Research Article

Multiple Positive Solutions for Second-Order p-Laplacian Dynamic Equations with Integral Boundary Conditions

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We are concerned with the following second-order p-Laplacian dynamic equations on time scales $(\varphi_p(x^\Delta(t)))^\nabla + \lambda f(t, x(t), x^\Delta(t)) = 0$, $t \in (0,T)_\mathbb{T}$, with integral boundary conditions $x^\Delta(0) = 0$, $\alpha x(T) - \beta x(0) = \int_0^T g(s)x(s)\nabla s$. By using Legget-Williams fixed point theorem, some criteria for the existence of at least three positive solutions are established. An example is presented to illustrate the main result.

1. Introduction

Boundary value problems with p-Laplacian have received a lot of attention in recent years. They often occur in the study of the n-dimensional p-Laplacian equation, non-Newtonian fluid theory, and the turbulent flow of gas in porous medium [1–7]. Many works have been carried out to discuss the existence of solutions or positive solutions and multiple solutions for the local or nonlocal boundary value problems.

On the other hand, the study of dynamic equations on time scales goes back to its founder Stefan Hilger [8] and is a new area of still fairly theoretical exploration in mathematics. Motivating the subject is the notion that dynamic equations on time scales can build bridges between continuous and discrete equations. Further, the study of time scales has led to several important applications, for example, in the study of insect population models, neural networks, heat transfer, and epidemic models, we refer to [8–10]. In addition, the study of BVPs on time scales has received a lot of attention in the literature, with the pioneering existence results to be found in [11–16].

However, existence results are not available for dynamic equations on time scales with integral boundary conditions. Motivated by above, we aim at studying the second-order

p-Laplacian dynamic equations on time scales in the form of

$$\left(\varphi_p\left(x^\Delta(t)\right)\right)^\nabla + \lambda f\left(t, x(t), x^\Delta(t)\right) = 0, \quad t \in (0, T)_\mathbb{T} \tag{1.1}$$

with integral boundary condition

$$x^\Delta(0) = 0, \quad \alpha x(T) - \beta x(0) = \int_0^T g(s)x(s)\nabla s, \tag{1.2}$$

where λ is positive parameter, $\varphi_p(s) = |s|^{p-2}s$ for $p > 1$ with $\varphi_p^{-1} = \varphi_q$ and $1/p + 1/q = 1$, Δ is the delta derivative, ∇ is the nabla derivative, $\mathbb{T}$ is a time scale which is a nonempty closed subset of $\mathbb{R}$ with the topology and ordering inherited from $\mathbb{R}$, 0 and T are points in $\mathbb{T}$, an interval $[0, T]_\mathbb{T} := [0, T] \cap \mathbb{T}$, $f \in C([0, T]_\mathbb{T} \times \mathbb{R}^2, [0, \infty))$ with $f(t, 0, 0) \neq 0$ for all $t \in [0, T]_\mathbb{T}$, $g \in C_{\mathrm{ld}}([0, T]_\mathbb{T}, [0, \infty))$, $\alpha, \beta > 0$ with $\alpha - g_0 > \beta$, and where $g_0 = \int_0^T g(s)\nabla s$.

      The main purpose of this paper is to establish some sufficient conditions for the existence of at least three positive solutions for BVPs (1.1)-(1.2) by using Legget-Williams fixed point theorem. This paper is organized as follows. In Section 2, some useful lemmas are established. In Section 3, by using Legget-Williams fixed point theorem, we establish sufficient conditions for the existence of at least three positive solutions for BVPs (1.1)-(1.2). An illustrative example is given in Section 4.

2. Preliminaries

In this section, we will first recall some basic definitions and lemmas which are used in what follows.

Definition 2.1 (see [8]). A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real set $\mathbb{R}$ with the topology and ordering inherited from $\mathbb{R}$. The forward and backward jump operators $\sigma, \rho : \mathbb{T} \to \mathbb{T}$ and the graininess $\mu, \nu : \mathbb{T} \to \mathbb{R}^+$ are defined, respectively, by

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}, \quad \rho(t) := \sup\{s \in \mathbb{T} : s < t\}, \quad \mu(t) := \sigma(t) - t, \quad \nu(t) := t - \rho(t). \tag{2.1}$$

The point $t \in \mathbb{T}$ is called left-dense, left-scattered, right-dense, or right-scattered if $\rho(t) = t$, $\rho(t) < t$, and $\sigma(t) = t$ or $\sigma(t) > t$, respectively. Points that are right-dense and left-dense at the same time are called dense. If $\mathbb{T}$ has a left-scattered maximum m_1, defined $\mathbb{T}^\kappa = \mathbb{T} - \{m_1\}$; otherwise, set $\mathbb{T}^\kappa = \mathbb{T}$. If $\mathbb{T}$ has a right-scattered minimum m_2, defined $\mathbb{T}_\kappa = \mathbb{T} - \{m_2\}$; otherwise, set $\mathbb{T}_\kappa = \mathbb{T}$.

