Research Article

A Note on a Beam Equation with Nonlinear Boundary Conditions

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We present new results on the existence of multiple positive solutions of a fourth-order differential equation subject to nonlocal and nonlinear boundary conditions that models a particular stationary state of an elastic beam with nonlinear controllers. Our results are based on classical fixed point index theory. We improve and complement previous results in the literature. This is illustrated in some examples.

1. Introduction

The fourth-order differential equation

$$u^{(4)}(t) = g(t)f(t, u(t)), \quad t \in (0, 1), \tag{1.1}$$

arises naturally in the study of the displacement $u = u(t)$ of an elastic beam when we suppose that, along its length, a load is added to cause deformations. This classical problem has been widely studied under a variety of boundary conditions (BCs) that describe the controls at the ends of the beam. In particular, Gupta [1] studied, along other sets of local homogeneous BCs, the problem

$$u(0) = 0, \qquad u''(0) = 0, \qquad u'(1) = 0, \qquad u'''(1) = 0 \tag{1.2}$$

that models a bar with the left end being simply supported (hinged) and the right end being sliding clamped. This problem, and its generalizations, has been studied previously by Davies and coauthors [2], Graef and Henderson [3] and Yao [4].

Multipoint versions of this problem do have a physical interpretation. For example, the four-point boundary conditions

$$u(0) = 0, \qquad u''(0) = 0, \qquad u'(1) = H_1(u(\tau)), \qquad u'''(1) + H_2(u(\xi)) = 0 \qquad (1.3)$$

model a bar where the displacement u and the bending moment u'' at $t = 0$ are zero, and there are relations, not necessarily linear, between the shearing force u''' and the angular attitude u' at $t = 1$ and the displacement u in two other points of the beam.

In this paper we establish new results on the existence of positive solutions for the fourth-order differential equation (1.1) subject to the following nonlocal nonlinear boundary conditions:

$$u(0) = 0, \qquad u''(0) = 0, \qquad (1.4)$$

$$u'(1) = H_1(\alpha_1[u]), \qquad (1.5)$$

$$u'''(1) + H_2(\alpha_2[u]) = 0, \qquad (1.6)$$

where H_1, H_2 are nonnegative continuous functions and $\alpha_1[\cdot], \alpha_2[\cdot]$ are linear functionals given by

$$\alpha_1[u] = \int_0^1 u(s) dA_1(s), \qquad \alpha_2[u] = \int_0^1 u(s) dA_2(s), \qquad (1.7)$$

involving Riemann-Stieltjes integrals.

The conditions (1.5)-(1.6) cover a variety of cases and include, as special cases when $H_1(w) = H_2(w) = w$, multipoint and integral boundary conditions. These are widely studied objects in the case of fourth-order BVPs; see, for example, [5–14]. BCs of nonlinear type also have been studied before in the case of fourth-order equations; see, for example, [15–20] and references therein.

The study of positive solutions of BVPs that involve Stieltjes integrals has been done, in the case of *positive* measures, in [21–24]. *Signed* measures were used in [12, 25]; here, as in [21, 22], due to some inequalities involved in our theory, the functionals $\alpha_i[\cdot]$ are assumed to be given by positive measures.

A standard methodology to solve (1.1) subject to local BCs is to find the corresponding Green's function k and to rewrite the BVP as a Hammerstein integral equation of the form

$$u(t) = \int_0^1 k(t, s) g(s) f(s, u(s)) ds. \qquad (1.8)$$

However, for nonlocal and nonlinear BCs the form of Green's functions can become very complicated. In the case of linear, nonlocal BCs, an elegant approach is due to Webb and

Infante [12], where a unified method is given to study a large class of problems. This is done via an auxiliary perturbed Hammerstein integral equation of the form

$$u(t) = \gamma_1(t)\alpha_1[u] + \gamma_2(t)\alpha_2[u] + \int_0^1 k(t,s)g(t)f(s,u(s))ds, \qquad (1.9)$$

with suitable functions γ_1, γ_2.

