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Research Article

New Existence Results for Higher-Order Nonlinear Fractional Differential Equation with Integral Boundary Conditions

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This paper investigates the existence and multiplicity of positive solutions for a class of higher-order nonlinear fractional differential equations with integral boundary conditions. The results are established by converting the problem into an equivalent integral equation and applying Krasnoselskii's fixed-point theorem in cones. The nonexistence of positive solutions is also studied.

1. Introduction

Fractional differential equations arise in many engineering and scientific disciplines as the mathematical modelling of systems and processes in the fields of physics, chemistry, aerodynamics, electrodynamics of complex medium, polymer rheology, Bode's analysis of feedback amplifiers, capacitor theory, electrical circuits, electron-analytical chemistry, biology, control theory, fitting of experimental data, and so forth, and involves derivatives of fractional order. Fractional derivatives provide an excellent tool for the description of memory and hereditary properties of various materials and processes. This is the main advantage of fractional differential equations in comparison with classical integer-order models. An excellent account in the study of fractional differential equations can be found in [1–5]. For the basic theory and recent development of the subject, we refer a text by Lakshmikantham [6]. For more details and examples, see [7–23] and the references therein. However, the theory of boundary value problems for nonlinear fractional differential equations is still in the initial stages and many aspects of this theory need to be explored.

In [23], Zhang used a fixed-point theorem for the mixed monotone operator to show the existence of positive solutions to the following singular fractional differential equation.

$$\mathbf{D}_{0+}^{\alpha}u(t) + q(t)f\left(t, x(t), x'(t), \dots, x^{(n-2)}(t)\right) = 0, \quad 0 < t < 1, \tag{1.1}$$

subject to the boundary conditions

$$u(0) = u'(0) = u''(0) = \cdots = u^{(n-2)}(0) = u^{(n-1)}(1) = 0, \tag{1.2}$$

where $\mathbf{D}_{0+}^{\alpha}$ is the standard Rimann-Liouville fractional derivative of order $n-1 < \alpha \leq n,\ n \geq 2$, the nonlinearity f may be singular at $u = 0,\ u' = 0, \dots, u^{(n-2)} = 0$, and function $q(t)$ may be singular at $t = 0$. The author derived the corresponding Green's function named by fractional Green's function and obtained some properties as follows.

Proposition 1.1. *Green's function $G(t,s)$ satisfies the following conditions:*

(i) $G(t,s) \geq 0,\ G(t,s) \leq t^{\alpha-n+2}/\Gamma(\alpha-n+2),\ G(t,s) \leq G(s,s)$ *for all* $0 \leq t,\ s \leq 1$;

(ii) *there exists a positive function* $\rho \in C(0,1)$ *such that*

$$\min_{\gamma \leq t \leq \delta} G(t,s) \geq \rho(s)G(s,s), \quad s \in (0,1), \tag{1.3}$$

where $0 < \gamma < \delta < 1$ *and*

$$\rho(s) = \begin{cases} \dfrac{[\delta(1-s)]^{\alpha-n+1} - (\delta-s)^{\alpha-n+1}}{(s(1-s))^{\alpha-n+1}}, & s \in (0,r], \\ \left(\dfrac{\gamma}{s}\right)^{\alpha-n+1}, & s \in [r,1), \end{cases} \tag{1.4}$$

here $\gamma < r < \delta$.

It is well known that the cone theoretic techniques play a very important role in applying Green's function in the study of solutions to boundary value problems. In [23], the author cannot acquire a positive constant taking instead of the role of positive function $\rho(s)$ with $n-1 < \alpha \leq n,\ n \geq 2$ in (1.3). At the same time, we notice that many authors obtained the similar properties to that of (1.3), for example, see Bai [12], Bai and Lü [13], Jiang and Yuan [14], Li et al, [15], Kaufmann and Mboumi [19], and references therein. Naturally, one wishes to find whether there exists a positive constant ρ such that

$$\min_{\gamma \leq t \leq \delta} G(t,s) \geq \rho G(s,s), \quad s \in [0,1], \tag{1.5}$$

for the fractional order cases. In Section 2, we will deduce some new properties of Green's function.

Motivated by the above mentioned work, we study the following higher-order singular boundary value problem of fractional differential equation.

$$\begin{gathered}\mathbf{D}_{0^+}^{\alpha}x(t)+g(t)f(t,x(t))=0,\quad 0<t<1\\ x(0)=x'(0)=\cdots=x^{(n-2)}(0)=0,\\ x(1)=\int_0^1 h(t)x(t)dt,\end{gathered}\tag{P}$$

where $\mathbf{D}_{0+}^{\alpha}$ is the standard Rimann-Liouville fractional derivative of order $n-1<\alpha\leq n$, $n\geq 3$, $g\in C((0,1),[0,+\infty))$ and g may be singular at $t=0$ or/and at $t=1$, $h\in L^1[0,1]$ is nonnegative, and $f\in C([0,1]\times[0,+\infty),[0,+\infty))$.

For the case of $\alpha=n$, $\int_0^1 h(t)x(t)dt=ax(\eta)$, $0<\eta<1$, $0<a\eta^{n-1}<1$, the boundary value problems (P) reduces to the problem studied by Eloe and Ahmad in [24]. In [24], the authors used the Krasnosel'skii and Guo [25] fixed-point theorem to show the existence of at least one positive solution if f is either superlinear or sublinear to problem (P). For the case of $\alpha=n$, $\int_0^1 h(t)x(t)dt=\Sigma_{i=1}^{m-2}\xi_i x(\eta_i)$, $\xi_i\in(0,\infty)$, $\eta_i\in(0,1)$, $i=1,2,\ldots,n-2$, the boundary value problems (P) is related to a m-point boundary value problems of integer-order differential equation. Under this case, a great deal of research has been devoted to the existence of solutions for problem (P), for example, see Pang et al. [26], Yang and Wei [27], Feng and Ge [28], and references therein. All of these results are based upon the fixed-point index theory, the fixed-point theorems and the fixed-point theory in cone for strict set contraction operator.

The organization of this paper is as follows. We will introduce some lemmas and notations in the rest of this section. In Section 2, we present the expression and properties of Green's function associated with boundary value problem (P). In Section 3, we discuss some characteristics of the integral operator associated with the problem (P) and state a fixed-point theorem in cones. In Section 4, we discuss the existence of at least one positive solution of boundary value problem (P). In Section 5, we will prove the existence of two or m positive solutions, where m is an arbitrary natural number. In Section 6, we study the nonexistence of positive solution of boundary value problem (P). In Section 7, one example is also included to illustrate the main results. Finally, conclusions in Section 8 close the paper.