Definition 2.2 (see [8]). For $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^\kappa$, then the delta derivative of f at the point t is defined to be the number $f^\Delta(t)$ (provided it exists) with the property that for each $\epsilon > 0$, there is a neighborhood U of t such that

$$\left| f(\sigma(t)) - f(s) - f^\Delta(t)(\sigma(t) - s) \right| \le \epsilon |\sigma(t) - s| \quad \forall s \in U. \tag{2.2}$$

For $f : \mathbb{T} \rightarrow \mathbb{R}$ and $t \in \mathbb{T}_\kappa$, then the nabla derivative of f at the point t is defined to be the number $f^\nabla(t)$ (provided it exists) with the property that for each $\epsilon > 0$, there is a neighborhood U of t such that

$$\left| f(\rho(t)) - f(s) - f^\nabla(t)(\rho(t) - s) \right| \leq \epsilon |\rho(t) - s| \quad \forall s \in U. \tag{2.3}$$

Definition 2.3 (see [8]). A function f is rd-continuous provided it is continuous at each right-dense point in $\mathbb{T}$ and has a left-sided limit at each left-dense point in $\mathbb{T}$. The set of rd-continuous functions f will be denoted by $C_{rd}(\mathbb{T})$. A function g is left-dense continuous (i.e., ld-continuous) if g is continuous at each left-dense point in $\mathbb{T}$ and its right-sided limit exists (finite) at each right-dense point in $\mathbb{T}$. The set of left-dense continuous functions g will be denoted by $C_{ld}(\mathbb{T})$.

Definition 2.4 (see [8]). If $F^\Delta(t) = f(t)$, then we define the delta integral by

$$\int_a^b f(s)\Delta s = F(b) - F(a). \tag{2.4}$$

If $G^\nabla(t) = g(t)$, then we define the nabla integral by

$$\int_a^b g(s)\nabla s = G(b) - G(a). \tag{2.5}$$

Lemma 2.5 (see [8]). *If $f \in C_{rd}(\mathbb{T})$ and $t \in \mathbb{T}^\kappa$, then*

$$\int_t^{\sigma(t)} f(s)\Delta s = \mu(t)f(t). \tag{2.6}$$

If $g \in C_{ld}(\mathbb{T})$ and $t \in \mathbb{T}_\kappa$, then

$$\int_{\rho(t)}^t g(s)\nabla s = \nu(t)g(t). \tag{2.7}$$

Let the Banach space

$$\mathbb{B} = C_{ld}^1([0,T]_\mathbb{T})$$
$$= \left\{ x : [0,T]_\mathbb{T} \longrightarrow \mathbb{R} \mid x \text{ is } \Delta\text{-differentiable on } [0,T]_\mathbb{T}, \text{ and } x^\Delta \text{is ld-continuous on} [0,T]_\mathbb{T} \right\} \tag{2.8}$$

be endowed with the norm $\|x\| = \max\{\|x\|_0, \|x^\Delta\|_0\}$, where

$$\|x\|_0 = \sup_{t \in [0,T]_\mathbb{T}} |x(t)|, \quad \left\| x^\Delta \right\|_0 = \sup_{t \in [0,T]_\mathbb{T}} \left| x^\Delta(t) \right| \tag{2.9}$$

and choose a cone $\mathbb{P} \subset \mathbb{B}$ defined by

$$
\mathbb{P} = \left\{ \begin{matrix} x \in \mathbb{B} : x(t) \geq 0, \quad x^{\Delta}(t) \leq 0, \quad x^{\Delta\nabla}(t) \leq 0 \quad \forall t \in [0,T]_{\mathbb{T}}, \\ \\ \alpha x(T) - \beta x(0) = \displaystyle\int_0^T g(s) x(s) \nabla s \end{matrix} \right\}. \tag{2.10}
$$

Lemma 2.6. *If $x \in \mathbb{P}$, then $x(t) \geq \beta/(\alpha - g_0)\|x\|_0$ for all $t \in [0,T]_{\mathbb{T}}$.*

Proof. If $x \in \mathbb{P}$, then $x^{\Delta} \leq 0$. It follows that

$$
x(T) = \min_{t \in [0,T]_{\mathbb{T}}} x(t), \qquad \|x\|_0 = x(0) = \max_{t \in [0,T]_{\mathbb{T}}} x(t). \tag{2.11}
$$

With $\alpha x(T) - \beta x(0) = \int_0^T g(s) x(s) \nabla s$ and $x^{\Delta} \leq 0$, one obtains

$$
\alpha x(T) = \beta x(0) + \int_0^T g(s) x(s) \nabla s \geq \beta x(0) + \int_0^T g(s) \nabla s\, x(T) = \beta x(0) + g_0 x(T). \tag{2.12}
$$