Infante studied in [26, 27] the case of one nonlinear BC and in [21] a thermostat model with two nonlinear controllers. The approach used in [21] relied on an extension of the results of [25], valid for equations of the type

$$u(t) = \gamma_1(t)H_1(\alpha_1[u]) + \gamma_2(t)H_2(\alpha_2[u]) + \int_0^1 k(t,s)g(s)f(s,u(s))ds, \qquad (1.10)$$

and gives a simple general method to avoid long technical calculations.

In our paper the approach of [21] is applied to BVP (1.1)–(1.6): we rewrite this BVP as a perturbed Hammerstein integral equation, and we prove the existence of multiple positive solutions under a suitable oscillatory behavior of the nonlinearity f. We observe that our results are new even for the local BCs, when $H_1(\alpha_1[u]) = H_2(\alpha_2[u]) = 0$. We illustrate our theory with some examples. We also point out that this approach may be utilized for other sets of nonlinear BCs that have a physical interpretation, this is done in the last section.

2. The Boundary Value Problem

We begin by considering the homogeneous BVP

$$u^{(4)}(t) = g(t)f(t,u(t)), \quad u(0) = u''(0) = u'(1) = u'''(1) = 0, \quad t \in (0,1), \qquad (2.1)$$

of which we seek an equivalent integral formulation of the form

$$u(t) = \int_0^1 k(t,s)g(s)f(s,u(s))ds. \qquad (2.2)$$

Due to the nature of these particular BCs, the Green's function k can be constructed (as in [4]) by means of an auxiliary second-order BVP, namely,

$$u''(t) + g(t)f(t,u(t)) = 0, \quad u(0) = 0, \quad u'(1) = 0, \quad t \in (0,1). \qquad (2.3)$$

The solutions of the BVP (2.3) can be written in the form

$$u(t) = \int_0^1 G(t,s)g(s)f(s,u(s))ds, \qquad (2.4)$$

where

$$G(t,s) = \begin{cases} s, & s \le t, \\ t, & s > t. \end{cases} \tag{2.5}$$

Therefore the function k in (2.2) is given by

$$k(t,s) = \int_0^1 G(t,v)G(v,s)dv. \tag{2.6}$$

In order to use the approach of [21, 25, 28], we need to use some monotonicity properties of k. Now, since

$$G(t,v)G(v,s) = v^2 \chi_{[0,t]}(v)\chi_{[0,s]}(v) + vs\chi_{[0,t]}(v)\chi_{[s,1]}(v)$$
$$+ tv\chi_{[t,1]}(v)\chi_{[0,s]}(v) + ts\chi_{[t,1]}(v)\chi_{[s,1]}(v), \tag{2.7}$$

we obtain the following formulation for the Green's function:

$$k(t,s) = \begin{cases} \dfrac{s}{6}(-s^2 - 3t^2 + 6t), & s \le t, \\[2mm] \dfrac{t}{6}(-t^2 - 3s^2 + 6s), & s > t. \end{cases} \tag{2.8}$$

We now look for a suitable interval $[a,b] \subset [0,1]$, a function $\Phi(s)$, and a constant $c_\Phi > 0$ such that

$$k(t,s) \le \Phi(s), \quad \text{for every } (t,s) \in [0,1] \times [0,1],$$
$$k(t,s) \ge c_\Phi \Phi(s), \quad \text{for every } (t,s) \in [a,b] \times [0,1]. \tag{2.9}$$

Since k is continuous on $[0,1] \times [0,1]$ and $k(t,s) > 0$ for $(t,s) \in (0,1] \times (0,1]$, a natural choice could be

$$\Phi(s) = \max_{(t,s) \in [0,1] \times [0,1]} k(t,s), \qquad c_\Phi = \min_{(t,s) \in [a,b] \times [0,1]} \frac{k(t,s)}{\Phi(s)}; \tag{2.10}$$

here we look for a better Φ, since this enables us to weaken the growth requirements on the nonlinearity f.

An upper bound for k is obtained by finding $\max_{t \in [0,1]} k(t,s)$ for each fixed s. Since $(\partial/\partial t)k(t,s) \ge 0$ for $(t,s) \in [0,1] \times [0,1]$, k is a nondecreasing function of t that attains its maximum, for each fixed s, when $t = 1$.