The fractional differential equations related notations adopted in this paper can be found, if not explained specifically, in almost all literature related to fractional differential equations. The readers who are unfamiliar with this area can consult, for example, [1–6] for details.

Definition 1.2 (see [4]). The integral

$$I_{0+}^{\alpha}f(x)=\frac{1}{\Gamma(\alpha)}\int_0^x\frac{f(t)}{(x-t)^{1-\alpha}}dt,\quad x>0,\tag{1.6}$$

where $\alpha>0$, is called Riemann-Liouville fractional integral of order α.

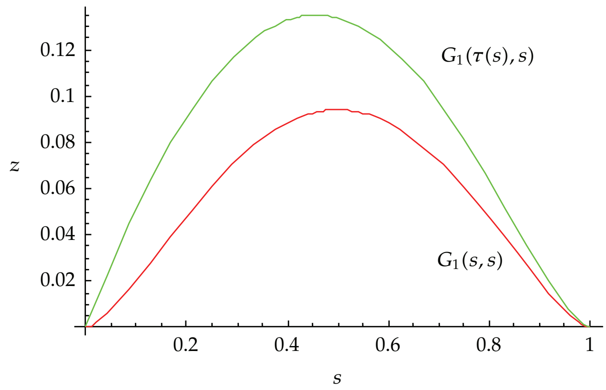

Figure 1: Graph of functions $G_1(\tau(s), s)$ $G_1(s, s)$ for $\alpha = 5/2$.

Definition 1.3 (see [4]). For a function $f(x)$ given in the interval $[0, 1)$, the expression

$$D_{0+}^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dx}\right)^n \int_0^x \frac{f(t)}{(x-t)^{\alpha-n+1}} dt, \tag{1.7}$$

where $n = [\alpha] + 1$, $[\alpha]$ denotes the integer part of number α, is called the Riemann-Liouville fractional derivative of order α.

Lemma 1.4 (see [13]). *Assume that $u \in C(0,1) \cap L(0,1)$ with a fractional derivative of order $\alpha > 0$ that belongs to $u \in C(0,1) \cap L(0,1)$. Then*

$$I_{0+}^{\alpha} D_{0+}^{\alpha} u(t) = u(t) + C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + \cdots + C_N t^{\alpha-N}, \tag{1.8}$$

for some $C_i \in R$, $i = 1, 2, \ldots, N$, where N is the smallest integer greater than or equal to α.

2. Expression and Properties of Green's Function

In this section, we present the expression and properties of Green's function associated with boundary value problem (P).

Lemma 2.1. *Assume that $\int_0^1 h(t) t^{\alpha-1} dt \neq 1$. Then for any $y \in C[0,1]$, the unique solution of boundary value problem*

$$\begin{gathered} \mathbf{D}_{0+}^{\alpha} x(t) + y(t) = 0, \quad 0 < t < 1 \\ x(0) = x'(0) = \ldots = x^{(n-2)}(0) = 0, \\ x(1) = \int_0^1 h(t) x(t) dt \end{gathered} \tag{2.1}$$

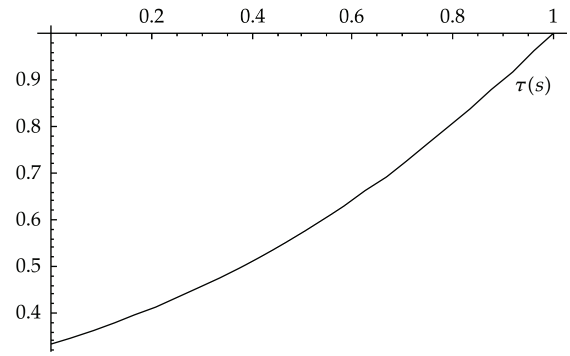

Figure 2: Graph of function $\tau(s)$ for $\alpha = 5/2$.

is given by

$$x(t) = \int_0^1 G(t,s)y(s)ds, \tag{2.2}$$

where

$$G(t,s) = G_1(t,s) + G_2(t,s), \tag{2.3}$$

$$G_1(t,s) = \begin{cases} \dfrac{t^{\alpha-1}(1-s)^{\alpha-1} - (t-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \le s \le t \le 1, \\[2ex] \dfrac{t^{\alpha-1}(1-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \le t \le s \le 1, \end{cases} \tag{2.4}$$

$$G_2(t,s) = \frac{t^{\alpha-1}}{1 - \int_0^1 h(t)t^{\alpha-1}dt} \int_0^1 h(t)G_1(t,s)dt. \tag{2.5}$$

Proof. By Lemma 1.4, we can reduce the equation of problem (2.1) to an equivalent integral equation

$$\begin{aligned} x(t) &= -I_{0+}^{\alpha}y(t) + c_1t^{\alpha-1} + c_2t^{\alpha-2} + \cdots + c_nt^{\alpha-n} \\ &= -\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}y(s)ds + c_1t^{\alpha-1} + c_2t^{\alpha-2} + \cdots + c_nt^{\alpha-n}. \end{aligned} \tag{2.6}$$

By $x(0) = 0$, there is $c_n = 0$. Thus,

$$x(t) = -I_{0+}^{\alpha}y(t) + c_1t^{\alpha-1} + c_2t^{\alpha-2} + \cdots + c_{n-1}t^{\alpha-n+1}. \tag{2.7}$$

Differentiating (2.7), we have

$$x'(t) = -\frac{\alpha-1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-2}y(s)ds + c_1(\alpha-1)t^{\alpha-2} + \cdots + c_{n-1}(\alpha-n+1)t^{\alpha-n}. \tag{2.8}$$

By (2.8) and $x'(0) = 0$, we have $c_{n-1} = 0$. Similarly, we can obtain that $c_2 = c_3 = \cdots = c_{n-2} = 0$. Then

$$x(t) = -\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}y(s)ds + c_1 t^{\alpha-1}. \tag{2.9}$$

By $x(1) = \int_0^1 h(t)x(t)dt$, we have

$$c_1 = \int_0^1 h(t)x(t)dt + \frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}y(s)ds. \tag{2.10}$$

Therefore, the unique solution of BVP (2.1) is

$$\begin{aligned} x(t) &= -\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}y(s)ds + t^{\alpha-1}\left(\int_0^1 h(t)x(t)dt + \frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}y(s)ds\right) \\ &= \int_0^1 G_1(t,s)y(s)ds + t^{\alpha-1}\int_0^1 h(t)x(t)dt, \end{aligned} \tag{2.11}$$

where $G_1(t,s)$ is defined by (2.4).