Therefore,

$$
x(T) \geq \frac{\beta}{\alpha - g_0} x(0) = \frac{\beta}{\alpha - g_0} \|x\|_0. \tag{2.13}
$$

From (2.11)–(2.13), we can get that

$$
x(t) \geq \min_{t \in [0,T]_{\mathbb{T}}} x(t) = x(T) \geq \frac{\beta}{\alpha - g_0} x(0) = \frac{\beta}{\alpha - g_0} \|x\|_0. \tag{2.14}
$$

So Lemma 2.6 is proved. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

Lemma 2.7. *$x \in \mathbb{B}$ is a solution of BVPs (1.1)-(1.2) if and only if $x \in \mathbb{B}$ is a solution of the following integral equation:*

$$
\begin{aligned}
x(t) = {} & \int_0^T \Theta(\beta + V(s)) \varphi_q \left(\int_0^s \lambda f\left(r, x(r), x^{\Delta}(r)\right) \nabla r \right) \Delta s \\
& + \int_t^T \varphi_q \left(\int_0^s \lambda f\left(r, x(r), x^{\Delta}(r)\right) \nabla r \right) \Delta s,
\end{aligned} \tag{2.15}
$$

where

$$
\begin{aligned}
\Theta &= \frac{1}{\alpha - \beta - \int_0^T g(s) \nabla s} = \frac{1}{\alpha - \beta - g_0}, \\
V(t) &= \int_0^t g(s) \nabla s \quad \forall t \in [0,T]_{\mathbb{T}}.
\end{aligned} \tag{2.16}
$$

Proof. First assume $x \in \mathbb{B}$ is a solution of BVPs (1.1)-(1.2); then we have

$$\varphi_p\left(x^\Delta(t)\right) = \varphi_p\left(x^\Delta(0)\right) - \int_0^t \lambda f\left(s, x(s), x^\Delta(s)\right) \nabla s = -\int_0^t \lambda f\left(s, x(s), x^\Delta(s)\right) \nabla s. \quad (2.17)$$

That is,

$$x^\Delta(t) = -\varphi_q\left(\int_0^t \lambda f\left(s, x(s), x^\Delta(s)\right) \nabla s\right) = -H(t). \quad (2.18)$$

Integrating (2.18) from t to T, it follows that

$$x(t) = x(T) + \int_t^T H(s)\Delta s. \quad (2.19)$$

Together with (2.19) and $\alpha x(T) - \beta x(0) = \int_0^T g(s)x(s)\nabla s$, we obtain

$$\alpha x(T) - \beta\left(x(T) + \int_0^T H(s)\Delta s\right) = \int_0^T g(s)\left(x(T) + \int_s^T H(r)\Delta r\right)\nabla s. \quad (2.20)$$

Thus,

$$\begin{aligned}
\left(\alpha - \beta - \int_0^T g(s)\nabla s\right)x(T) &= \beta\int_0^T H(s)\Delta s + \int_0^T g(s)\left(\int_s^T H(r)\Delta r\right)\nabla s \\
&= \beta\int_0^T H(s)\Delta s + \int_0^T \left(\int_s^T (V(s)-V(r))H(r)\Delta r\right)^\nabla \nabla s \\
&= \beta\int_0^T H(s)\Delta s - \int_0^T (V(0)-V(s))H(s)\Delta s \\
&= \beta\int_0^T H(s)\Delta s + \int_0^T V(s)H(s)\Delta s,
\end{aligned} \quad (2.21)$$

namely,

$$x(T) = \beta\Theta\int_0^T H(s)\Delta s + \Theta\int_0^T V(s)H(s)\Delta s. \quad (2.22)$$

Substituting (2.22) into (2.19), we obtain

$$
\begin{aligned}
x(t) &= \beta\Theta \int_0^T H(s)\Delta s + \Theta \int_0^T V(s)H(s)\Delta s + \int_t^T H(s)\Delta s \\
&= \int_0^T \Theta(\beta + V(s))\varphi_q\left(\int_0^s \lambda f\left(r, x(r), x^\Delta(r)\right)\nabla r\right)\Delta s \\
&\quad + \int_t^T \varphi_q\left(\int_0^s \lambda f\left(r, x(r), x^\Delta(r)\right)\nabla r\right)\Delta s.
\end{aligned}
\tag{2.23}
$$

The proof of sufficiency is complete.