Therefore, for $(t,s) \in [0,1] \times [0,1]$, we have

$$k(t,s) \le k(1,s) = \frac{s}{6}(-s^2 + 3) := \Phi(s). \tag{2.11}$$

Now, one can see that the derivative of the function $k(t,s)/\Phi(s)$ with respect to s is non-positive for all $s \in [0,1]$, that is, the function $k(t,s)/\Phi(s)$ is a non-increasing function of s. Therefore, for an arbitrary $[a,b] \subset (0,1]$, we have

$$k(t,s) \geq c_\Phi \Phi(s), \quad \text{for every } (t,s) \in [a,b] \times [0,1], \tag{2.12}$$

where

$$c_\Phi := \min_{a \leq t \leq b} \frac{k(t,1)}{\Phi(1)} = \frac{k(a,1)}{\Phi(1)} = \frac{1}{2}a(3-a^2). \tag{2.13}$$

We now turn our attention to the BVP (1.1)–(1.6)

$$u^{(4)}(t) = g(t)f(t,u(t)), \qquad u(0) = u''(0) = u'(1) - H_1(\alpha_1[u]) = u'''(1) + H_2(\alpha_2[u]) = 0, \tag{2.14}$$

and we show that we can study this problem by means of a perturbation of the Hammerstein integral equation (2.2).

In order to do this, we look for the (unique) solutions of the linear problems

$$\gamma_1^{(4)}(t) = 0, \qquad \gamma_1(0) = \gamma_1''(0) = 0, \qquad \gamma_1'(1) = 1, \qquad \gamma_1'''(1) = 0,$$
$$\gamma_2^{(4)}(t) = 0, \qquad \gamma_2(0) = \gamma_2''(0) = \gamma_2'(1) = 0, \qquad \gamma_2'''(1) + 1 = 0 \tag{2.15}$$

that are

$$\gamma_1(t) = t, \qquad \gamma_2(t) = -\frac{1}{6}t^3 + \frac{1}{2}t. \tag{2.16}$$

We observe that, for an arbitrary $[a,b] \subset (0,1]$, we have

$$\|\gamma_1\| = 1, \qquad \min_{t \in [a,b]} \gamma_1(t) = \gamma_1(a) = a,$$
$$\|\gamma_2\| = \frac{1}{3}, \qquad \min_{t \in [a,b]} \gamma_2(t) = \gamma_2(a) = -\frac{1}{6}a^3 + \frac{1}{2}a, \tag{2.17}$$

where $\|u\| := \sup\{|u(t)|: t \in [0,1]\}$, and therefore

$$\gamma_1(t) \geq c_{\gamma_1}\|\gamma_1\|, \quad \gamma_2(t) \geq c_{\gamma_2}\|\gamma_2\|, \quad \text{for every } t \in [a,b], \tag{2.18}$$

with

$$c_{\gamma_1} := \frac{\min_{t \in [a,b]} \gamma_1(t)}{\|\gamma_1\|} = a, \qquad c_{\gamma_2} := \frac{\min_{t \in [a,b]} \gamma_2(t)}{\|\gamma_2\|} = -\frac{a^3}{2} + \frac{3a}{2}. \tag{2.19}$$

By a solution of the BVP (1.1)–(1.6) we mean a solution of the perturbed integral equation

$$u(t) = \gamma_1(t)H_1(\alpha_1[u]) + \gamma_2(t)H_2(\alpha_2[u]) + \int_0^1 k(t,s)g(t)f(s,u(s))ds =: Tu(t), \qquad (2.20)$$

and we work in a suitable cone in the Banach space $C[0,1]$ of continuous functions defined on the interval $[0,1]$ endowed with the usual supremum norm.