From (2.11), we have

$$\int_0^1 h(t)x(t)dt = \int_0^1 h(t)\int_0^1 G_1(t,s)y(s)ds\,dt + \int_0^1 h(t)t^{\alpha-1}dt\int_0^1 h(t)x(t)dt. \tag{2.12}$$

It follows that

$$\int_0^1 h(t)x(t)dt = \frac{1}{1-\int_0^1 h(t)t^{\alpha-1}dt}\int_0^1 h(t)\int_0^1 G_1(t,s)y(s)ds\,dt. \tag{2.13}$$

Substituting (2.13) into (2.11), we obtain

$$\begin{aligned} x(t) &= \int_0^1 G_1(t,s)y(s)ds + \frac{t^{\alpha-1}}{1-\int_0^1 h(t)t^{\alpha-1}dt}\int_0^1 h(t)\int_0^1 G_1(t,s)y(s)ds\,dt \\ &= \int_0^1 G_1(t,s)y(s)ds + \int_0^1 G_2(t,s)y(s)ds \\ &= \int_0^1 G(t,s)y(s)ds, \end{aligned} \tag{2.14}$$

where $G(t,s)$, $G_1(t,s)$, and $G_2(t,s)$ are defined by (2.3), (2.4), and (2.5), respectively. The proof is complete. □

From (2.3), (2.4), and (2.5), we can prove that $G(t,s)$, $G_1(t,s)$, and $G_2(t,s)$ have the following properties.

Proposition 2.2. *The function $G_1(t,s)$ defined by* (2.4) *satisfies*

(i) $G_1(t,s) \geq 0$ *is continuous for all* $t,s \in [0,1]$, $G_1(t,s) > 0$, *for all* $t,s \in (0,1)$;

(ii) *for all* $t \in [0,1]$, $s \in (0,1)$, *one has*

$$G_1(t,s) \leq G_1(\tau(s),s) = \frac{(\tau(s))^{\alpha-1}(1-s)^{\alpha-1} - (\tau(s)-s)^{\alpha-1}}{\Gamma(\alpha)}, \tag{2.15}$$

where

$$\tau(s) = \frac{s}{1-(1-s)^{(\alpha-1)/(\alpha-2)}}. \tag{2.16}$$

Proof. (i) It is obvious that $G_1(t,s)$ is continuous on $[0,1]\times[0,1]$ and $G_1(t,s) \geq 0$ when $s \geq t$. For $0 \leq s < t \leq 1$, we have

$$t^{\alpha-1}(1-s)^{\alpha-1} - (t-s)^{\alpha-1} = (1-s)^{\alpha-1}\left[t^{\alpha-1} - \left(\frac{t-s}{1-s}\right)^{\alpha-1}\right] \geq 0. \tag{2.17}$$

So, by (2.4), we have

$$G_1(t,s) \geq 0, \quad \forall t,\ s \in [0,1]. \tag{2.18}$$

Similarly, for $t,s \in (0,1)$, we have $G_1(t,s) > 0$.

(ii) Since $n-1 < \alpha \leq n, n \geq 3$, it is clear that $G_1(t,s)$ is increasing with respect to t for $0 \leq t \leq s \leq 1$.

On the other hand, from the definition of $G_1(t,s)$, for given $s \in (0,1), s < t \leq 1$, we have

$$\frac{\partial G_1(t,s)}{\partial t} = \frac{\alpha-1}{\Gamma(\alpha)}\left\{t^{\alpha-2}(1-s)^{\alpha-1} - (t-s)^{\alpha-2}\right\}. \tag{2.19}$$

Let

$$\frac{\partial G_1(t,s)}{\partial t} = 0. \tag{2.20}$$

Then, we have

$$t^{\alpha-2}(1-s)^{\alpha-1} = (t-s)^{\alpha-2}, \tag{2.21}$$

and so,

$$(1-s)^{\alpha-1} = \left(1-\frac{s}{t}\right)^{\alpha-2}. \tag{2.22}$$

Noticing $\alpha > 2$, from (2.22), we have

$$t = \frac{s}{1-(1-s)^{(\alpha-1)/(\alpha-2)}} =: \tau(s). \tag{2.23}$$

Then, for given $s \in (0,1)$, we have $G_1(t,s)$ arrives at maximum at $(\tau(s),s)$ when $s < t$. This together with the fact that $G_1(t,s)$ is increasing on $s \geq t$, we obtain that (2.15) holds. □

Remark 2.3. From Figure 1, we can see that $G_1(s,s) \leq G_1(\tau(s),s)$ for $\alpha > 2$. If $1 < \alpha \leq 2$, then

$$G_1(t,s) \leq G_1(s,s) = \frac{s^{\alpha-1}(1-s)^{\alpha-1}}{\Gamma(\alpha)}. \tag{2.24}$$

Remark 2.4. From Figure 2, we can see that $\tau(s)$ is increasing with respect to s.

Remark 2.5. From Figure 3, we can see that $G_1(\tau(s),s) > 0$ for $s \in J_\theta = [\theta, 1-\theta]$, where $\theta \in (0,1/2)$.

Remark 2.6. Let $\overline{G}_1(\tau(s),s) = (\tau(s))^{\alpha-1}(1-s)^{\alpha-1} - (\tau(s)-s)^{\alpha-1}$. From (2.15), for $s \in (0,1)$, we have

$$\begin{aligned}
\frac{d\overline{G}_1(\tau(s),s)}{ds} &= -(\alpha-1)(1-s)^{\alpha-2}(\tau(s))^{\alpha-1} - (\alpha-1)(\tau(s)-s)^{\alpha-2} \\
&\quad \times \left(-1 + \frac{1}{1-(1-s)^{(\alpha-1)/(\alpha-2)}} - \frac{(\alpha-1)(1-s)^{-1+(\alpha-1)/(\alpha-2)}s}{(\alpha-2)\left(1-(1-s)^{(\alpha-1)/(\alpha-2)}\right)^2}\right) \\
&\quad + (\alpha-1)(1-s)^{(\alpha-1)}(\tau(s))^{(\alpha-2)} \\
&\quad \times \left(\frac{1}{1-(1-s)^{(\alpha-1)/(\alpha-2)}} - \frac{(\alpha-1)(1-s)^{(\alpha-1)/(\alpha-2)}s}{(\alpha-2)\left(1-(1-s)^{(\alpha-1)/(\alpha-2)}\right)^2}\right).
\end{aligned} \tag{2.25}$$

Remark 2.7. From (2.25), we have

$$\lim_{s\to 0}\frac{d\overline{G}_1(\tau(s),s)}{ds} = (\alpha-1)\left[-\left(\frac{\alpha-2}{\alpha-1}\right)^{\alpha-1} + \left(\frac{\alpha-2}{\alpha-1}\right)^{\alpha-2}\right] := f(\alpha). \tag{2.26}$$

Remark 2.8. From Figure 4, it is easy to obtain that $f(\alpha)$ is decreasing with respect to α, and

$$\lim_{\alpha\to 2} f(\alpha) = 1, \qquad \lim_{\alpha\to\infty} f(\alpha) = \frac{1}{e}. \tag{2.27}$$

Proposition 2.9. *There exists $\gamma > 0$ such that*

$$\min_{t\in[\theta,1-\theta]} G_1(t,s) \geq \gamma G_1(\tau(s),s), \quad \forall s \in [0,1]. \tag{2.28}$$

Proof. For $t \in J_\theta$, we divide the proof into the following three cases for $s \in [0,1]$.

