Section 2 and then prove the blow-up criterion of smooth solutions to the magneto-micropolar fluid equations (1.2) in Section 3.

2. Preliminaries

Let $\mathcal{S}(\mathbb{R}^n)$ be the Schwartz class of rapidly decreasing functions. Given $f \in \mathcal{S}(\mathbb{R}^n)$, its Fourier transform $\mathcal{F}f = \widehat{f}$ is defined by

$$\widehat{f}(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)dx \tag{2.1}$$

and for any given $g \in \mathcal{S}(\mathbb{R}^n)$, its inverse Fourier transform $\mathcal{F}^{-1}g = \check{g}$ is defined by

$$\check{g}(x) = \int_{\mathbb{R}^n} e^{ix\cdot\xi} g(\xi)d\xi. \tag{2.2}$$

Next, let us recall the Littlewood-Paley decomposition. Choose a nonnegative radial functions $\phi \in \mathcal{S}(\mathbb{R}^n)$, supported in $\mathcal{C} = \{\xi \in \mathbb{R}^n : (3/4) \le |\xi| \le 8/3\}$ such that

$$\sum_{k=-\infty}^{\infty} \phi\left(2^{-k}\xi\right) = 1, \quad \forall \xi \in \mathbb{R}^n \setminus \{0\}. \tag{2.3}$$

The frequency localization operator is defined by

$$\Delta_k f = \int_{\mathbb{R}^n} \check{\phi}(y) f\left(x - 2^{-k}y\right)dy. \tag{2.4}$$

Let us now define homogeneous function spaces (see e.g., [33, 34]). For $(p,q) \in [1,\infty]^2$ and $s \in \mathbb{R}$, the homogeneous Triebel-Lizorkin space $\dot{F}^s_{p,q}$ as the set of tempered distributions f such that

$$\|f\|_{\dot{F}^s_{p,q}} \triangleq \left\| \left(\sum_{k\in\mathbb{Z}} 2^{sqk} |\Delta_k f|^q \right)^{1/q} \right\|_{L^p} < \infty. \tag{2.5}$$

BMO denotes the homogenous space of bounded mean oscillations associated with the norm

$$\|f\|_{\mathrm{BMO}} \triangleq \sup_{x\in\mathbb{R}^n, R>0} \frac{1}{|B_R(x)|} \int_{B_R(x)} \left| f(y) - \frac{1}{|B_R(y)|} \int_{B_R(y)} f(z)dz \right| dy. \tag{2.6}$$

Thereafter, we will use the fact $\mathrm{BMO} = \dot{F}^0_{\infty,2}$.

In what follows, we will make continuous use of Bernstein inequalities, which comes from [35].

Lemma 2.1. *For any $s \in \mathbb{N}$, $1 \le p \le q \le \infty$ and $f \in L^p(\mathbb{R}^n)$, then*

$$c2^{km}\|\Delta_k f\|_{L^p} \le \|\nabla^m \Delta_k f\|_{L^p} \le C2^{km}\|\Delta_k f\|_{L^p},$$
$$\|\Delta_k f\|_{L^q} \le C2^{n(1/p-1/q)k}\|\Delta_k f\|_{L^p} \tag{2.7}$$

hold, where c and C are positive constants independent of f and k.

The following inequality is well-known Gagliardo-Nirenberg inequality.

Lemma 2.2. *There exists a uniform positive constant $C > 0$ such that*

$$\left\|\nabla^i u\right\|_{L^{2m/i}} \le C\|u\|_{L^\infty}^{1-i/m}\|\nabla^m u\|_{L^2}^{i/m}, \quad 0 \le i \le m \tag{2.8}$$

holds for all $u \in L^\infty(\mathbb{R}^n) \cap H^m(\mathbb{R}^n)$.

The following lemma comes from [36].