Our assumptions are the following:

(C_1) $f : [0,1] \times [0,\infty) \to [0,\infty)$ satisfies Carathéodory conditions, that is, for each u, $t \mapsto f(t,u)$ is measurable, for almost every t, $u \mapsto f(t,u)$ is continuous, and for every $r > 0$, there exists an L^∞-function $\phi_r : [0,1] \to [0,\infty)$ such that

$$f(t,u) \le \phi_r(t) \quad \text{for almost all } t \in [0,1] \text{ and all } u \in [0,r]; \qquad (2.21)$$

(C_2) $g \ \Phi \in L^1[0,1]$, $g \ge 0$ almost everywhere, and $\int_0^1 \Phi(s)g(s)ds > 0$;

(C_3) H_1, H_2 are positive continuous functions such that there exist $h_{11}, h_{12}, h_{21}, h_{22} \in [0,\infty)$ with

$$h_{11}v \le H_1(v) \le h_{12}v, \qquad h_{21}v \le H_2(v) \le h_{22}v, \qquad (2.22)$$

for every $v \ge 0$;

(C_4) $\alpha_1[\cdot], \ \alpha_2[\cdot]$ are positive bounded linear functionals on $C[0,1]$ given by

$$\alpha_i[u] = \int_0^1 u(s)dA_i(s), \quad i = 1, 2, \qquad (2.23)$$

involving Stieltjes integrals with *positive* measures dA_i;

(C_5) $D_2 := (1 - h_{12}\alpha_1[\gamma_1])(1 - h_{22}\alpha_2[\gamma_2]) - h_{12}h_{22}\alpha_1[\gamma_2]\alpha_2[\gamma_1] > 0$, $h_{12}\alpha_1[\gamma_1] < 1$ and $h_{22}\alpha_2[\gamma_2] < 1$.

It follows from this last hypothesis that

$$D_1 := (1 - h_{11}\alpha_1[\gamma_1])(1 - h_{21}\alpha_2[\gamma_2]) - h_{11}h_{21}\alpha_1[\gamma_2]\alpha_2[\gamma_1] \ge D_2 > 0. \qquad (2.24)$$

The above hypotheses enable us to utilize the cone

$$K = \left\{ u \in C[0,1] : u \geq 0, \ \min_{t \in [a,b]} u(t) \geq c\|u\| \right\}, \tag{2.25}$$

for an arbitrary $[a,b] \subset (0,1]$ and

$$c := \min\{c_\Phi, c_{\gamma_1}, c_{\gamma_2}\} = a, \tag{2.26}$$

and to use the classical fixed point index for compact maps (see e.g., [29] or [30]).

We observe, as in [21], that T leaves K invariant and is compact. We give the proof in the Carathéodory case for completeness.

Lemma 2.1. *If the hypotheses (C_1)–(C_4) hold, then T maps K into K. Moreover, T is a compact map.*

Proof. Take $u \in K$ such that $\|u\| \leq r$. Then we have, for $t \in [0,1]$,

$$Tu(t) = \gamma_1(t)H_1(\alpha_1[u]) + \gamma_2(t)H_2(\alpha_2[u]) + \int_0^1 k(t,s)g(s)f(s,u(s))ds$$

$$\leq \gamma_1(t)H_1(\alpha_1[u]) + \gamma_2(t)H_2(\alpha_2[u]) + \int_0^1 \Phi(s)g(s)\phi_r(s)ds, \tag{2.27}$$

therefore

$$\|Tu\| \leq \|\gamma_1\|H_1(\alpha_1[u]) + \|\gamma_2\|H_2(\alpha_2[u]) + \int_0^1 \Phi(s)g(s)\phi_r(s)ds. \tag{2.28}$$

Then we obtain

$$\min_{t \in [a,b]} Tu(t) \geq c_{\gamma_1}\|\gamma_1\|H_1(\alpha_1[u]) + c_{\gamma_2}\|\gamma_2\|H_2(\alpha_2[u]) + c_\Phi \int_0^1 \Phi(s)g(s)\phi_r(s) \geq c\|Tu\|. \tag{2.29}$$

Hence we have $Tu \in K$. Moreover, the map T is compact since it is a sum of two compact maps: the compactness of $\int_0^1 k(t,s)g(s)f(s,u(s))ds$ is well known, and since γ_1, γ_2, H_1, and H_2 are continuous, the perturbation $\gamma_1(t)H_1(\alpha_1[u]) + \gamma_2(t)H_2(\alpha_2[u])$ maps bounded sets into bounded subsets of a 1-dimensional space. $\qquad\square$

For $\rho > 0$, we use, as in [23, 31], the following bounded open subsets of K:

$$K_\rho = \{u \in K : \|u\| < \rho\}, \qquad V_\rho = \left\{ u \in K : \min_{t \in [a,b]} u(t) < \rho \right\}. \tag{2.30}$$

Note that $K_\rho \subset V_\rho \subset K_{\rho/c}$.