Case 1. If $s \in J_\theta$, then from (i) of Proposition 2.2 and Remark 2.5, we have

$$G_1(t,s) > 0, \quad G_1(\tau(s),s) > 0, \quad \forall t,s \in J_\theta. \tag{2.29}$$

It is obvious that $G_1(t,s)$ and $G_1(\tau(s),s)$ are bounded on J_θ. So, there exists a constant $\gamma_1 > 0$ such that

$$G_1(t,s) \geq \gamma_1 G_1(\tau(s),s), \quad \forall t,s \in J_\theta. \tag{2.30}$$

Case 2. If $s \in [1-\theta,1]$, then from (2.4), we have

$$G_1(t,s) = \frac{t^{\alpha-1}(1-s)^{\alpha-1}}{\Gamma(\alpha)}. \tag{2.31}$$

On the other hand, from the definition of $\tau(s)$, we obtain that $\tau(s)$ takes its maximum 1 at $s = 1$. So

$$\begin{aligned} G_1(\tau(s),s) &= \frac{(\tau(s))^{\alpha-1}(1-s)^{\alpha-1} - (\tau(s)-s)^{\alpha-1}}{\Gamma(\alpha)} \\ &\leq \frac{(\tau(s))^{\alpha-1}(1-s)^{\alpha-1}}{\Gamma(\alpha)} \\ &= \frac{(\tau(s))^{\alpha-1}}{t^{\alpha-1}} \frac{(1-s)^{\alpha-1}t^{\alpha-1}}{\Gamma(\alpha)} \\ &\leq \frac{1}{\theta^{\alpha-1}} G_1(t,s). \end{aligned} \tag{2.32}$$

Therefore, $G_1(t,s) \geq \theta^{\alpha-1} G_1(\tau(s),s)$. Letting $\theta^{\alpha-1} = \gamma_2$, we have

$$G_1(t,s) \geq \gamma_2 G_1(\tau(s),s). \tag{2.33}$$

Case 3. If $s \in [0,\theta]$, from (i) of Proposition 2.2, it is clear that

$$G_1(t,s) > 0, \quad G_1(\tau(s),s) > 0, \quad \forall t \in J_\theta, \ s \in (0,\theta]. \tag{2.34}$$

In view of Remarks 2.6–2.8, we have

$$\begin{aligned}\lim_{s\to 0}\frac{G_1(t,s)}{G_1(\tau(s),s)} &= \lim_{s\to 0}\frac{t^{\alpha-1}(1-s)^{\alpha-1}-(t-s)^{\alpha-1}}{(\tau(s))^{\alpha-1}(1-s)^{\alpha-1}-(\tau(s)-s)^{\alpha-1}}\\ &= \lim_{s\to 0}\frac{-(\alpha-1)t^{\alpha-1}(1-s)^{\alpha-2}-(\alpha-1)(t-s)^{\alpha-2}}{d\overline{G}_1(\tau(s),s)/ds}\\ &> 0.\end{aligned} \tag{2.35}$$

From (2.35), there exists a constant γ_3 such that

$$G_1(t,s)\geq \gamma_3 G_1(\tau(s),s). \tag{2.36}$$

Letting $\gamma=\min\{\gamma_1,\gamma_2,\gamma_3\}$ and using (2.30), (2.33), and (2.36), it follows that (2.28) holds. This completes the proof. □

Let

$$\mu=\int_0^1 h(t)t^{\alpha-1}dt. \tag{2.37}$$

Proposition 2.10. *If $\mu\in[0,1)$, then one has*

(i) $G_2(t,s)\geq 0$ *is continuous for all* $t,s\in[0,1]$, $G_2(t,s)>0$, *for all* t, $s\in(0,1)$;

(ii) $G_2(t,s)\leq (1/(1-\mu))\int_0^1 h(t)G_1(t,s)dt$, *for all* $t\in[0,1]$, $s\in(0,1)$.

Proof. Using the properties of $G_1(t,s)$, definition of $G_2(t,s)$, it can easily be shown that (i) and (ii) hold. □

Theorem 2.11. *If $\mu\in[0,1)$, the function $G(t,s)$ defined by* (2.3) *satisfies*

(i) $G(t,s)\geq 0$ *is continuous for all* $t,s\in[0,1], G(t,s)>0$, *for all* $t,s\in(0,1)$;

(ii) $G(t,s)\leq G(s)$ *for each* $t,s\in[0,1]$, *and*

$$\min_{t\in[\theta,1-\theta]}G(t,s)\geq \gamma^{*}G(s),\quad \forall s\in[0,1], \tag{2.38}$$

where

$$\gamma^{*}=\min\left\{\gamma,\theta^{\alpha-1}\right\},\qquad G(s)=G_1(\tau(s),s)+G_2(1,s), \tag{2.39}$$

$\tau(s)$ *is defined by* (2.16), γ *is defined in Proposition 2.9.*

Proof. (i) From Propositions 2.2 and 2.10, we obtain that $G(t,s) \geq 0$ is continuous for all $t,s \in [0,1]$, and $G(t,s) > 0$, for all $t,s \in (0,1)$.

(ii) From (ii) of Proposition 2.2 and (ii) of Proposition 2.10, we have that $G(t,s) \leq G(s)$ for each t, $s \in [0,1]$.

Now, we show that (2.38) holds.

In fact, from Proposition 2.9, we have

$$\begin{aligned}\min_{t\in J_\theta} G(t,s) &\geq \gamma G_1(\tau(s),s) + \frac{\theta^{\alpha-1}}{1-\mu}\int_0^1 h(t)G_1(t,s)dt \\ &\geq \gamma^*\left[G_1(\tau(s),s) + \frac{1}{1-\mu}\int_0^1 h(t)G_1(t,s)dt\right] \\ &= \gamma^* G(s), \quad \forall s \in [0,1].\end{aligned} \tag{2.40}$$

Then the proof of Theorem 2.11 is completed. □

Remark 2.12. From the definition of γ^*, it is clear that $0 < \gamma^* < 1$.