Lemma 2.3. *The following calculus inequality holds:*

$$\|\nabla^m(u \cdot \nabla v) - u \cdot \nabla\nabla^m v\|_{L^2} \le C(\|\nabla u\|_{L^\infty}\|\nabla^m v\|_{L^2} + \|\nabla v\|_{L^\infty}\|\nabla^m u\|_{L^2}). \tag{2.9}$$

Lemma 2.4. *There is a uniform positive constant C, such that*

$$\|\nabla u\|_{L^\infty} \le C\left(1 + \|u\|_{L^2} + \|\nabla \times u\|_{BMO}\sqrt{\ln(e + \|u\|_{H^3})}\right) \tag{2.10}$$

holds for all vectors $u \in H^3(\mathbb{R}^n)$ $(n = 2, 3)$ with $\nabla \cdot u = 0$.

Proof. The proof can be found in [37]. For completeness, the proof will be also sketched here. It follows from Littlewood-Paley decomposition that

$$\nabla u = \sum_{k=-\infty}^{0} \Delta_k \nabla u + \sum_{k=1}^{A} \Delta_k \nabla u + \sum_{k=A+1}^{\infty} \Delta_k \nabla u. \tag{2.11}$$

Using (2.7) and (2.11), we obtain

$$\begin{aligned}
\|\nabla u\|_{L^\infty} &\le \sum_{k=-\infty}^{0} \|\Delta_k \nabla u\|_{L^\infty} + \left\|\sum_{k=1}^{A} \Delta_k \nabla u\right\|_{L^\infty} + \sum_{k=A+1}^{\infty} \|\Delta_k \nabla u\|_{L^\infty} \\
&\le C\sum_{k=-\infty}^{0} 2^{(1+n/2)k}\|\Delta_k u\|_{L^2} + A^{1/2}\left\|\left(\sum_{k=1}^{A}|\Delta_k \nabla u|^2\right)^{1/2}\right\|_{L^\infty} + \sum_{k=A+1}^{\infty} 2^{-(2-n/2)k}\left\|\Delta_k \nabla^3 u\right\|_{L^2} \\
&\le C\left(\|u\|_{L^2} + A^{1/2}\|\nabla u\|_{\mathrm{BMO}} + 2^{-(2-n/2)A}\left\|\nabla^3 u\right\|_{L^2}\right).
\end{aligned} \tag{2.12}$$

By the Biot-Savard law, we have a representation of ∇u in terms of $\nabla \times u$ as

$$u_{x_j} = R_j(R \times \nabla u), \quad j = 1, 2, \ldots, n. \tag{2.13}$$

where $R = (R_1, \ldots, R_n)$, $R_j = (\partial/\partial x_j)(-\Delta)^{-1/2}$ denote the Riesz transforms. Since R is a bounded operator in BMO, this yields

$$\|\nabla u\|_{\mathrm{BMO}} \le C\|\nabla \times u\|_{\mathrm{BMO}} \tag{2.14}$$

with $C = C(n)$. Taking

$$A = \left[\frac{1}{(2 - n/2)\ln 2}\ln(e + \|u\|_{H^3})\right] + 1. \tag{2.15}$$

It follows from (2.12), (2.14), and (2.15) that (2.10) holds. Thus, the lemma is proved. □

In order to prove Theorem 1.1, we need the following interpolation inequalities in two and three space dimensions.

Lemma 2.5. *In three space dimensions, the following inequalities*

$$\begin{aligned} \|\nabla u\|_{L^2} &\le C\|u\|_{L^2}^{2/3}\left\|\nabla^3 u\right\|_{L^2}^{1/3}, \\ \|u\|_{L^\infty} &\le C\|u\|_{L^2}^{1/4}\left\|\nabla^2 u\right\|_{L^2}^{3/4}, \\ \|u\|_{L^4} &\le C\|u\|_{L^2}^{3/4}\left\|\nabla^3 u\right\|_{L^2}^{1/4} \end{aligned} \tag{2.16}$$

hold, and in two space dimensions, the following inequalities

$$\begin{aligned} \|\nabla u\|_{L^2} &\le C\|u\|_{L^2}^{2/3}\left\|\nabla^3 u\right\|_{L^2}^{1/3}, \\ \|u\|_{L^\infty} &\le C\|u\|_{L^2}^{1/2}\left\|\nabla^2 u\right\|_{L^2}^{1/2}, \\ \|u\|_{L^4} &\le C\|u\|_{L^2}^{5/6}\left\|\nabla^3 u\right\|_{L^2}^{1/6} \end{aligned} \tag{2.17}$$

hold.