We employ the following numbers:

$$f^{0,\rho} := \sup_{0 \leq u \leq \rho, \ 0 \leq t \leq 1} \frac{f(t,u)}{\rho}, \qquad f_{\rho,\rho/c} := \inf_{\rho \leq u \leq \rho/c, \ a \leq t \leq b} \frac{f(t,u)}{\rho},$$

$$\frac{1}{m} := \sup_{t \in [0,1]} \int_0^1 k(t,s)g(s)ds, \qquad \frac{1}{M(a,b)} := \inf_{t \in [a,b]} \int_a^b k(t,s)g(s)ds, \tag{2.31}$$

and we note

$$\mathcal{K}_i(s) := \int_0^1 k(t,s)dA_i(t), \quad i = 1, 2. \tag{2.32}$$

The proofs of the following results can be immediately deduced from the analogous results in [21], where the proofs involve a careful analysis of fixed point index and utilize order-preserving matrices. The only difference here is that we allow nonlinearity f to be Carathéodory instead of continuous. The lines of proof are not effected and therefore the proofs are omitted.

Firstly we give conditions which imply that the fixed point index is 0 on the set V_ρ.

Lemma 2.2. *Suppose that* (C_1)–(C_5) *hold. Assume that there exist* $\rho > 0$ *such that*

$$f_{\rho,\rho/c}\left(\left(\frac{c_{\gamma_1}\|\gamma_1\|}{D_1}(1 - h_{21}\alpha_2[\gamma_2]) + \frac{c_{\gamma_2}\|\gamma_2\|}{D_1}h_{11}\alpha_2[\gamma_1]\right)\int_0^1 \mathcal{K}_1(s)g(s)ds\right.$$

$$\left. + \left(\frac{c_{\gamma_1}\|\gamma_1\|}{D_1}h_{21}\alpha_1[\gamma_2] + \frac{c_{\gamma_2}\|\gamma_2\|}{D_1}(1 - h_{11}\alpha_1[\gamma_1])\right)\int_0^1 \mathcal{K}_2(s)g(s)ds + \frac{1}{M(a,b)}\right) > 1. \tag{2.33}$$

Then the fixed point index, $i_K(T, V_\rho)$, *is 0.*

Now, we give conditions which imply that the fixed point index is 1 on the set K_ρ.

Lemma 2.3. *Suppose* (C_1)–(C_5) *hold. Assume that there exists* $\rho > 0$ *such that*

$$f^{0,\rho}\left(\left(\frac{\|\gamma_1\|}{D_2}(1 - h_{22}\alpha_2[\gamma_2]) + \frac{\|\gamma_2\|}{D_2}h_{12}\alpha_2[\gamma_1]\right)\int_0^1 \mathcal{K}_1(s)g(s)ds\right.$$

$$\left. + \left(\frac{\|\gamma_1\|}{D_2}h_{22}\alpha_1[\gamma_2] + \frac{\|\gamma_2\|}{D_2}(1 - h_{12}\alpha_1[\gamma_1])\right)\int_0^1 \mathcal{K}_2(s)g(s)ds + \frac{1}{m}\right) < 1. \tag{2.34}$$

Then $i_K(T, K_\rho) = 1$.

The two lemmas above lead to the following result on existence of one or two positive solutions for the integral equation (2.20). Note that, if the nonlinearity f has a suitable oscillatory behavior, it is possible to state, with the same arguments as in [23], a theorem on the existence of more than two positive solutions.

Theorem 2.4. *Suppose (C_1)–(C_5) hold. Let $[a,b] \subset (0,1)$ and c be as in (2.26). Then (2.20) has one positive solution in K if either of the following conditions holds:*

(S_1) *there exist $\rho_1, \rho_2 \in (0, \infty)$ with $\rho_1 < \rho_2$ such that (2.34) is satisfied for ρ_1 and (2.33) is satisfied for ρ_2;*

(S_2) *there exist $\rho_1, \rho_2 \in (0, \infty)$ with $\rho_1 < c\rho_2$ such that (2.33) is satisfied for ρ_1 and (2.34) is satisfied for ρ_2.*

Equation (2.20) has at least two positive solutions in K if one of the following conditions hold.