3. Preliminaries

Let $J = [0,1]$ and $E = C[0,1]$ denote a real Banach space with the norm $\|\cdot\|$ defined by $\|x\| = \max_{0\leq t\leq 1}|x(t)|$. Let

$$\begin{gathered}K = \left\{x \in E : x \geq 0, \min_{t\in J_\theta} x(t) \geq \gamma^*\|x\|\right\}, \\ K_r = \{x \in K : \|x\| \leq r\}, \qquad \partial K_r = \{x \in K : \|x\| = r\}.\end{gathered} \tag{3.1}$$

To prove the existence of positive solutions for the boundary value problem (P), we need the following assumptions:

(H_1) $g \in C((0,1),[0,+\infty))$, $g(t) \not\equiv 0$ on any subinterval of (0,1) and $0 < \int_0^1 G(s)g(s)ds < +\infty$, where $G(s)$ is defined in Theorem 2.11;

(H_2) $f \in C([0,1]\times[0,+\infty),[0,+\infty))$ and $f(t,0) = 0$ uniformly with respect to t on $[0,1]$;

(H_3) $\mu \in [0,1)$, where μ is defined by (2.37).

From condition (H_1), it is not difficult to see that g may be singular at $t = 0$ or/and at $t = 1$, that is, $\lim_{t\to 0^+} g(t) = \infty$ or/and $\lim_{t\to 1^-} g(t) = \infty$.

Define $T : K \to K$ by

$$(Tx)(t) = \int_0^1 G(t,s)g(s)f(s,x(s))ds, \tag{3.2}$$

where $G(t,s)$ is defined by (2.3).

Lemma 3.1. *Let* (H_1)–(H_3) *hold. Then boundary value problems* (P) *has a solution* x *if and only if* x *is a fixed point of* T.

Proof. From Lemma 2.1, we can prove the result of this lemma. □

Lemma 3.2. *Let* (H_1)–(H_3) *hold. Then* $TK \subset K$ *and* $T : K \to K$ *is completely continuous.*

Proof. For any $x \in K$, by (3.2), we can obtain that $Tx \geq 0$. On the other hand, by (ii) of Theorem 2.11, we have

$$(Tx)(t) \leq \int_0^1 G(s)g(s)f(s,x(s))ds. \tag{3.3}$$

Similarly, by (2.38), we obtain

$$\begin{aligned}(Tx)(t) &\geq \gamma^* \int_0^1 G(s)g(s)f(s,x(s))ds \\ &= \gamma^* \|Tx\|, \quad t \in J_\theta.\end{aligned} \tag{3.4}$$

So, $Tx \in K$ and hence $T(K) \subset K$. Next by similar proof of Lemma 3.1 in [13] and Ascoli-Arzela theorem one can prove $T : K \to K$ is completely continuous. So it is omitted. □

To obtain positive solutions of boundary value problem (P), the following fixed-point theorem in cones is fundamental which can be found in [25, page 94].

Lemma 3.3 (Fixed-point theorem of cone expansion and compression of norm type). *Let* P *be a cone of real Banach space* E, *and let* Ω_1 *and* Ω_2 *be two bounded open sets in* E *such that* $0 \in \Omega_1$ *and* $\overline{\Omega}_1 \subset \Omega_2$. *Let operator* $A : P \cap (\overline{\Omega}_2 \setminus \Omega_1) \to P$ *be completely continuous. Suppose that one of the two conditions*

(i) $\|Ax\| \leq \|x\|$, *for all* $x \in P \cap \partial\Omega_1$ *and* $\|Ax\| \geq \|x\|$, *for all* $x \in P \cap \partial\Omega_2$

or

(ii) $\|Ax\| \geq \|x\|$, *for all* $x \in P \cap \partial\Omega_1$, *and* $\|Ax\| \leq \|x\|$, *for all* $x \in P \cap \partial\Omega_2$

is satisfied. Then A *has at least one fixed point in* $P \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

4. Existence of Positive Solution

In this section, we impose growth conditions on f which allow us to apply Lemma 3.3 to establish the existence of one positive solution of boundary value problem (P), and we begin by introducing some notations:

$$f^\beta = \limsup_{x \to \beta} \max_{t \in [0,1]} \frac{f(t,x)}{x}, \qquad f_\beta = \liminf_{x \to \beta} \min_{t \in [0,1]} \frac{f(t,x)}{x}, \tag{4.1}$$

where β denotes 0 or ∞, and

$$\sigma = \int_0^1 G(s)g(s)ds. \tag{4.2}$$

Theorem 4.1. *Assume that* (H_1)–(H_3) *hold. In addition, one supposes that one of the following conditions is satisfied:*

(C_1) $f_0 > 1/\int_\theta^{1-\theta} G(s)g(s)ds(\gamma^*)^2$ *and* $f^\infty < 1/\sigma$ *(particularly,* $f_0 = \infty$ *and* $f^\infty = 0$*).*

(C_2) *there exist two constants* r_2, R_2 *with* $0 < r_2 \le R_2$ *such that* $f(t,\cdot)$ *is nondecreasing on* $[0, R_2]$

for all $t \in [0,1]$, *and* $f(t, \gamma^* r_2) \ge r_2/\gamma^* \int_\theta^{1-\theta} G(s)g(s)ds$, *and* $f(t, R_2) \le R_2/\sigma$ *for all* $t \in [0,1]$.
Then boundary value problem (P) *has at least one positive solution.*

Proof. Let T be cone preserving completely continuous that is defined by (3.2).

Case 1. The condition (C_1) holds. Considering $f_0 > 1/\int_\theta^{1-\theta} G(s)g(s)ds(\gamma^*)^2$, there exists $r_1 > 0$ such that $f(t,x) \ge (f_0 - \varepsilon_1)x$, for $t \in [0,1]$, $x \in [0, r_1]$, where $\varepsilon_1 > 0$ satisfies $\int_\theta^{1-\theta} G(s)g(s)ds(\gamma^*)^2(f^0 - \varepsilon_1) \ge 1$. Then, for $t \in [0,1]$, $x \in \partial K_{r_1}$, we have

$$\begin{aligned}(Tx)(t) &= \int_0^1 G(t,s)g(s)f(s,x(s))ds \\ &\ge \gamma^* \int_0^1 G(s)g(s)f(s,x(s))ds \\ &\ge \gamma^* \int_0^1 G(s)g(s)\left(f^0 - \varepsilon_1\right)x(s)ds \\ &\ge (\gamma^*)^2\left(f^0 - \varepsilon_1\right)\int_\theta^{1-\theta} G(s)g(s)ds\|x\| \\ &\ge \|x\|,\end{aligned} \tag{4.3}$$

that is, $x \in \partial K_{r_1}$ imply that

$$\|Tx\| \ge \|x\|. \tag{4.4}$$

Next, turning to $f^\infty < 1/\sigma$, there exists $\overline{R}_1 > 0$ such that

$$f(t,x) \le (f^\infty + \varepsilon_2)x, \quad \text{for } t \in [0,1],\ x \in \left(\overline{R}_1, \infty\right), \tag{4.5}$$

where $\varepsilon_2 > 0$ satisfies $\sigma(f^\infty + \varepsilon_2) \le 1$.
Set

$$M = \max_{0 \le x \le \overline{R}_1,\ t \in [0,1]} f(t,x), \tag{4.6}$$

then $f(t,x) \le M + (f^\infty + \varepsilon_2)x$.