Proof. (2.16) and (2.17) are of course well known. In fact, we can obtain them by Sobolev embedding and the scaling techniques. In what follows, we only prove the last inequality in (2.16) and (2.17). Sobolev embedding implies that $H^3(\mathbb{R}^n) \hookrightarrow L^4(\mathbb{R}^n)$ for $n = 2, 3$. Consequently, we get

$$\|u\|_{L^4} \le C\left(\|u\|_{L^2} + \left\|\nabla^3 u\right\|_{L^2}\right). \tag{2.18}$$

For any given $0 \neq u \in H^3(\mathbb{R}^n)$ and $\delta > 0$, let

$$u_\delta(x) = u(\delta x). \tag{2.19}$$

By (2.18) and (2.19), we obtain

$$\|u_\delta\|_{L^4} \le C\left(\|u_\delta\|_{L^2} + \left\|\nabla^3 u_\delta\right\|_{L^2}\right), \tag{2.20}$$

which is equivalent to

$$\|u\|_{L^4} \le C\left(\delta^{-n/4}\|u\|_{L^2} + \delta^{3-n/4}\left\|\nabla^3 u\right\|_{L^2}\right). \tag{2.21}$$

Taking $\delta = \|u\|_{L^2}^{1/3}\|\nabla^3 u\|_{L^2}^{-1/3}$ and $n = 3$ and $n = 2$, respectively. From (2.21), we immediately get the last inequality in (2.16) and (2.17). Thus, we have completed the proof of Lemma 2.5. □

3. Proof of Main Results

Proof of Theorem 1.1. Multiplying (1.2) by (u, v, b), respectively, then integrating the resulting equation with respect to x on $\mathbb{R}^n$ and using integration by parts, we get

$$\frac{1}{2}\frac{d}{dt}\left(\|u(t)\|_{L^2}^2 + \|v(t)\|_{L^2}^2 + \|b(t)\|_{L^2}^2\right) + \|\nabla v(t)\|_{L^2}^2 + \|\operatorname{div} v(t)\|_{L^2}^2 + \|\nabla b(t)\|_{L^2}^2 = 0, \tag{3.1}$$

where we have used $\nabla \cdot u = 0$ and $\nabla \cdot b = 0$.

Integrating with respect to t, we obtain

$$\begin{aligned}\|u(t)\|_{L^2}^2 + \|v(t)\|_{L^2}^2 + \|b(t)\|_{L^2}^2 + 2\int_0^t \|\nabla v(\tau)\|_{L^2}^2 d\tau + 2\int_0^t \|\operatorname{div} v(\tau)\|_{L^2}^2 d\tau \\ + 2\int_0^t \|\nabla b(\tau)\|_{L^2}^2 d\tau = \|u_0\|_{L^2}^2 + \|v_0\|_{L^2}^2 + \|b_0\|_{L^2}^2.\end{aligned} \tag{3.2}$$

Applying ∇ to (1.2) and taking the L^2 inner product of the resulting equation with $(\nabla u, \nabla v, \nabla b)$, with help of integration by parts, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}&\left(\|\nabla u(t)\|_{L^2}^2 + \|\nabla v(t)\|_{L^2}^2 + \|\nabla b(t)\|_{L^2}^2\right) + \left\|\nabla^2 v(t)\right\|_{L^2}^2 + \|\operatorname{div} \nabla v(t)\|_{L^2}^2 + \left\|\nabla^2 b(t)\right\|_{L^2}^2 \\ &= -\int_{\mathbb{R}^n} \nabla(u \cdot \nabla u)\nabla u \, dx + \int_{\mathbb{R}^n} \nabla(b \cdot \nabla b)\nabla u \, dx - \int_{\mathbb{R}^n} \nabla(u \cdot \nabla v)\nabla v \, dx \\ &\quad - \int_{\mathbb{R}^n} \nabla(u \cdot \nabla b)\nabla b \, dx + \int_{\mathbb{R}^n} \nabla(b \cdot \nabla u)\nabla b \, dx.\end{aligned} \tag{3.3}$$