(D_1) *there exist $\rho_1, \rho_2, \rho_3 \in (0, \infty)$, with $\rho_1 < \rho_2 < c\rho_3$, such that (2.34) is satisfied for ρ_1, (2.33) is satisfied for ρ_2, and (2.34) is satisfied for ρ_3;*

(D_2) *there exist $\rho_1, \rho_2, \rho_3 \in (0, \infty)$, with $\rho_1 < c\rho_2$ and $\rho_2 < \rho_3$, such that (2.33) is satisfied for ρ_1, (2.34) is satisfied for ρ_2, and (2.33) is satisfied for ρ_3.*

3. Optimal Constants and Examples

Consider the differential equation

$$u^{(4)}(t) = f(t, u(t)), \quad t \in (0,1), \tag{3.1}$$

with the BCs (1.4)–(1.6).

The value $1/m$ is given by direct calculation as follows:

$$\frac{1}{m} = \sup_{t \in [0,1]} \int_0^1 k(t,s)ds = \max_{t \in [0,1]} \frac{t}{24}\left(t^3 - 4t^2 + 8\right) = \frac{5}{24}. \tag{3.2}$$

We seek the "optimal" $[a,b]$ for which $M(a,b)$ is a minimum. This type of problems has been tackled in the past, for example, in the second-order case for heat-flow problems in [32] and for beam equations (under different BCs) in [9, 12, 13].

The kernel k is a positive, nondecreasing function of t, thus

$$\frac{1}{M(a,b)} = \min_{t \in [a,b]} \int_a^b k(t,s)ds = \int_a^b k(a,s)ds = \int_a^b \frac{a}{6}\left(-a^2 - 3s^2 + 6s\right)ds. \tag{3.3}$$

Since k is a nondecreasing function of t, we have

$$\max_{0<a<b\le 1}\left\{\frac{1}{M(a,b)}\right\} = \max_{0<a\le 1} \int_a^1 \frac{a}{6}\left(-a^2 - 3s^2 + 6s\right)ds = \max_{0<a\le 1}\left\{\frac{a}{6}\left(2a^3 - 4a^2 + 2\right)\right\}. \tag{3.4}$$

Such maximum is attained at $a = 1/2$. Thus the "optimal" interval $[a,b]$, for which $M(a,b)$ is a minimum, is the interval $[1/2, 1]$; this gives $M(1/2, 1) = 48/5$ and $c = 1/2$.

Remark 3.1. From Theorem 2.4, it is possible to state results for the existence of several nonnegative solutions for the homogeneous BVP

$$u^{(4)}(t) = f(t, u(t)), \quad u(0) = u''(0) = u'(1) = u'''(1) = 0, \quad t \in (0,1). \tag{3.5}$$

For example, with $[a, b] = [1/2, 1]$ and $c = 11/16$, the BVP (3.5) has at least two positive solutions in K if there exist $\rho_1 < \rho_2 < c\rho_3$, such that $f^{0,\rho_1} < 4.8$, $f_{\rho_2,\rho_2/c} > 9.6$ and $f^{0,\rho_3} < 4.8$.

These results are new and improve and complement the previous ones. Gupta [1] and Yao [4] studied the problem with more general nonlinearity but established existence results only. Davies and co-authors [2] and Graef and Henderson [3] obtain the existence of multiple positive solutions for a $2n$th-order differential equation subject to our boundary conditions in the case $n = 2$. In [2] the choice $[a, b] = [1/4, 3/4]$ gives the values $\eta = 4$ and $\mu = 16^4$ which replace our constants m and $M(1/2, 1)$ in the growth conditions of f. The growth conditions of the nonlinearity $f = f(u)$ in Theorem 3.1 in [3] cannot be compared with ours, but we do not require the restriction $f(0) \neq 0$.

The next examples illustrate the applicability of our results. Firstly we consider, as an illustrative example, the case of a nonlinear 4-point problem.