Conversely, assume $x \in \mathbb{B}$ is a solution of the following integral equation:

$$
\begin{aligned}
x(t) &= \int_0^T \Theta(\beta + V(s))\varphi_q\left(\int_0^s \lambda f\left(r, x(r), x^\Delta(r)\right)\nabla r\right)\Delta s \\
&\quad + \int_t^T \varphi_q\left(\int_0^s \lambda f\left(r, x(r), x^\Delta(r)\right)\nabla r\right)\Delta s \\
&= \int_0^T \Theta(\beta + V(s))H(s)\Delta s + \int_t^T H(s)\Delta s.
\end{aligned}
\tag{2.24}
$$

It follows that

$$
\begin{gathered}
x^\Delta(t) = -\varphi_q\left(\int_0^t \lambda f\left(s, x(s), x^\Delta(s)\right)\nabla s\right) = -H(t), \\
\left(\varphi_p\left(x^\Delta(t)\right)\right)^\nabla + \lambda f\left(t, x(t), x^\Delta(t)\right) = 0.
\end{gathered}
\tag{2.25}
$$

So $x^\Delta(0) = 0$. Furthermore, we have

$$
\begin{aligned}
\alpha x(T) - \beta x(0) &= \alpha \int_0^T \Theta(\beta + V(s))H(s)\Delta s - \beta \int_0^T \Theta(\beta + V(s))H(s)\Delta s - \beta \int_0^T H(s)\Delta s \\
&= (\alpha - \beta) \int_0^T \Theta(\beta + V(s))H(s)\Delta s - \beta \int_0^T H(s)\Delta s, \\
\int_0^T g(s)x(s)\nabla s &= \int_0^T g(s)\left(\int_0^T \Theta(\beta + V(r))H(r)\Delta r + \int_s^T H(r)\Delta r\right)\nabla s \\
&= \int_0^T g(s)\nabla s \int_0^T \Theta(\beta + V(s))H(s)\Delta s + \int_0^T \int_s^T g(s)H(r)\Delta r \nabla s
\end{aligned}
$$

$$
= \int_0^T g(s)\nabla s \int_0^T \Theta(\beta + V(s))H(s)\Delta s
$$

$$
+ \int_0^T \left(\int_s^T (V(s) - V(r))H(r)\Delta r \right)^\nabla \nabla s
$$

$$
= \int_0^T g(s)\nabla s \int_0^T \Theta(\beta + V(s))H(s)\Delta s + \int_0^T V(s)H(s)\Delta s,
$$

$$(2.26)$$

which imply that

$$
\alpha x(T) - \beta x(0) - \int_0^T g(s)x(s)\nabla s = (\alpha - \beta) \int_0^T \Theta(\beta + V(s))H(s)\Delta s
$$

$$
- \beta \int_0^T H(s)\Delta s - \int_0^T g(s)\nabla s \int_0^T \Theta(\beta + V(s))H(s)\Delta s
$$

$$
- \int_0^T V(s)H(s)\Delta s
$$

$$
= 0.
$$

$$(2.27)$$

The proof of Lemma 2.7 is complete. □

Define the operator $\Psi : \mathbb{P} \to \mathbb{B}$ by

$$
(\Psi x)(t) = \int_0^T \Theta(\beta + V(s))\varphi_q \left(\int_0^s \lambda f\left(r, x(r), x^\Delta(r)\right) \nabla r \right) \Delta s
$$

$$
+ \int_t^T \varphi_q \left(\int_0^s \lambda f\left(r, x(r), x^\Delta(r)\right) \nabla r \right) \Delta s
$$

$$(2.28)$$

for all $t \in [0, T]_\mathbb{T}$. Obviously, $\Psi x(t) \geq 0$ for all $t \in [0, T]_\mathbb{T}$.

Lemma 2.8. *If $x \in \mathbb{P}$, then $\Psi x \in \mathbb{P}$.*

Proof. It is easily obtained from the second part of the proof in Lemma 2.7. The proof is complete. □

Lemma 2.9. $\Psi : \mathbb{P} \to \mathbb{P}$ *is complete continuous.*

Proof. First, we show that Ψ maps bounded set into itself. Assume c is a positive constant and $x \in \overline{\mathbb{P}}_c = \{x \in \mathbb{P} : \|x\| \leq c\}$. Note that the continuity of $f(t, x, x^\Delta)$ guarantees that there is a

$C > 0$ such that $f(t, x, x^{\Delta}) \leq \varphi_p(C)$ for all $t \in [0, T]_{\mathbb{T}}$. So we get from $\Psi^{\Delta} x \leq 0$ and $\Psi^{\Delta \nabla} x \leq 0$ that

$$
\begin{aligned}
\|\Psi x\|_0 &= \Psi x(0) \\
&= \int_0^T \Theta(\beta + V(s)) \varphi_q\left(\int_0^s \lambda f\left(r, x(r), x^{\Delta}(r)\right) \nabla r\right) \Delta s \\
&\quad + \int_0^T \varphi_q\left(\int_0^s \lambda f\left(r, x(r), x^{\Delta}(r)\right) \nabla r\right) \Delta s \\
&\leq C \lambda^{q-1} T^{q-1} \int_0^T \Theta(\beta + V(s)) \Delta s + C \lambda^{q-1} T^q,
\end{aligned} \tag{2.29}
$$

$$
\begin{aligned}
\left\|\Psi^{\Delta} x\right\|_0 &= \left|\Psi^{\Delta} x(T)\right| \\
&= \varphi_q\left(\int_0^T \lambda f\left(r, x(r), x^{\Delta}(r)\right) \nabla r\right) \\
&\leq C \lambda^{q-1} T^{q-1}.
\end{aligned} \tag{2.30}
$$