It follows from (3.3) and $\nabla \cdot u = 0$, $\nabla \cdot b = 0$ that

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left(\|\nabla u(t)\|_{L^2}^2+\|\nabla v(t)\|_{L^2}^2+\|\nabla b(t)\|_{L^2}^2\right)+\left\|\nabla^2 v(t)\right\|_{L^2}^2+\|\operatorname{div}\nabla v(t)\|_{L^2}^2+\left\|\nabla^2 b(t)\right\|_{L^2}^2 \\ &\quad\le 3\|\nabla u(t)\|_{L^\infty}\left(\|\nabla u(t)\|_{L^2}^2+\|\nabla v(t)\|_{L^2}^2+\|\nabla b(t)\|_{L^2}^2\right). \end{aligned} \tag{3.4}$$

By Gronwall inequality, we get

$$\begin{aligned} &\|\nabla u(t)\|_{L^2}^2+\|\nabla v(t)\|_{L^2}^2+\|\nabla b(t)\|_{L^2}^2+2\int_{t_1}^t\left\|\nabla^2 v(\tau)\right\|_{L^2}^2 d\tau \\ &\quad+2\int_{t_1}^t\|\operatorname{div}\nabla v(\tau)\|_{L^2}^2 d\tau+2\int_{t_1}^t\left\|\nabla^2 b(\tau)\right\|_{L^2}^2 d\tau \\ &\quad\le\left(\|\nabla u(t_1)\|_{L^2}^2+\|\nabla v(t_1)\|_{L^2}^2+\|\nabla b(t_1)\|_{L^2}^2\right)\exp\left\{C\int_{t_1}^t\|\nabla u(\tau)\|_{L^\infty}d\tau\right\}. \end{aligned} \tag{3.5}$$

Thanks to (1.3), we know that for any small constant $\varepsilon > 0$, there exists $T_\star < T$ such that

$$\int_{T_\star}^T \frac{\|\nabla\times u(t)\|_{\mathrm{BMO}}}{\sqrt{\ln(e+\|\nabla\times u(t)\|_{\mathrm{BMO}})}}dt\le\varepsilon. \tag{3.6}$$

Let

$$A(t)=\sup_{T_\star\le\tau\le t}\left(\left\|\nabla^3 u(\tau)\right\|_{L^2}^2+\left\|\nabla^3 v(\tau)\right\|_{L^2}^2+\left\|\nabla^3 b(\tau)\right\|_{L^2}^2\right),\quad T_\star\le t<T. \tag{3.7}$$

It follows from (3.5), (3.6), (3.7), and Lemma 2.4 that

$$\begin{aligned} &\|\nabla u(t)\|_{L^2}^2+\|\nabla v(t)\|_{L^2}^2+\|\nabla b(t)\|_{L^2}^2+2\int_{T_\star}^t\left\|\nabla^2 v(\tau)\right\|_{L^2}^2 d\tau \\ &\quad+2\int_{T_\star}^t\|\operatorname{div}\nabla v(\tau)\|_{L^2}^2 d\tau+2\int_{T_\star}^t\left\|\nabla^2 b(\tau)\right\|_{L^2}^2 d\tau \\ &\quad\le C_1\exp\left\{C_0\int_{T_\star}^t\|\nabla\times u\|_{\mathrm{BMO}}\sqrt{\ln(e+\|u\|_{H^3})}d\tau\right\} \\ &\quad\le C_1\exp\{C_0\varepsilon\ln(e+A(t))\} \\ &\quad\le C_1(e+A(t))^{C_0\varepsilon},\quad T_\star\le t<T, \end{aligned} \tag{3.8}$$

where C_1 depends on $\|\nabla u(T_\star)\|_{L^2}^2+\|\nabla v(T_\star)\|_{L^2}^2+\|\nabla b(T_\star)\|_{L^2}^2$, while C_0 is an absolute positive constant.