Example 3.2. Consider the differential equation

$$u^{(4)}(t) = f(t, u(t)), \quad t \in (0, 1), \tag{3.6}$$

with the BCs

$$u(0) = 0, \qquad u''(0) = 0, \qquad u'(1) = H_1(u(\tau)), \qquad u'''(1) + H_2(u(\xi)) = 0, \tag{3.7}$$

where $\tau, \xi \in (0, 1)$ and, as in [22], for $i = 1, 2$

$$H_i(w) = \begin{cases} \dfrac{1}{4i}w, & 0 \le w \le 1, \\[2mm] \dfrac{1}{8i}w + \dfrac{1}{8i}, & w \ge 1. \end{cases} \tag{3.8}$$

In this case we have $h_{11} = 1/8$, $h_{12} = 1/4$, $h_{21} = 1/16$, $h_{22} = 1/8$,

$$\alpha_1[\gamma_1] = \tau, \qquad \alpha_1[\gamma_2] = -\frac{1}{6}\left(\tau^3 - 3\tau\right), \qquad \alpha_2[\gamma_1] = \xi, \qquad \alpha_2[\gamma_2] = -\frac{1}{6}\left(\xi^3 - 3\xi\right),$$

$$\int_0^1 \mathcal{K}_1(s)ds = \int_0^1 k(\tau, s)ds = \frac{\tau}{24}\left(\tau^3 - 4\tau^2 + 8\right), \tag{3.9}$$

$$\int_0^1 \mathcal{K}_2(s)ds = \int_0^1 k(\xi, s)ds = \frac{\xi}{24}\left(\xi^3 - 4\xi^2 + 8\right).$$

We now fix $\tau = 1/4$, $\xi = 1/2$ and show that all the constants that appear in (2.33) and in (2.34) can be computed. This choice leads to

$$\alpha_1[\gamma_1] = \frac{1}{4}, \qquad \alpha_1[\gamma_2] = \frac{47}{384}, \qquad \alpha_2[\gamma_1] = \frac{1}{2}, \qquad \alpha_2[\gamma_2] = \frac{11}{48}, \qquad D_2 = \frac{45373}{49152}. \qquad (3.10)$$

Moreover we have

$$\int_0^1 \mathcal{K}_1(s)ds = \frac{497}{6144}, \qquad \int_0^1 \mathcal{K}_2(s)ds = \frac{19}{128}, \qquad (3.11)$$

and conditions (2.34) and (2.33) read $f^{0,\rho_1} < 2.8587$ and $f_{\rho_2,\rho_2/c} > 6.9470$. Since (S_1) holds, from Theorem 2.4 it follows that this BVP has a nontrivial solution in K. A nonlinearity that easily verifies (S_1), for example, is the function $f(t,u) = u^3$ for every $0 < \rho_1 < \sqrt{2.8587}$ and every $\rho_2 > \sqrt{6.9470}$.

We now give an example with continuously distributed positive measures.

Example 3.3. Consider the differential equation

$$u^{(4)}(t) = f(t, u(t)), \quad t \in (0,1), \qquad (3.12)$$

with the BCs

$$u(0) = 0, \qquad u''(0) = 0, \qquad u'(1) = H_1\left(\int_0^1 \alpha_1 u(s)ds\right), \qquad u'''(1) + H_2\left(\int_0^1 \alpha_2 u(s)ds\right) = 0, \qquad (3.13)$$

with $\alpha_1, \alpha_2 > 0$ and H_i, $i = 1,2$, as in the previous example. In this case, we obtain

$$\alpha_1[\gamma_1] = \frac{\alpha_1}{2}, \qquad \alpha_1[\gamma_2] = \frac{5\alpha_1}{24}, \qquad \alpha_2[\gamma_1] = \frac{\alpha_2}{2}, \qquad \alpha_2[\gamma_2] = \frac{5\alpha_2}{24}, \qquad (3.14)$$

and for $i = 1,2$

$$\mathcal{K}_i(s) = \int_0^1 k(t,s)dA_i(t) = \int_0^1 \alpha_i k(t,s)dt = \alpha_i\left(\frac{1}{24}s^4 - \frac{1}{6}s^3 + \frac{1}{3}s\right),$$

$$\int_0^1 \mathcal{K}_i(s)ds = \frac{2\alpha_i}{15}. \qquad (3.15)$$

The Condition (C_5) becomes

$$\alpha_1 < 8, \qquad \alpha_2 < \frac{192}{5}, \qquad 1 - \frac{5}{192}\alpha_2 - \frac{1}{8}\alpha_1 > 0. \tag{3.16}$$