Chose $R_1 > \max\{r_1, \overline{R}_1, M\sigma(1-\sigma(f^\infty+\varepsilon_2))^{-1}\}$. Then, for $x \in \partial K_{R_1}$, we have

$$\begin{aligned}
(Tx)(t) &= \int_0^1 G(t,s)g(s)f(s,x(s))ds \\
&\le \int_0^1 G(s)g(s)f(s,x(s))ds \\
&\le \int_0^1 G(s)g(s)\left[M+(f^\infty+\varepsilon_2)x(s)\right]ds \\
&\le M\int_0^1 G(s)g(s)ds + \left(f^\infty+\varepsilon_2\right)\int_0^1 G(s)g(s)ds\|x\| \\
&< R_1 - \sigma\left(f^\infty+\varepsilon_2\right)R_1 + \left(f^\infty+\varepsilon_2\right)\sigma\|x\| \\
&= R_1,
\end{aligned} \tag{4.7}$$

that is, $x \in \partial K_{R_1}$ imply that

$$\|Tx\| < \|x\|. \tag{4.8}$$

Case 2. The Condition (C_2) satisfies. For $x \in K$, from (3.1) we obtain that $\min_{t\in J_\theta} x(t) \ge \gamma^*\|x\|$. Therefore, for $x \in \partial K_{r_2}$, we have $x(t) \ge \gamma^*\|x\| = \gamma^* r_2$ for $t \in J_\theta$, this together with (C_2), we have

$$\begin{aligned}
(Tx)(t) &= \int_0^1 G(t,s)g(s)f(s,x(s))ds \\
&\ge \gamma^*\int_\theta^{1-\theta} G(s)g(s)f(s,\gamma^* r_2)ds \\
&\ge \gamma^*\frac{1}{\gamma^*\int_\theta^{1-\theta}G(s)g(s)ds}r_2\int_\theta^{1-\theta} G(s)g(s)ds \\
&= r_2 = \|x\|,
\end{aligned} \tag{4.9}$$

that is, $x \in \partial K_{r_2}$ imply that

$$\|Tx\| \ge \|x\|. \tag{4.10}$$

On the other hand, for $x \in \partial K_{R_2}$, we have that $x(t) \le R_2$ for $t \in [0,1]$, this together with (C_2), we have

$$\begin{aligned}(Tx)(t) &= \int_0^1 G(t,s)g(s)f(s,x(s))ds \\ &\le \int_0^1 G(s)g(s)f(s,x(s))ds \\ &\le \frac{R_2}{\sigma}\int_0^1 G(s)g(s)ds \\ &= R_2,\end{aligned} \tag{4.11}$$

that is, $x \in \partial K_{R_2}$ imply that

$$\|Tx\| \le \|x\|. \tag{4.12}$$

Applying Lemma 3.3 to (4.4) and (4.8), or (4.10) and (4.12), yields that T has a fixed point $x^* \in \overline{K}_{r,R}$ or $x^* \in \overline{K}_{r_i,R_i}$ $(i = 1,2)$ with $x^*(t) \ge \gamma^*\|x^*\| > 0, t \in [0,1]$. Thus it follows that boundary value problems (P) has a positive solution x^*, and the theorem is proved. □

Theorem 4.2. *Assume that* (H_1)–(H_3) *hold. In addition, one supposes that the following condition is satisfied:*

(C_3) $f^0 < 1/\sigma$ *and* $f_\infty > 1/\int_\theta^{1-\theta} G(s)g(s)ds(\gamma^*)^2$ *(particularly,* $f^0 = 0$ *and* $f_\infty = \infty$*).*

Then boundary value problem (P) *has at least one positive solution.*

5. The Existence of Multiple Positive Solutions

Now we discuss the multiplicity of positive solutions for boundary value problem (P). We obtain the following existence results.

Theorem 5.1. *Assume* (H_1)–(H_3)*, and the following two conditions:*

(C_4) $f_0 > 1/\int_\theta^{1-\theta} G(s)g(s)ds(\gamma^*)^2$ *and* $f_\infty > 1/\int_\theta^{1-\theta} G(s)g(s)ds(\gamma^*)^2$ *(particularly,* $f_0 = f_\infty = \infty$*);*

(C_5) *there exists* $b > 0$ *such that* $\max_{t\in[0,1],x\in\partial K_b} f(t,x) < b/\sigma$.

Then boundary value problem (P) *has at least two positive solutions* $x^*(t)$, $x^{**}(t)$*, which satisfy*

$$0 < \|x^{**}\| < b < \|x^*\|. \tag{5.1}$$

Proof. We consider condition (C_4). Choose r, R with $0 < r < b < R$.

If $f_0 > 1/\int_\theta^{1-\theta} G(s)g(s)ds(\gamma^*)^2$, then by the proof of (4.4), we have

$$\|Tx\| \ge \|x\|, \quad \text{for } x \in \partial K_r. \tag{5.2}$$

If $f_\infty > 1/\int_\theta^{1-\theta} G(s)g(s)ds(\gamma^*)^2$, then similar to the proof of (4.4), we have

$$\|Tx\| \geq \|x\|, \quad \text{for } x \in \partial K_R. \tag{5.3}$$

On the other hand, by (C_5), for $x \in \partial K_b$, we have

$$\begin{aligned}(Tx)(t) &= \int_0^1 G(t,s)g(s)f(s,x(s))ds \\ &\leq \int_0^1 G(s)g(s)f(s,x(s))ds \\ &\leq \frac{b}{\sigma}\int_0^1 G(s)g(s)ds \\ &= b.\end{aligned} \tag{5.4}$$

By (5.4), we have

$$\|(Tx)\| < b = \|x\|. \tag{5.5}$$

Applying Lemma 3.3 to (5.2), (5.3), and (5.5) yields that T has a fixed point $x^{**} \in \overline{K}_{r,b}$, and a fixed point $x^* \in \overline{K}_{b,R}$. Thus it follows that boundary value problem (P) has at least two positive solutions x^* and x^{**}. Noticing (5.5), we have $\|x^*\| \neq b$ and $\|x^{**}\| \neq b$. Therefore (5.1) holds, and the proof is complete. □

Theorem 5.2. *Assume (H_1)–(H_3), and the following two conditions:*

(C_6) $f^0 < 1/\sigma$ *and* $f^\infty < 1/\sigma$;

(C_7) *there exists* $B > 0$ *such that* $\min_{t\in J_\theta, x\in\partial K_B} f(t,x) > B/\int_\theta^{1-\theta} G(s)g(s)ds\gamma^*$.