That is, $\Psi \overline{\mathbb{P}}_c$ is uniformly bounded. In addition, notice that

$$
\begin{aligned}
|(\Psi x)(t_1) - (\Psi x)(t_2)| &= \left|\int_{t_2}^{t_1} \varphi_q\left(\int_0^s \lambda f\left(r, x(r), x^{\Delta}(r)\right) \nabla r\right) \Delta s\right| \\
&\leq C \lambda^{q-1} T^{q-1} |t_1 - t_2|,
\end{aligned} \tag{2.31}
$$

which implies that

$$
|(\Psi x)(t_1) - (\Psi x)(t_2)| \longrightarrow 0 \quad \text{as } t_1 - t_2 \longrightarrow 0,
$$

$$
\begin{aligned}
\left|\left((\Psi x)^{\Delta}(t_1)\right)^{p-1} - \left((\Psi x)^{\Delta}(t_2)\right)^{p-1}\right| &= \left|\varphi_p\left((\Psi x)^{\Delta}(t_1)\right) - \varphi_p\left((\Psi x)^{\Delta}(t_2)\right)\right| \\
&= \left|\int_{t_2}^{t_1} \lambda f\left(r, x(r), x^{\Delta}(r)\right) \nabla r\right| \\
&\leq \lambda \varphi_p(C) |t_1 - t_2|,
\end{aligned} \tag{2.32}
$$

which implies that

$$
\left|\left((\Psi x)^{\Delta}(t_1)\right)^{p-1} - \left((\Psi x)^{\Delta}(t_2)\right)^{p-1}\right| \longrightarrow 0 \quad \text{as } t_1 - t_2 \longrightarrow 0. \tag{2.33}
$$

That is,

$$
\left|(\Psi x)^{\Delta}(t_1) - (\Psi x)^{\Delta}(t_2)\right| \longrightarrow 0 \quad \text{as } t_1 - t_2 \longrightarrow 0. \tag{2.34}
$$

So Ψx is equicontinuous for any $x \in \overline{\mathbb{P}}_c$. Using Arzela-Ascoli theorem on time scales [17], we obtain that $\Psi \overline{\mathbb{P}}_c$ is relatively compact. In view of Lebesgue's dominated convergence theorem on time scales [18], it is easy to prove that Ψ is continuous. Hence, Ψ is complete continuous. The proof of this lemma is complete. □

Let υ and ω be nonnegative continuous convex functionals on a pone $\mathbb{P}$, ψ a nonnegative continuous concave functional on $\mathbb{P}$, and r, a, L positive numbers with $r > a$ we defined the following convex sets:

$$
\begin{aligned}
\mathbb{P}(\upsilon, r; \omega, l) &= \{x \in \mathbb{P} : \upsilon(x) < r, \omega(x) < l\}, \\
\overline{\mathbb{P}}(\upsilon, r; \omega, l) &= \{x \in \mathbb{P} : \upsilon(x) \le r, \omega(x) \le l\}, \\
\mathbb{P}(\upsilon, r; \omega, l; \psi, a) &= \{x \in \mathbb{P} : \upsilon(x) < r, \omega(x) < l, \psi(x) > a\}, \\
\overline{\mathbb{P}}(\upsilon, r; \omega, l; \psi, a) &= \{x \in \mathbb{P} : \upsilon(x) \le r, \omega(x) \le l, \psi(x) \ge a\}
\end{aligned}
\tag{2.35}
$$

and introduce two assumptions with regard to the functionals υ, ω as follows:

(H1) there exists $M > 0$ such that $\|x\| \le M \ \max\{\upsilon(x), \omega(x)\}$ for all $x \in \mathbb{P}$;

(H2) $\mathbb{P}(\upsilon, r; \omega, l) \ne \emptyset$ for any $r > 0$ and $l > 0$.

The following fixed point theorem duo to Bai and Ge is crucial in the arguments of our main result.