We now fix $\alpha_1 = 4$, $\alpha_2 = 8$. This choice leads to

$$\alpha_1[\gamma_1] = 2, \qquad \alpha_1[\gamma_2] = \frac{5}{6}, \qquad \alpha_2[\gamma_1] = 4, \qquad \alpha_2[\gamma_2] = \frac{5}{3}, \qquad D_2 = \frac{7}{24},$$
$$\int_0^1 \mathcal{K}_1(s)ds = \frac{8}{15}, \qquad \int_0^1 \mathcal{K}_2(s)ds = \frac{16}{15} \tag{3.17}$$

and conditions (2.34) and (2.33) read $f^{0,\rho_1} < 0.30713$ and $f_{\rho_2,\rho_2/c} > 1.1166$.

4. Other Nonlinear BCs

So far we have discussed in detail the case of the BCs (1.4)–(1.6), but the same approach may be applied to (1.1) subject to the nonlinear BCs

$$u(0) = H_1(\alpha_1[u]), \qquad u''(0) + H_2(\alpha_2[u]) = 0, \qquad u'(1) = 0, \qquad u'''(1) = 0, \tag{4.1}$$

or

$$u(0) = H_1(\alpha_1[u]), \qquad u''(0) = 0, \qquad u'(1) = H_2(\alpha_2[u]), \qquad u'''(1) = 0, \tag{4.2}$$

or

$$u(0) = H_1(\alpha_1[u]), \qquad u''(0) = 0, \qquad u'(1) = 0, \qquad u'''(1) + H_2(\alpha_2[u]) = 0, \tag{4.3}$$

or

$$u(0) = 0, \qquad u''(0) + H_1(\alpha_1[u]) = 0, \qquad u'(1) = H_2(\alpha_2[u]), \qquad u'''(1) = 0, \tag{4.4}$$

or

$$u(0) = 0, \qquad u''(0) + H_1(\alpha_1[u]) = 0, \qquad u'(1) = 0, \qquad u'''(1) + H_2(\alpha_2[u]) = 0. \tag{4.5}$$

As in [12], where a different set of BCs were investigated, we point out that these nonlocal boundary conditions can be interpreted as feedback controls: for example, the BCs (4.1) can be seen as a control on the displacement at the left end and a device handling the shear force at $t = 0$.

Table 1

	BCs (1.4)–(1.6)	BCs (4.1)	BCs (4.2)	BCs (4.3)	BCs (4.4)	BCs (4.5)
$\gamma_1(t)$	t	1	1	1	$-\dfrac{t^2}{2}+t$	$-\dfrac{t^2}{2}+t$
$\gamma_2(t)$	$-\dfrac{t^3}{6}+\dfrac{t}{2}$	$-\dfrac{t^2}{2}+t$	t	$-\dfrac{t^3}{6}+\dfrac{t}{2}$	t	$-\dfrac{t^3}{6}+\dfrac{t}{2}$
c_{γ_1}	a	1	1	1	$2a-a^2$	$2a-a^2$
c_{γ_2}	$-\dfrac{a^3}{2}+\dfrac{3a}{2}$	$2a-a^2$	a	$-\dfrac{a^3}{2}+\dfrac{3a}{2}$	a	$-\dfrac{a^3}{2}+\dfrac{3a}{2}$

Table 1 illustrates how the choice of the BCs affects the functions γ_1, γ_2 and the constants $c_{\gamma_1}, c_{\gamma_2}$.

Since one can see that

$$1 > 2a - a^2 > -\frac{a^3}{2} + \frac{3a}{2} > a, \quad a \in (0,1], \tag{4.6}$$

the cone K, given by the constant c, varies according to the nonhomogeneous BCs considered. This affects also, in a natural manner, conditions (2.33) and (2.34).

Acknowledgments

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