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Research Article

Iterative Solutions of Singular Boundary Value Problems of Third-Order Differential Equation

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By using the cone theory and the Banach contraction mapping principle, the existence and uniqueness results are established for singular third-order boundary value problems. The theorems obtained are very general and complement previous known results.

1. Introduction

Third-order differential equations arise in a variety of different areas of applied mathematics and physics, such as the deflection of a curved beam having a constant or varying cross section, three-layer beam, electromagnetic waves, or gravity-driven flows [1]. Recently, third-order boundary value problems have been studied extensively in the literature (see, e.g., [2–13], and their references). In this paper, we consider the following third-order boundary value problem:

$$\begin{gathered} u'''(t) + f(t, u(t)) = 0, \quad t \in (0,1), \\ u(0) = u'(0) = 0, \qquad u'(1) = \alpha u'(\eta), \end{gathered} \tag{1.1}$$

where $f(t,x) \in C((0,1) \times (-\infty,+\infty),(-\infty,+\infty))$, $0 < \eta < 1$.

Three-point boundary value problems (BVPs for short) have been also widely studied because of both practical and theoretical aspects. There have been many papers investigating the solutions of three-point BVPs, see [2–5, 10, 12] and references therein. Recently, the existence of solutions of third-order three-point BVP (1.1) has been studied in [2, 3]. Guo et al. [2] show the existence of positive solutions for BVP (1.1) when $1 < \alpha < 1/\eta$ and

$f(t,x)$ is separable by using cone expansion-compression fixed point theorem. In [3], the singular third-order three-point BVP (1.1) is considered under some conditions concerning the first eigenvalues corresponding to the relevant linear operators, where $1 < \alpha < 1/\eta$, $f(t,x)$ is separable and is not necessary to be nonnegative, and the existence results of nontrivial solutions and positive solutions are given by means of the topological degree theory. Motivated by the above works, we consider the singular third-order three-point BVP (1.1). Here, we give the unique solution of BVP (1.1) under the conditions that $\alpha\eta \neq 1$ and $f(t,x)$ is mixed nonmonotone in x and does not need to be separable by using the cone theory and the Banach contraction mapping principle.

2. Preliminaries

Let $J = (0,1)$, $I = [0,1]$. By [2, Lemma 2.1], we have that x is a solution of (1.1) if and only if

$$x(t) = \int_0^1 G(t,s) f(s,x(s))ds, \quad t \in I, \tag{2.1}$$

where

$$G(t,s) = \frac{1}{2(1-\alpha\eta)} \begin{cases} (1-\alpha\eta)(s^2-2ts) - (1-\alpha)t^2 s, & s \le \min\{\eta,t\}, \\ (\alpha\eta - 1)t^2 + (1-\alpha)t^2 s, & t \le s \le \eta, \\ (1-\alpha\eta)(s^2-2ts) + (s-\alpha\eta)t^2, & \eta \le s \le t, \\ (s-1)t^2, & \max\{\eta,t\} \le s. \end{cases} \tag{2.2}$$

It is shown in [2] that $G(t,s)$ is the Green's function to $-u''' = 0$, $u(0) = u'(0) = 0$, and $u'(1) = \alpha u'(\eta)$.

Let

$$\begin{gathered} h(t,s) = \frac{1}{s(1-s)} |G(t,s)|, \\ I_1(t) = \int_0^1 h(t,s)ds, \\ I_{n+1}(t) = \int_0^1 h(t,s) I_n(s)ds, \quad (n = 1,2,\ldots), \\ r(G) = \lim_{n\to\infty} \left(\sup_{t\in J} I_n(t) \right)^{-1/n}. \end{gathered} \tag{2.3}$$

It is easy to see that $r(G) > 0$.

Lemma 2.1 (Guo [14, 15]). *P is generating if and only if there exists a constant $\tau > 0$ such that every element $x \in C(I)$ can be represented in the form $x = y - z$, where $y, z \in P$ and $\|y\| \le \tau\|x\|$, $\|z\| \le \tau\|x\|$.*

3. Singular Third-Order Boundary Value Problem

This section discusses singular third-order boundary value problem (1.1).