Applying ∇^m to the first equation of (1.2), then taking L^2 inner product of the resulting equation with $\nabla^m u$, using integration by parts, we get

$$\frac{1}{2}\frac{d}{dt}\|\nabla^m u(t)\|_{L^2}^2 = -\int_{\mathbb{R}^n} \nabla^m (u\cdot\nabla u)\nabla^m u\,dx + \int_{\mathbb{R}^n} \nabla^m (b\cdot\nabla b)\nabla^m u\,dx. \tag{3.9}$$

Similarly, we obtain

$$\begin{gathered}\frac{1}{2}\frac{d}{dt}\|\nabla^m v(t)\|_{L^2}^2 + \|\nabla^m\nabla v(t)\|_{L^2}^2 + \|\operatorname{div}\nabla^m v(t)\|_{L^2}^2 = -\int_{\mathbb{R}^n} \nabla^m (u\cdot\nabla v)\nabla^m v\,dx,\\ \frac{1}{2}\frac{d}{dt}\|\nabla^m b(t)\|_{L^2}^2 + \|\nabla^m\nabla b(t)\|_{L^2}^2 = -\int_{\mathbb{R}^n} \nabla^m (u\cdot\nabla b)\nabla^m b\,dx + \int_{\mathbb{R}^n} \nabla^m (b\cdot\nabla u)\nabla^m b\,dx.\end{gathered} \tag{3.10}$$

Using (3.9), (3.10), $\nabla\cdot u = 0$, $\nabla\cdot b = 0$, and integration by parts, we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\|\nabla^m u(t)\|_{L^2}^2 + \|\nabla^m v(t)\|_{L^2}^2 + \|\nabla^m b(t)\|_{L^2}^2\right)\\ &\quad + \|\nabla^m\nabla v(t)\|_{L^2}^2 + \|\operatorname{div}\nabla^m v(t)\|_{L^2}^2 + \|\nabla^m\nabla b(t)\|_{L^2}^2\\ &= -\int_{\mathbb{R}^n} [\nabla^m (u\cdot\nabla u) - u\cdot\nabla\nabla^m u]\nabla^m u\,dx + \int_{\mathbb{R}^n} [\nabla^m (b\cdot\nabla b) - b\cdot\nabla\nabla^m b]\nabla^m u\,dx\\ &\quad - \int_{\mathbb{R}^n} [\nabla^m (u\cdot\nabla v) - u\cdot\nabla\nabla^m v]\nabla^m v\,dx - \int_{\mathbb{R}^n} [\nabla^m (u\cdot\nabla b) - u\cdot\nabla\nabla^m b]\nabla^m b\,dx\\ &\quad + \int_{\mathbb{R}^n} [\nabla^m (b\cdot\nabla u) - b\cdot\nabla\nabla^m u]\nabla^m b\,dx.\end{aligned} \tag{3.11}$$

In what follows, for simplicity, we will set $m = 3$.

From Hölder inequality and Lemma 2.3, we get

$$\left|-\int_{\mathbb{R}^n}\left[\nabla^3(u\cdot\nabla u) - u\cdot\nabla\nabla^3 u\right]\nabla^3 u\,dx\right| \le C\|\nabla u(t)\|_{L^\infty}\left\|\nabla^3 u(t)\right\|_{L^2}^2. \tag{3.12}$$

Using integration by parts and Hölder inequality, we obtain

$$\begin{aligned}&\left|-\int_{\mathbb{R}^n}\left[\nabla^3(u\cdot\nabla v) - u\cdot\nabla\nabla^3 v\right]\nabla^3 v\,dx\right|\\ &\quad \le 7\|\nabla u(t)\|_{L^\infty}\left\|\nabla^3 v(t)\right\|_{L^2}^2 + 4\|\nabla u(t)\|_{L^\infty}\left\|\nabla^2 v(t)\right\|_{L^2}\left\|\nabla^4 v(t)\right\|_{L^2}\\ &\qquad + \left\|\nabla^2 u(t)\right\|_{L^4}\|\nabla v(t)\|_{L^4}\left\|\nabla^4 v(t)\right\|_{L^2}.\end{aligned} \tag{3.13}$$