Lemma 2.10 (see [19]). *Let $\mathbb{B}$ be Banach space, $\mathbb{P} \subset \mathbb{B}$ a cone, and $r_2 \ge d > b > r_1 > 0, l_2 \ge l_1 > 0$. Assume that υ and ω are nonnegative continuous convex functionals satisfying (H1) and (H2), ψ is a nonnegative continuous concave functional on $\mathbb{P}$ such that $\psi(x) \le \upsilon(x)$ for all $x \in \overline{\mathbb{P}}(\upsilon, r_2; \omega, l_2)$, and $\Psi : \overline{\mathbb{P}}(\upsilon, r_2; \omega, l_2) \to \overline{\mathbb{P}}(\upsilon, r_2; \omega, l_2)$ is a complete continuous operator. Suppose*

(C1) $\{x \in \overline{\mathbb{P}}(\upsilon, d; \omega, l_2; \psi, b)\} \ne \emptyset$, $\psi(\Psi x) > b$ for $x \in \overline{\mathbb{P}}(\upsilon, d; \omega, l_2; \psi, b)$;

(C2) $\upsilon(\Psi x) < r_1, \omega(\Psi x) < l_1$ for $x \in \overline{\mathbb{P}}(\upsilon, r_1; \omega, l_1)$;

(C3) $\psi(\Psi x) > b$ for $x \in \overline{\mathbb{P}}(\upsilon, r_2; \omega, l_2; \psi, b)$ with $\upsilon(\Psi x) > d$.

Then Ψ has at least three fixed points $x_1, x_2, x_3 \in \overline{\mathbb{P}}(\upsilon, r_2; \omega, l_2)$ with

$$
\begin{aligned}
x_1 &\in \mathbb{P}(\upsilon, r_1; \omega, l_1), \\
x_2 &\in \left\{x \in \overline{\mathbb{P}}(\upsilon, r_2; \omega, l_2; \psi, b) : \psi(x) > b\right\}, \\
x_3 &\in \overline{\mathbb{P}}(\upsilon, r_2; \omega, l_2) \setminus \left(\overline{\mathbb{P}}(\upsilon, r_2; \omega, l_2; \psi, b) \cup \overline{\mathbb{P}}(\upsilon, r_1; \omega, l_1)\right).
\end{aligned}
\tag{2.36}
$$

3. Main Result

In this section, we will give sufficient conditions for the existence of at least three positive solutions to BVPs (1.1)-(1.2).

Theorem 3.1. *Suppose that there are positive numbers $0 < \epsilon_0 < \epsilon < T$, $l_2 \geq l_1 > 0$, and $r_2 > b > r_1 > 0$ with $\epsilon_0, \epsilon \in [0, T]_{\mathbb{T}}$, $b/N \leq \min\{r_2/K, l_2/L\}$ and $\alpha b - g_0 b \leq r_2 \beta$ such that the following conditions are satisfied.*

(H3) $f(t, u, v) \leq \min\{\varphi_p(r_2/K), \varphi_p(l_2/L)\}$ *for all* $(t, u, v) \in [0, T]_{\mathbb{T}} \times [0, r_2] \times [-l_2, l_2]$, *where*

$$K = \lambda^{q-1}\left(\int_0^T \Theta(\beta + V(s))s^{q-1}\Delta s + \int_0^T s^{q-1}\Delta s\right), \quad L = \lambda^{q-1}T^{q-1}. \tag{3.1}$$

(H4) $f(t, u, v) < \min\{\varphi_p(r_1/K), \varphi_p(l_1/L)\}$ *for all* $(t, u, v) \in [0, T]_{\mathbb{T}} \times [0, r_1] \times [-l_1, l_1]$.

(H5) $f(t, u, v) > \varphi_p(b/N)$ *for all* $(t, u, v) \in [\epsilon_0, \epsilon]_{\mathbb{T}} \times [b, (\alpha b - g_0 b)/\beta] \times [-l_2, l_2]$, *where*

$$N = \lambda^{q-1}(\epsilon - \epsilon_0)^{q-1}\int_\epsilon^T \Theta(\beta + V(s))\Delta s. \tag{3.2}$$

Then BVPs (1.1)-(1.2) *have at least three positive solutions.*

Proof. By the definition of the operator Ψ and its properties, it suffices to show that the conditions of Lemma 2.10 hold with respect to the operator Ψ.

Let the nonnegative continuous convex functionals v, ω and the nonnegative continuous concave functional ψ be defined on the cone $\mathbb{P}$ by

$$v(x) = \max_{t \in [0,T]_{\mathbb{T}}} |x(t)| = x(0), \quad \omega(x) = \max_{t \in [0,T]_{\mathbb{T}}}\left|x^\Delta(t)\right| = x^\Delta(T), \quad \psi(x) = \min_{t \in [\epsilon, T]_{\mathbb{T}}} x(t) = x(T). \tag{3.3}$$

Then it is easy to see that $\|x\| = \max\{v(x), \omega(x)\}$ and (H1)-(H2) hold.