Let $P = \{x \in C(I) \mid x(t) \geq 0, \ \forall\, t \in [0,1]\}$. Obviously, P is a normal solid cone of Banach space $C(I)$; by [16, Lemma 2.1.2], we have that P is a generating cone in $C(I)$.

Theorem 3.1. *Suppose that $f(t,x) = g(t,x,x)$, and there exist two positive linear bounded operators $B : C(I) \to C(I)$ and $C : C(I) \to C(I)$ with $r(B+C) < r(G)$ such that for any $t \in I$, $x_1, x_2, y_1, y_2 \in C(I)$, $x_1 \geq x_2$, $y_1 \leq y_2$, we have*

$$-B(x_1 - x_2) - C(y_2 - y_1) \leq t(1-t)g(t, x_1, y_1) - t(1-t)g(t, x_2, y_2) \leq B(x_1 - x_2) + C(y_2 - y_1), \tag{3.1}$$

and there exists $x_0, y_0 \in C(I)$, such that

$$\int_0^1 t(1-t)_g(t, x_0(t), y_0(t))dt \text{ can converge to } \sigma \in \mathbb{R}. \tag{3.2}$$

Then (1.1) has a unique solution x^* in $C(I)$. And moreover, for any $x_0 \in C(I)$, the iterative sequence

$$x_n(t) = \int_0^1 G(t,s)f(t, x_{n-1}(s))ds \quad (n = 1, 2, \ldots) \tag{3.3}$$

converges to x^* $(n \to \infty)$.

Remark 3.2. Recently, in the study of BVP (1.1), almost all the papers have supposed that the Green's function $G(t,s)$ is nonnegative. However, the scope of α is not limited to $1 < \alpha < 1/\eta$ in Theorem 3.1, so, we do not need to suppose that $G(t,s)$ is nonnegative.

Remark 3.3. The function f in Theorem 3.1 is not monotone or convex; the conclusions and the proof used in this paper are different from the known papers in essence.

Proof. It is easy to see that, for any $t \in J$, $h(t,s)$ can be divided into finite partitioned monotone and bounded function on $(0,1)$, and then by (3.2), we have

$$\int_0^1 G(t,s)g(s, x_0(s), y_0(s))ds \text{ converges to } \sigma(t) \in \mathbb{R}. \tag{3.4}$$

For any $x, y \in C(I)$, let $u = |x_0| + |x|$, $v = -|y_0| - |y|$, then $u \geq x_0$, $v \leq y_0$, by (3.1), we have

$$-B(u - x_0)(t) - C(y_0 - v)(t) \leq t(1-t)g(t, u(t), v(t)) - t(1-t)g(t, x_0(t), y_0(t)) \leq B(u - x_0)(t) + C(y_0 - v)(t). \tag{3.5}$$

Hence

$$\begin{aligned}\left\|t(1-t)g(t,u(t),v(t))-t(1-t)g\big(t,x_0(t),y_0(t)\big)\right\| &\le \|B(u-x_0)(t)\|+\|C\big(y_0-v\big)(t)\| \\ &\le \|B\|\|u-x_0\|+\|C\|\|y_0-v\|.\end{aligned} \tag{3.6}$$

Following the former inequality, we can easily have

$$\int_0^1 G(t,s)\big[g(s,u(s),v(s))-g\big(s,x_0(s),y_0(s)\big)\big]ds \text{ converges to some element } \sigma_1(t)\in\mathbb{R}; \tag{3.7}$$

thus

$$\begin{aligned}\int_0^1 G(t,s)g(s,u(s),v(s))d &= \int_0^1 G(t,s)g\big(s,x_0(s),y_0(s)\big)ds \\ &\quad+\int_0^1 G(t,s)\big[g(s,u(s),v(s))-g\big(s,x_0(s),y_0(s)\big)\big]ds \text{ is converged.}\end{aligned} \tag{3.8}$$