By Lemma 2.5, Young inequality, and (3.8), we deduce that

$$
\begin{aligned}
&4\|\nabla u(t)\|_{L^\infty}\left\|\nabla^2 v(t)\right\|_{L^2}\left\|\nabla^4 v(t)\right\|_{L^2} \\
&\quad\le C\|\nabla u(t)\|_{L^\infty}\|\nabla v(t)\|_{L^2}^{2/3}\left\|\nabla^4 v(t)\right\|_{L^2}^{4/3} \\
&\quad\le \frac{1}{4}\left\|\nabla^4 v(t)\right\|_{L^2}^2 + C\|\nabla u(t)\|_{L^\infty}^3\|\nabla v(t)\|_{L^2}^2 \\
&\quad\le \frac{1}{4}\left\|\nabla^4 v(t)\right\|_{L^2}^2 + C\|\nabla u(t)\|_{L^\infty}\|\nabla u(t)\|_{L^2}^{1/2}\left\|\nabla^3 u(t)\right\|_{L^2}^{3/2}\|\nabla v(t)\|_{L^2}^2 \\
&\quad\le \frac{1}{4}\left\|\nabla^4 v(t)\right\|_{L^2}^2 + C\|\nabla u(t)\|_{L^\infty}(e+A(t))^{(5/4)C_0\varepsilon}A^{3/4}(t)
\end{aligned}
\tag{3.14}
$$

in 3D and

$$
\begin{aligned}
&4\|\nabla u(t)\|_{L^\infty}\left\|\nabla^2 v(t)\right\|_{L^2}\left\|\nabla^4 v(t)\right\|_{L^2} \\
&\quad\le C\|\nabla u(t)\|_{L^\infty}\|\nabla v(t)\|_{L^2}^{2/3}\left\|\nabla^4 v(t)\right\|_{L^2}^{4/3} \\
&\quad\le \frac{1}{4}\left\|\nabla^4 v(t)\right\|_{L^2}^2 + C\|\nabla u(t)\|_{L^\infty}^3\|\nabla v(t)\|_{L^2}^2 \\
&\quad\le \frac{1}{4}\left\|\nabla^4 v(t)\right\|_{L^2}^2 + C\|\nabla u(t)\|_{L^\infty}\|\nabla u(t)\|_{L^2}\left\|\nabla^3 u(t)\right\|_{L^2}\|\nabla v(t)\|_{L^2}^2 \\
&\quad\le \frac{1}{4}\left\|\nabla^4 v(t)\right\|_{L^2}^2 + C\|\nabla u(t)\|_{L^\infty}(e+A(t))^{(3/2)C_0\varepsilon}A^{1/2}(t)
\end{aligned}
\tag{3.15}
$$

in 2D.

From Lemmas 2.2 and 2.5, Young inequality, and (3.8), we have

$$
\begin{aligned}
&\left\|\nabla^2 u(t)\right\|_{L^4}\|\nabla v(t)\|_{L^4}\left\|\nabla^4 v(t)\right\|_{L^2} \\
&\quad\le C\|\nabla u(t)\|_{L^\infty}^{1/2}\left\|\nabla^3 u(t)\right\|_{L^2}^{1/2}\|\nabla v(t)\|_{L^2}^{3/4}\left\|\nabla^4 v(t)\right\|_{L^2}^{5/4} \\
&\quad\le \frac{1}{4}\left\|\nabla^4 v(t)\right\|_{L^2}^2 + C\|\nabla u(t)\|_{L^\infty}^{4/3}\left\|\nabla^3 u(t)\right\|_{L^2}^{4/3}\|\nabla v(t)\|_{L^2}^2 \\
&\quad\le \frac{1}{4}\left\|\nabla^4 v(t)\right\|_{L^2}^2 + C\|\nabla u(t)\|_{L^\infty}\|\nabla u(t)\|_{L^2}^{1/12}\left\|\nabla^3 u(t)\right\|_{L^2}^{19/12}\|\nabla v(t)\|_{L^2}^2 \\
&\quad\le \frac{1}{4}\left\|\nabla^4 v(t)\right\|_{L^2}^2 + C\|\nabla u(t)\|_{L^\infty}(e+A(t))^{(25/24)C_0\varepsilon}A^{19/24}(t)
\end{aligned}
\tag{3.16}
$$