First of all, we show that $\Psi : \overline{\mathbb{P}}(v, r_2; \omega, l_2) \to \overline{\mathbb{P}}(v, r_2; \omega, l_2)$. In fact, if $x \in \overline{\mathbb{P}}(v, r_2; \omega, l_2)$, then

$$v(x) = \max_{t \in [0,T]_{\mathbb{T}}} |x(t)| \leq r_2, \quad \omega(x) = \max_{t \in [0,T]_{\mathbb{T}}}\left|x^\Delta(t)\right| \leq l_2 \tag{3.4}$$

and assumption (H3) implies that

$$f\left(t, x(t), x^\Delta(t)\right) \leq \min\left\{\varphi_p\left(\frac{r_2}{K}\right), \varphi_p\left(\frac{l_2}{L}\right)\right\} \quad \forall t \in [0, T]_{\mathbb{T}}. \tag{3.5}$$

On the other hand, for $x \in \mathbb{P}$, there is $\Psi x \in \mathbb{P}$; thus

$$
\begin{aligned}
\upsilon(\Psi x) &= \max_{t \in [0,T]_{\mathbb{T}}} |(\Psi x)(t)| \\
&= \max_{t \in [0,T]_{\mathbb{T}}} \left| \int_0^T \Theta(\beta + V(s)) \varphi_q \left(\int_0^s \lambda f\big(r, x(r), x^\Delta(r)\big) \nabla r \right) \Delta s \right. \\
&\quad \left. + \int_t^T \varphi_q \left(\int_0^s \lambda f\big(r, x(r), x^\Delta(r)\big) \nabla r \right) \Delta s \right| \\
&= \left| \int_0^T \Theta(\beta + V(s)) \varphi_q \left(\int_0^s \lambda f\big(r, x(r), x^\Delta(r)\big) \nabla r \right) \Delta s \right. \\
&\quad \left. + \int_0^T \varphi_q \left(\int_0^s \lambda f\big(r, x(r), x^\Delta(r)\big) \nabla r \right) \Delta s \right| \\
&\leq \int_0^T \Theta(\beta + V(s)) \varphi_q \left(\int_0^s \lambda \varphi_p \left(\frac{r_2}{K} \right) \nabla r \right) \Delta s \\
&\quad + \int_0^T \varphi_q \left(\int_0^s \lambda \varphi_p \left(\frac{r_2}{K} \right) \nabla r \right) \Delta s \\
&= \frac{r_2}{K} \int_0^T \Theta(\beta + V(s)) \varphi_q \left(\int_0^s \lambda \nabla r \right) \Delta s + \frac{r_2}{K} \int_0^T \varphi_q \left(\int_0^s \lambda \nabla r \right) \Delta s \\
&= \frac{r_2}{K} \lambda^{q-1} \left(\int_0^T \Theta(\beta + V(s)) s^{q-1} \Delta s + \int_0^T s^{q-1} \Delta s \right) \\
&= \frac{r_2}{K} \cdot K \\
&= r_2, \\
\omega(\Psi x) &= \max_{t \in [0,T]_{\mathbb{T}}} \left| (\Psi x)^\Delta(t) \right| \\
&= \max_{t \in [0,T]_{\mathbb{T}}} \left| -\varphi_q \left(\int_0^t \lambda f\big(r, x(r), x^\Delta(r)\big) \nabla r \right) \right| \\
&= \varphi_q \left(\int_0^T \lambda f\big(r, x(r), x^\Delta(r)\big) \nabla r \right) \\
&\leq \varphi_q \left(\int_0^T \lambda \varphi_p \left(\frac{l_2}{L} \right) \nabla r \right) \\
&= \frac{l_2}{L} \varphi_q \left(\int_0^T \lambda \nabla r \right) \\
&= l_2.
\end{aligned}
\tag{3.6}
$$

Therefore, $\Psi : \overline{\mathbb{P}}(\upsilon, r_2; \omega, l_2) \rightarrow \overline{\mathbb{P}}(\upsilon, r_2; \omega, l_2)$.

In the same way, if $x \in \overline{\mathbb{P}}(v, r_1; \omega, l_1)$, then assumption (H4) implies

$$f\left(t, x(t), x^\Delta(t)\right) < \min\left\{\varphi_p\left(\frac{r_1}{K}\right), \varphi_p\left(\frac{l_1}{L}\right)\right\} \quad \forall t \in [0, T]_\mathbb{T}. \tag{3.7}$$

As in the argument above, we can get that $\Psi : \overline{\mathbb{P}}(v, r_1; \omega, l_1) \to \overline{\mathbb{P}}(v, r_1; \omega, l_1)$. Thus, condition (C2) of Lemma 2.10 holds.