Similarly, by $u\ge x,\ v\le y$ and $\int_0^1 G(t,s)g(s,u(s),v(s))ds$ being converged, we have that

$$\int_0^1 G(t,s)g(s,x(s),y(s))ds \text{ converges to some element } \sigma_2(t)\in\mathbb{R}. \tag{3.9}$$

Define the operator $A: C(I)\times C(I)\to C(I)$ by

$$A\big(x,y\big)(t)=\int_0^1 G(t,s)g\big(s,x(s),y(s)\big)ds,\quad \forall t\in I. \tag{3.10}$$

Then x is the solution of BVP (1.1) if and only if $x=A(x,x)$. Let

$$(Sx)(t)=\int_0^1 h(t,s)(Bx)(s)ds,\qquad (Ty)(t)=\int_0^1 h(t,s)\big(Cy\big)(s)ds. \tag{3.11}$$

By (3.1) and (3.10), for any $x_1, x_2, y_1, y_2 \in C(I)$, $x_1 \geq x_2$, $y_1 \leq y_2$, we have

$$-S(x_1 - x_2) - T(y_2 - y_1) \leq A(x_1, y_1) - A(x_2, y_2) \leq S(x_1 - x_2) + T(y_2 - y_1), \tag{3.12}$$

$$\begin{aligned}
(S+T)(x)(t) &= \int_0^1 h(t,s)(B+C)(x)(s)ds, \\
(S+T)^{n+1}(x)(t) &= \int_0^1 h(t,s)(B+C)(S+T)^n(x)(s)ds \\
&= (B+C)^{n+1} I_{n+1}(t), \quad n = 1, 2, \ldots, \\
\|(S+T)^n\| &\leq \|(B+C)^n\| \sup_{t\in J} I_n(t), \\
r(S+T) &\leq \frac{r(B+C)}{r(G)} < 1,
\end{aligned} \tag{3.13}$$

so we can choose an α, which satisfies $\lim_{n\to\infty} \|(S+T)^n\|^{1/n} = r(S+T) < \alpha < 1$, and so there exists a positive integer n_0 such that

$$\|(S+T)^n\| < \alpha^n < 1, \quad n \geq n_0. \tag{3.14}$$

Since P is a generating cone in $C(I)$, from Lemma 2.1, there exists $\tau > 0$ such that every element $x \in C(I)$ can be represented in

$$x = y - z, \quad y, z \in P, \quad \|y\| \leq \tau\|x\|, \quad \|z\| \leq \tau\|x\|. \tag{3.15}$$

This implies

$$-(y+z) \leq x \leq y + z. \tag{3.16}$$

Let

$$\|x\|_0 = \inf\{\|u\| \mid u \in P, -u \leq x \leq u\}. \tag{3.17}$$

By (3.16), we know that $\|x\|_0$ is well defined for any $x \in C(I)$. It is easy to verify that $\|\cdot\|_0$ is a norm in $C(I)$. By (3.15)–(3.17), we get

$$\|x\|_0 \leq \|y+z\| \leq 2\tau\|x\|, \quad \forall x \in C(I). \tag{3.18}$$

On the other hand, for any $u \in P$ which satisfies $-u \leq x \leq u$, we have $\theta \leq x + u \leq 2u$. Thus $\|x\| \leq \|x+u\| + \|-u\| \leq (2N+1)\|u\|$, where N denotes the normal constant of P. Since u is arbitrary, we have

$$\|x\| \leq (2N+1)\|x\|_0, \quad \forall x \in C(I). \tag{3.19}$$

It follows from (3.18) and (3.19) that the norms $\|\cdot\|_0$ and $\|\cdot\|$ are equivalent.

Now, for any $x, y \in C(I)$ and $u \in P$ which satisfies $-u \leq x - y \leq u$, let

$$u_1 = \frac{1}{2}(x + y - u), \qquad u_2 = \frac{1}{2}(x - y + u), \qquad u_3 = \frac{1}{2}(-x + y + u); \tag{3.20}$$

then $x \geq u_1$, $y \geq u_1$, $x - u_1 = u_2$, $y - u_1 = u_3$, and $u_2 + u_3 = u$.