in 3D and

$$
\begin{aligned}
&\left\|\nabla^2 u(t)\right\|_{L^4}\|\nabla v(t)\|_{L^4}\left\|\nabla^4 v(t)\right\|_{L^2} \\
&\quad\le C\|\nabla u(t)\|_{L^\infty}^{1/2}\left\|\nabla^3 u(t)\right\|_{L^2}^{1/2}\|\nabla v(t)\|_{L^2}^{5/6}\left\|\nabla^4 v(t)\right\|_{L^2}^{7/6} \\
&\quad\le \frac{1}{4}\left\|\nabla^4 v(t)\right\|_{L^2}^2 + C\|\nabla u(t)\|_{L^\infty}^{6/5}\left\|\nabla^3 u(t)\right\|_{L^2}^{6/5}\|\nabla v(t)\|_{L^2}^2 \\
&\quad\le \frac{1}{4}\left\|\nabla^4 v(t)\right\|_{L^2}^2 + C\|\nabla u(t)\|_{L^\infty}\|\nabla u(t)\|_{L^2}^{1/10}\left\|\nabla^3 u(t)\right\|_{L^2}^{13/10}\|\nabla v(t)\|_{L^2}^2 \\
&\quad\le \frac{1}{4}\left\|\nabla^4 v(t)\right\|_{L^2}^2 + C\|\nabla u(t)\|_{L^\infty}(e + A(t))^{(21/20)C_0\varepsilon}A^{13/20}(t)
\end{aligned}
\tag{3.17}
$$

in 2D.

Consequently, we get

$$
\begin{aligned}
&4\|\nabla u(t)\|_{L^\infty}\left\|\nabla^2 v(t)\right\|_{L^2}\left\|\nabla^4 v(t)\right\|_{L^2} \\
&\quad\le \frac{1}{4}\left\|\nabla^4 v(t)\right\|_{L^2}^2 + C\|\nabla u(t)\|_{L^\infty}(e + A(t)), \\
&\left\|\nabla^2 u(t)\right\|_{L^4}\|\nabla v(t)\|_{L^4}\left\|\nabla^4 v(t)\right\|_{L^2} \\
&\quad\le \frac{1}{4}\left\|\nabla^4 v(t)\right\|_{L^2}^2 + C\|\nabla u(t)\|_{L^\infty}(e + A(\mathrm{t}))
\end{aligned}
\tag{3.18}
$$

provided that

$$
\varepsilon \le \frac{1}{5C_0}. \tag{3.19}
$$

It follows from (3.13) and (3.18) that

$$
\begin{aligned}
&\left| -\int_{\mathbb{R}^n}\left[\nabla^3(u\cdot\nabla v) - u\cdot\nabla\nabla^3 v\right]\nabla^3 v\,dx\right| \\
&\quad\le \frac{1}{2}\left\|\nabla^4 v(t)\right\|_{L^2}^2 + C\|\nabla u(t)\|_{L^\infty}(e + A(t)).
\end{aligned}
\tag{3.20}
$$