Then boundary value problem (P) has at least two positive solutions $x^*(t), x^{**}(t)$, which satisfy

$$0 < \|x^{**}\| < B < \|x^*\|. \tag{5.6}$$

Theorem 5.3. *Assume that (H_1), (H_2), and (H_3) hold. If there exist $2m$ positive numbers $d_k, D_k, k = 1,2,\ldots,m$ with $d_1 < \gamma^* D_1 < D_1 < d_2 < \gamma^* D_2 < D_2 < \cdots < d_m < \gamma^* D_m < D_m$ such that*

(C_8) $f(t,x) \geq (1/\int_\theta^{1-\theta} G(s)g(s)ds\gamma^*)d_k$ *for* $(t,x) \in [0,1]\times[\gamma^* d_k, d_k]$ *and* $f(t,x) \leq \sigma^{-1}D_k$ *for* $(t,x) \in [0,1]\times[\gamma^* D_k, D_k], k = 1,2,\ldots,m$.

Then boundary value problem (P) has at least m positive solutions x_k satisfying $d_k \leq \|x_k\| \leq D_k$, $k = 1,2,\ldots,m$.

Theorem 5.4. *Assume that* (H_1), (H_2), *and* (H_3) *hold. If there exist* $2m$ *positive numbers* $d_k, D_k, k = 1, 2, \ldots, m$ *with* $d_1 < D_1 < d_2 < D_2 < \cdots < d_m < D_m$ *such that*

(C_9) $f(t, \cdot)$ *is nondecreasing on* $[0, D_m]$ *for all* $t \in [0, 1]$;

(C_{10}) $f(t, \gamma^* d_k) \geq d_2 / \int_\theta^{1-\theta} G(s)g(s)ds\gamma^*$, *and* $f(t, D_k) \leq \sigma^{-1} D_k$, $k = 1, 2, \ldots, m$.

Then boundary value problem (P) *has at least* m *positive solutions* x_k *satisfying* $d_k \leq \|x_k\| \leq D_k$, $k = 1, 2, \ldots, m$.

6. The Nonexistence of Positive Solution

Our last results corresponds to the case when boundary value problem (P) has no positive solution.

Theorem 6.1. *Assume* (H_1)–(H_3) *and* $f(t, x) < \sigma^{-1} x$, *for all* $t \in J, x > 0$, *then boundary value problem* (P) *has no positive solution.*

Proof. Assume to the contrary that $x(t)$ is a positive solution of the boundary value problem (P). Then,$x \in K$, $x(t) > 0$ for $t \in (0, 1)$, and

$$\begin{aligned} \|x\| &= \max_{t \in J} |x(t)| \\ &= \int_0^1 G(t, s)g(s)f(s, x(s))ds \\ &\leq \int_0^1 G(s)g(s)f(s, x(s))ds \\ &< \int_0^1 G(s)g(s)\frac{1}{\sigma}\|x\|ds \\ &= \frac{1}{\sigma}\int_0^1 G(s)g(s)ds\|x\| \\ &= \|x\|, \end{aligned} \tag{6.1}$$

which is a contradiction, and complete the proof. □

Similarly, we have the following results.

Theorem 6.2. *Assume* $(H_1) - (H_3)$ *and* $f(t, x) > (1 / \int_\theta^{1-\theta} G(s)g(s)ds(\gamma^*)^2)x$, *for all* $x > 0$, $t \in J$, *then boundary value problem* (P) *has no positive solution.*

7. Example

To illustrate how our main results can be used in practice we present an example.

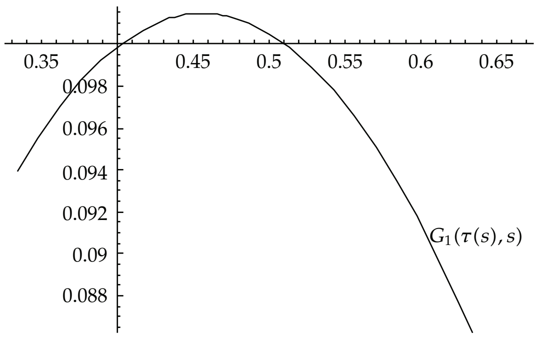

Figure 3: Graph of function $G_1(\tau(s), s)$ for $\theta = 1/3$, $\alpha = 5/2$.

Example 7.1. Consider the following boundary value problem of nonlinear fractional differential equations:

$$-D_{0^+}^{5/2}x = \frac{1}{\sqrt{t}}\left(t + x^{1/3}\tanh x + x^{1/3}\right),$$
$$x(0) = 0, \qquad x'(0) = 0, \tag{7.1}$$
$$x(1) = \int_0^1 \frac{1}{6|t - 1/2|^{2/3}}x(t)dt,$$

where

$$\alpha = \frac{5}{2}, \qquad g(t) = \frac{1}{\sqrt{t}}, \qquad h(t) = \frac{1}{6|t - 1/2|^{2/3}},$$
$$f(t, x) = t + x^{1/3}\tanh x + x^{1/3}. \tag{7.2}$$

It is easy to see that (H_1)–(H_3) hold. By simple computation, we have

$$f_0 = \infty, \qquad f^\infty = 0, \tag{7.3}$$

thus it follows that problem (7.1) has a positive solution by (C_1).

8. Conclusions

In this paper, by using the famous Guo-Krasnoselskii fixed-point theorem, we have investigated the existence and multiplicity of positive solutions for a class of higher-order nonlinear fractional differential equations with integral boundary conditions and obtained some easily verifiable sufficient criteria. The interesting point is that we obtain some new positive properties of Green's function, which significantly extend and improve many known results for fractional order cases, for example, see [12–15, 19]. The methodology which we employed in studying the boundary value problems of integer-order differential equation

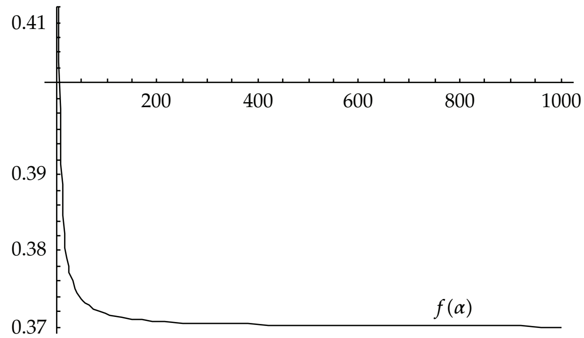

Figure 4: Graph of function $f(\alpha)$ for $\alpha > 2$.

in [28] can be modified to establish similar sufficient criteria for higher-order nonlinear fractional differential equations. It is worth mentioning that there are still many problems that remain open in this vital field except for the results obtained in this paper: for example, whether or not we can obtain the similar results of fractional differential equations with p-Laplace operator by employing the same technique of this paper, and whether or not our concise criteria can guarantee the existence of positive solutions for higher-order nonlinear fractional differential equations with impulses. More efforts are still needed in the future.