It follows from (3.12) that

$$-Su_2 \leq A(x, x) - A(u_1, x) \leq Su_2, \tag{3.21}$$

$$-Su_3 - Tu_2 \leq A(y, u_1) - A(u_1, x) \leq Su_2 + Tu_3, \tag{3.22}$$

$$-Tu_3 \leq A(y, u_1) - A(y, y) \leq Tu_3. \tag{3.23}$$

Subtracting (3.22) from (3.21) + (3.23), we obtain

$$-(S + T)u \leq A(x, x) - A(y, y) \leq (S + T)u. \tag{3.24}$$

Let $\widetilde{A}(x) = A(x, x)$; then we have

$$-(S + T)u \leq \widetilde{A}(x) - \widetilde{A}(y) \leq (S + T)u. \tag{3.25}$$

As S and T are both positive linear bounded operators, so, $S + T$ is a positive linear bounded operator, and therefore $(S + T)u \in P$. Hence, by mathematical induction, it is easy to know that for natural number n_0 in (3.14), we have

$$-(S + T)^{n_0}u \leq \widetilde{A}^{n_0}(x) - \widetilde{A}^{n_0}(y) \leq (S + T)^{n_0}u, \quad (S + T)^{n_0}u \in P. \tag{3.26}$$

Since $(S + T)^{n_0}u \in P$, we see that

$$\left\|\widetilde{A}^{n_0}(x) - \widetilde{A}^{n_0}(y)\right\|_0 \leq \|(S + T)^{n_0}\|\|u\|, \tag{3.27}$$

which implies by virtue of the arbitrariness of u that

$$\left\|\widetilde{A}^{n_0}x - \widetilde{A}^{n_0}y\right\|_0 \leq \|(S + T)^{n_0}\|\left\|x - y\right\|_0 \leq \alpha^{n_0}\left\|x - y\right\|_0. \tag{3.28}$$

By $0 < \alpha < 1$, we have $0 \leq \alpha^{n_0} < 1$. Thus the Banach contraction mapping principle implies that $\widetilde{A}^{n_0}$ has a unique fixed point x^* in $C(I)$, and so $\widetilde{A}$ has a unique fixed point x^* in $C(I)$; by the definition of $\widetilde{A}, A$ has a unique fixed point x^* in $C(I)$, that is, x^* is the unique solution of (1.1). And, for any $x_0 \in C(I)$, let $x_n = A(x_{n-1}, x_{n-1})$ $(n = 1, 2, \ldots)$; we have $\|x_n - x^*\|_0 \to 0$ $(n \to \infty)$. By the equivalence of $\|\cdot\|_0$ and $\|\cdot\|$ again, we get $\|x_n - x^*\| \to 0$ $(n \to \infty)$. This completes the proof. $\square$

Example 3.4. In this paper, the results apply to a very wide range of functions, we are following only one example to illustrate.

Consider the following singular third-order boundary value problem:

$$\begin{gathered} u'''(t)=\frac{k_1t+m_1}{(1-t)\tan t}\sqrt{u^2(t)+(k_2t+m_2)^2}+\int_0^t\frac{p(s,u(s))}{t\tan(1-s)}ds,\quad t\in J,\\ u(0)=u'(0)=0,\qquad u'(1)=\alpha u'(\eta), \end{gathered}\tag{3.29}$$

where $k_1,m_1,k_2,m_2\in\mathbb{R}$ and there exists $M\geq 0$, such that for any $t\in I$, $y_1,y_2\in C(I)$, $y_1\leq y_2$, we have

$$-M(y_2-y_1)(t)\leq p(t,y_1(t))-p(t,y_2(t))\leq M(y_2-y_1)(t).\tag{3.30}$$

Applying Theorem 3.1, we can find that (3.29) has a unique solution $x^*(t)\in C^2(I)$ provided $N=\max\{|m_1|,|k_1+m_1|\}<r(G)$. And moreover, for any $w_0\in C(I)$, the iterative sequence

$$\begin{gathered} w_n(t)=\int_0^1G(t,s)\left[\frac{k_1s+m_1}{s\tan(1-s)}\sqrt{w_{n-1}^2(s)+(k_2s+m_2)^2}+\int_0^s\frac{p(\tau,w_{n-1}(\tau))}{(1-\tau)\tan s}d\tau\right]ds,\\ n=1,2,\ldots \end{gathered}\tag{3.31}$$

converges to x^* $(n\to\infty)$.