Similarly, we obtain

$$\begin{aligned}
&\left|-\int_{\mathbb{R}^n}\left[\nabla^3(u\cdot\nabla b)-u\cdot\nabla\nabla^3 b\right]\nabla^3 b\,dx\right| \\
&\qquad\le \frac{1}{6}\left\|\nabla^4 b(t)\right\|_{L^2}^2 + C\|\nabla u(t)\|_{L^\infty}(e+A(t)),\\
&\left|\int_{\mathbb{R}^n}\left[\nabla^3(b\cdot\nabla b)-b\cdot\nabla\nabla^3 b\right]\nabla^3 u\,dx\right| \\
&\qquad\le \frac{1}{6}\left\|\nabla^4 b(t)\right\|_{L^2}^2 + C\|\nabla u(t)\|_{L^\infty}(e+A(t)),\\
&\left|\int_{\mathbb{R}^n}\left[\nabla^3(b\cdot\nabla u)-b\cdot\nabla\nabla^3 u\right]\nabla^3 b\,dx\right| \\
&\qquad\le \frac{1}{6}\left\|\nabla^4 b(t)\right\|_{L^2}^2 + C\|\nabla u(t)\|_{L^\infty}(e+A(t)).
\end{aligned} \tag{3.21}$$

Combining (3.11), (3.12), (3.20), and (3.21) yields

$$\begin{aligned}
&\frac{d}{dt}\left(\left\|\nabla^3 u(t)\right\|_{L^2}^2+\left\|\nabla^3 v(t)\right\|_{L^2}^2+\left\|\nabla^3 b(t)\right\|_{L^2}^2\right)+\left\|\nabla^4 v(t)\right\|_{L^2}^2+\left\|\operatorname{div}\nabla^3 v(t)\right\|_{L^2}^2+\left\|\nabla^4 b(t)\right\|_{L^2}^2\\
&\quad\le C\|\nabla u(t)\|_{L^\infty}(e+A(t))
\end{aligned} \tag{3.22}$$

for all $T_\star \le t < T$.

Integrating (3.22) with respect to t from $T_\star$ to τ and using Lemma 2.4, we have

$$\begin{aligned}
&e+\left\|\nabla^3 u(\tau)\right\|_{L^2}^2+\left\|\nabla^3 v(\tau)\right\|_{L^2}^2+\left\|\nabla^3 b(\tau)\right\|_{L^2}^2\\
&\quad\le e+\left\|\nabla^3 u(T_\star)\right\|_{L^2}^2+\left\|\nabla^3 v(T_\star)\right\|_{L^2}^2+\left\|\nabla^3 b(T_\star)\right\|_{L^2}^2\\
&\qquad+C_2\int_{T_\star}^{\tau}\left[1+\|u\|_{L^2}+\|\nabla\times u(s)\|_{\mathrm{BMO}}\sqrt{\ln(e+A(s))}\right](e+A(s))ds,
\end{aligned} \tag{3.23}$$

which implies

$$\begin{aligned}
e+A(t)\le{}& e+\left\|\nabla^3 u(T_\star)\right\|_{L^2}^2+\left\|\nabla^3 v(T_\star)\right\|_{L^2}^2+\left\|\nabla^3 b(T_\star)\right\|_{L^2}^2\\
&+C_2\int_{T_\star}^{t}\left[1+\|u\|_{L^2}+\|\nabla\times u(\tau)\|_{\mathrm{BMO}}\sqrt{\ln(e+A(\tau))}\right](e+A(\tau))d\tau.
\end{aligned} \tag{3.24}$$

For all $T_\star \le t < T$, from Gronwall inequality and (3.24), we obtain

$$e + \left\|\nabla^3 u(t)\right\|_{L^2}^2 + \left\|\nabla^3 v(t)\right\|_{L^2}^2 + \left\|\nabla^3 b(t)\right\|_{L^2}^2 \le C, \tag{3.25}$$

where C depends on $\|\nabla u(T_\star)\|_{L^2}^2 + \|\nabla v(T_\star)\|_{L^2}^2 + \|\nabla b(T_\star)\|_{L^2}^2$.

Noting that (3.2) and the right hand side of (3.25) is independent of t for $T_\star \le t < T$, we know that $(u(T,\cdot), v(T,\cdot), b(T,\cdot)) \in H^3(\mathbb{R}^n)$. Thus, Theorem 1.1 is proved. □

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