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References

[1] K. S. Miller and B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations*, A Wiley-Interscience Publication, John Wiley & Sons, New York, NY, USA, 1993.
[2] K. B. Oldham and J. Spanier, *The Fractional Calculus: Theory and Applications of Differentiation and Integration to Arbitrary Order*, vol. 11 of *Mathematics in Science and Engineering*, Academic Press, London, UK, 1974.
[3] I. Podlubny, *Fractional Differential Equations: An Introduction to Fractional Derivatives, Fractional Differential Equations, to Methods of Their Solution and Some of Their Applications*, vol. 198 of *Mathematics in Science and Engineering*, Academic Press, San Diego, Calif, USA, 1999.
[4] S. G. Samko, A. A. Kilbas, and O. I. Marichev, *Fractional Integrals and Derivatives: Theory and Applications*, Gordon and Breach, Yverdon, Switzerland, 1993.
[5] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, vol. 204 of *North-Holland Mathematics Studies*, Elsevier, Amsterdam, The Netherlands, 2006.
[6] V. Lakshmikantham, S. Leela, and J. Vasundhara Devi, *Theory of Fractional Dynamic Systems*, Cambridge Academic, Cambridge, UK, 2009.

[7] V. Lakshmikantham and A. S. Vatsala, "Basic theory of fractional differential equations," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 69, no. 8, pp. 2677–2682, 2008.

[8] V. Lakshmikantham, "Theory of fractional functional differential equations," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 69, no. 10, pp. 3337–3343, 2008.

[9] V. Lakshmikantham and A. S. Vatsala, "General uniqueness and monotone iterative technique for fractional differential equations," *Applied Mathematics Letters*, vol. 21, no. 8, pp. 828–834, 2008.

[10] B. Ahmad and J. J. Nieto, "Existence of solutions for nonlocal boundary value problems of higher-order nonlinear fractional differential equations," *Abstract and Applied Analysis*, vol. 2009, Article ID 494720, 9 pages, 2009.

[11] B. Ahmad and J. J. Nieto, "Existence results for nonlinear boundary value problems of fractional integrodifferential equations with integral boundary conditions," *Boundary Value Problems*, vol. 2009, Article ID 708576, 11 pages, 2009.

[12] Z. Bai, "On positive solutions of a nonlocal fractional boundary value problem," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 72, no. 2, pp. 916–924, 2010.

[13] Z. Bai and H. Lü, "Positive solutions for boundary value problem of nonlinear fractional differential equation," *Journal of Mathematical Analysis and Applications*, vol. 311, no. 2, pp. 495–505, 2005.

[14] D. Jiang and C. Yuan, "The positive properties of the Green function for Dirichlet-type boundary value problems of nonlinear fractional differential equations and its application," *Nonlinear Analysis: Theory, Methods and Applications*, vol. 72, no. 2, pp. 710–719, 2009.

[15] C. F. Li, X. N. Luo, and Y. Zhou, "Existence of positive solutions of the boundary value problem for nonlinear fractional differential equations," *Computers and Mathematics with Applications*, vol. 59, no. 3, pp. 1363–1375, 2010.

[16] M. Benchohra, S. Hamani, and S. K. Ntouyas, "Boundary value problems for differential equations with fractional order and nonlocal conditions," *Nonlinear Analysis: Theory, Methods and Applications*, vol. 71, no. 7-8, pp. 2391–2396, 2009.

[17] H. A. H. Salem, "On the nonlinear Hammerstein integral equations in Banach spaces and application to the boundary value problem of fractional order," *Mathematical and Computer Modelling*, vol. 48, no. 7-8, pp. 1178–1190, 2008.

[18] H. A. H. Salem, "On the fractional order m-point boundary value problem in reflexive Banach spaces and weak topologies," *Journal of Computational and Applied Mathematics*, vol. 224, no. 2, pp. 565–572, 2009.

[19] E. R. Kaufmann and E. Mboumi, "Positive solutions of a boundary value problem for a nonlinear fractional differential equation," *Electronic Journal of Qualitative Theory of Differential Equations*, vol. 2008, no. 3, pp. 1–11, 2008.

[20] S. Zhang, "Positive solutions for boundary-value problems of nonlinear fractional differential equations," *Electronic Journal of Differential Equations*, vol. 2006, pp. 1–12, 2006.

[21] S. Zhang, "The existence of a positive solution for a nonlinear fractional differential equation," *Journal of Mathematical Analysis and Applications*, vol. 252, no. 2, pp. 804–812, 2000.

[22] S. Zhang, "Existence of positive solution for some class of nonlinear fractional differential equations," *Journal of Mathematical Analysis and Applications*, vol. 278, no. 1, pp. 136–148, 2003.

[23] S. Zhang, "Positive solutions to singular boundary value problem for nonlinear fractional differential equation," *Computers and Mathematics with Applications*, vol. 59, no. 3, pp. 1300–1309, 2010.

[24] P. W. Eloe and B. Ahmad, "Positive solutions of a nonlinear nth order boundary value problem with nonlocal conditions," *Applied Mathematics Letters*, vol. 18, no. 5, pp. 521–527, 2005.

[25] D. Guo, V. Lakshmikantham, and X. Liu, *Nonlinear Integral Equations in Abstract Spaces*, vol. 373 of *Mathematics and Its Applications*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1996.

[26] C. Pang, W. Dong, and Z. Wei, "Green's function and positive solutions of nth order m-point boundary value problem," *Applied Mathematics and Computation*, vol. 182, no. 2, pp. 1231–1239, 2006.

[27] J. Yang and Z. Wei, "Positive solutions of nth order m-point boundary value problem," *Applied Mathematics and Computation*, vol. 202, no. 2, pp. 715–720, 2008.

[28] M. Feng and W. Ge, "Existence results for a class of nth order m-point boundary value problems in Banach spaces," *Applied Mathematics Letters*, vol. 22, no. 8, pp. 1303–1308, 2009.