To see that, we put

$$\begin{gathered} g(t,x(t),y(t))=\frac{k_1t+m_1}{(1-t)\tan t}\sqrt{x^2(t)+(k_2t+m_2)^2}+\int_0^t\frac{p(s,y(s))}{t\tan(1-s)}ds,\\ (Bx)(t)=Nx(t),\qquad (Cy)(t)=M\int_0^t y(s)ds. \end{gathered}\tag{3.32}$$

Then (3.1) is satisfied for any $t\in I$, $x_1,x_2,y_1,y_2\in C[I]$, $x_1\geq x_2$, and $y_1\leq y_2$.

In fact, if $x_1(t)=x_2(t)$, then

$$\begin{aligned} &t(1-t)g(t,x_1(t),y_1(t))-t(1-t)g(t,x_2(t),y_2(t))\\ &\quad\leq(k_1t+m_1)\sqrt{x_1^2(t)+(k_2t+m_2)^2}-(k_1t+m_1)\sqrt{x_2^2(t)+(k_2t+m_2)^2}\\ &\qquad+\int_0^t[p(s,y_1(s))-p(s,y_2(s))]ds\\ &\quad=\int_0^t[p(s,y_1(s))-p(s,y_2(s))]ds\\ &\quad\leq\int_0^t[M(y_2(s)-y_1(s))]ds\\ &\quad=B(x_1-x_2)(t)+C(y_2-y_1)(t). \end{aligned}\tag{3.33}$$

If $x_1(t) > x_2(t)$, then

$$\begin{aligned}
&t(1-t)g(t,x_1(t),y_1(t)) - t(1-t)g(t,x_2(t),y_2(t)) \\
&\quad\le (k_1t+m_1)\sqrt{x_1^2(t)+(k_2t+m_2)^2} - (k_1t+m_1)\sqrt{x_2^2(t)+(k_2t+m_2)^2} \\
&\qquad+\int_0^t \left[p(s,y_1(s)) - p(s,y_2(s))\right]ds \\
&\quad= \frac{(k_1t+m_1)\left[x_1^2(t)-x_2^2(t)\right]}{\sqrt{x_1^2(t)+(k_2t+m_2)^2}+\sqrt{x_2^2(t)+(k_2t+m_2)^2}} \\
&\qquad+\int_0^t \left[p(s,y_1(s)) - p(s,y_2(s))\right]ds \\
&\quad\le (k_1t+m_1)[|x_1(t)|-|x_2(t)|] + \int_0^t M\left[y_2(s)-y_1(s)\right]ds \\
&\quad\le |k_1t+m_1|[x_1(t)-x_2(t)] + M\int_0^t \left[y_2(s)-y_1(s)\right]ds \\
&\quad\le B(x_1-x_2)(t) + C(y_2-y_1)(t).
\end{aligned} \tag{3.34}$$

Similarly,

$$t(1-t)g(t,x_1(t),y_1(t)) - t(1-t)g(t,x_2(t),y_2(t)) \ge -B(x_1-x_2)(t) - C(y_2-y_1)(t). \tag{3.35}$$

Next, for any $t \in I$, by (3.30) and (3.32), we get

$$\|(Tu)(t)\| \le Mt\|u\|_c. \tag{3.36}$$

Then, from (3.32) and (3.36), we have

$$\left\|\left(T^2u\right)(t)\right\| \le M\int_0^t \|(Tu)(s)\|ds \le M\|u\|_c\int_0^t s\,ds = \frac{M^2t^2}{2!}\|u\|_c, \quad \forall t \in I, \tag{3.37}$$

so it is easy to know by induction, for any n, we get

$$\|(T^nu)(t)\| \le \frac{M^nt^n}{n!}\|u\|_c, \quad \forall t \in I; \tag{3.38}$$

thus

$$\|(T^nu)\| = \max_{t\in I}\|(T^nu)(t)\| \le \frac{M^nt^n}{n!}\|u\|_c, \tag{3.39}$$

so

$$r(T) = \lim_{n\to\infty} \|(T^n\|^{1/n} = 0. \tag{3.40}$$

then we get

$$r(B + C) \le r(B) + r(C) = N + 0 < r(G). \tag{3.41}$$

Let $x_0 = y_0 = 1$; then

$$\int_0^1 t(1-t)g\big(t, x_0(t), y_0(t)\big)dt \text{ is converged.} \tag{3.42}$$

Thus all conditions in Theorem 3.1 are satisfied.

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References

[1] M. Gregus, "Third order linear Differential equations," in *Mathematics and Its Applications*, Reidel, Dordrecht, the Netherlands, 1987.

[2] L.-J. Guo, J.-P. Sun, and Y.-H. Zhao, "Existence of positive solutions for nonlinear third-order three-point boundary value problems," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 68, no. 10, pp. 3151–3158, 2008.

[3] F. Wang and Y. Cui, "On the existence of solutions for singular boundary value problem of third-order differential equations," *Mathematica Slovaca*, vol. 60, no. 4, pp. 485–494, 2010.

[4] Y. Sun, "Positive solutions for third-order three-point nonhomogeneous boundary value problems," *Applied Mathematics Letters*, vol. 22, no. 1, pp. 45–51, 2009.

[5] J. R. Graef and J. R. L. Webb, "Third order boundary value problems with nonlocal boundary conditions," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 71, no. 5-6, pp. 1542–1551, 2009.

[6] Z. Liu, L. Debnath, and S. M. Kang, "Existence of monotone positive solutions to a third order two-point generalized right focal boundary value problem," *Computers & Mathematics with Applications*, vol. 55, no. 3, pp. 356–367, 2008.

[7] F. M. Minhós, "On some third order nonlinear boundary value problems: existence, location and multiplicity results," *Journal of Mathematical Analysis and Applications*, vol. 339, no. 2, pp. 1342–1353, 2008.

[8] B. Hopkins and N. Kosmatov, "Third-order boundary value problems with sign-changing solutions," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 67, no. 1, pp. 126–137, 2007.

[9] Q. Yao, "Successive iteration of positive solution for a discontinuous third-order boundary value problem," *Computers & Mathematics with Applications*, vol. 53, no. 5, pp. 741–749, 2007.

[10] A. Boucherif and N. Al-Malki, "Nonlinear three-point third-order boundary value problems," *Applied Mathematics and Computation*, vol. 190, no. 2, pp. 1168–1177, 2007.

[11] S. Li, "Positive solutions of nonlinear singular third-order two-point boundary value problem," *Journal of Mathematical Analysis and Applications*, vol. 323, no. 1, pp. 413–425, 2006.

[12] Y. Sun, "Positive solutions of singular third-order three-point boundary value problem," *Journal of Mathematical Analysis and Applications*, vol. 306, no. 2, pp. 589–603, 2005.

[13] Z. Du, W. Ge, and X. Lin, "Existence of solutions for a class of third-order nonlinear boundary value problems," *Journal of Mathematical Analysis and Applications*, vol. 294, no. 1, pp. 104–112, 2004.

[14] D. Guo, *Semi-Ordered Method in Nonlinear Analysis*, Shandong Scientific Technical Press, Jinan, China, 2000.

[15] D. J. Guo and V. Lakshmikantham, *Nonlinear Problems in Abstract Cones*, vol. 5 of *Notes and Reports in Mathematics in Science and Engineering*, Academic Press, Boston, Mass, USA, 1988.

[16] D. Guo, V. Lakshmikantham, and X. Liu, *Nonlinear Integral Equations in Abstract Spaces*, vol. 373 of *Mathematics and Its Applications*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1996.