To check condition (C1) in Lemma 2.10. Let $d = (\alpha b - g_0 b)/\beta$. We choose $x(t) \equiv d > b$ for $t \in [0, T]_\mathbb{T}$. It is easy to see that

$$x(t) \equiv d \in \overline{\mathbb{P}}(v, d; \omega, l_2; \psi, b), \quad \psi(x) = d > b. \tag{3.8}$$

Consequently,

$$\left\{x \in \overline{\mathbb{P}}(v, d; \omega, l_2; \psi, b) : \psi(x) > b\right\} \neq \emptyset. \tag{3.9}$$

Hence, for $x \in \overline{\mathbb{P}}(v, d; \omega, l_2; \psi, b)$, there are

$$b \leq x(t) \leq d, \quad \left|x^\Delta(t)\right| \leq l_2 \quad \forall t \in [\epsilon, T]_\mathbb{T}. \tag{3.10}$$

In view of assumption (H5), we have

$$f\left(t, x(t), x^\Delta(t)\right) > \varphi_p\left(\frac{b}{N}\right) \quad \forall t \in [\epsilon_0, \epsilon]_\mathbb{T}. \tag{3.11}$$

It follows that

$$
\begin{aligned}
\psi(\Psi x) &= \min_{t \in [\epsilon, T]_\mathbb{T}} (\Psi x)(t) \\
&= (\Psi x)(T) \\
&= \int_0^T \Theta(\beta + V(s)) \varphi_q\left(\int_0^s \lambda f\left(r, x(r), x^\Delta(r)\right) \nabla r\right) \Delta s \\
&\geq \int_\epsilon^T \Theta(\beta + V(s)) \varphi_q\left(\int_0^s \lambda f\left(r, x(r), x^\Delta(r)\right) \nabla r\right) \Delta s \\
&\geq \int_\epsilon^T \Theta(\beta + V(s)) \varphi_q\left(\int_0^\epsilon \lambda f\left(r, x(r), x^\Delta(r)\right) \nabla r\right) \Delta s \\
&\geq \int_\epsilon^T \Theta(\beta + V(s)) \varphi_q\left(\int_{\epsilon_0}^\epsilon \lambda f\left(r, x(r), x^\Delta(r)\right) \nabla r\right) \Delta s \\
&> \int_\epsilon^T \Theta(\beta + V(s)) \varphi_q\left(\int_{\epsilon_0}^\epsilon \lambda \varphi_p\left(\frac{b}{N}\right) \nabla r\right) \Delta s
\end{aligned}
$$

$$= \lambda^{q-1} (\epsilon - \epsilon_0)^{q-1} \int_\epsilon^T \Theta(\beta + V(s)) \Delta s \frac{b}{N}$$

$$= N \cdot \frac{b}{N}$$

$$= b. \tag{3.12}$$

Therefore, $\psi(\Psi x) > b$ for $x \in \overline{\mathbb{P}}(v, d; \omega, l_2; \psi, b)$. So condition (C1) in Lemma 2.10 is satisfied.

Finally, we show that (C3) in Lemma 2.10 holds. In fact, for $x \in \overline{\mathbb{P}}(v, r_2; \omega, l_2; \psi, b)$ and $v(\Psi x) > d = (\alpha b - g_0 b)/\beta$, we have

$$\psi(\Psi x) = \min_{t \in [\epsilon, T]_{\mathbb{T}}} (\Psi x)(t) = (\Psi x)(T) \geq \frac{\beta}{\alpha - g_0} \max_{t \in [0,T]_{\mathbb{T}}} (\Psi x)(t) = \frac{\beta}{\alpha - g_0} v(\Psi x) > b. \tag{3.13}$$

Thus by Lemma 2.10 and the assumption that $f(t, 0, 0) \neq 0$ on $[0, T]_{\mathbb{T}}$, BVPs (1.1)-(1.2) have at least three positive solutions. The proof is complete. □

Theorem 3.2. *Suppose that there are positive numbers $0 < \xi < T$, $l_2 \geq l_1 > 0$, and $r_2 > b > r_1 > 0$ with $\xi \in [0, T]_{\mathbb{T}}$, $b/F \leq \min\{r_2/K, l_2/L\}$, and $\alpha b - g_0 b \leq r_2 \beta$ such that (H3)-(H4) and the following condition are satisfied.*

(H6) $f(t, u, v) > \varphi_p(b/F)$ *for all* $(t, u, v) \in [0, \xi]_{\mathbb{T}} \times [b, (\alpha b - g_0 b)/\beta] \times [-l_2, l_2]$, *where*

$$F = \lambda^{q-1} \xi^{q-1} (T - \xi). \tag{3.14}$$

Then BVPs (1.1)-(1.2) have at least three positive solutions.

Proof. Let the nonnegative continuous convex functionals v, ω be defined on the cone $\mathbb{P}$ as Theorem 3.1 and the nonnegative continuous concave functional ψ be defined on the cone $\mathbb{P}$ by

$$\psi(x) = \min_{t \in [0, \xi]_{\mathbb{T}}} x(t) = x(\xi). \tag{3.15}$$

We will show that condition (C1) in Lemma 2.10 holds. Let $d = (\alpha b - g_0 b)/\beta$. We choose $x(t) \equiv d > b$ for $t \in [0, T]_{\mathbb{T}}$. It is easy to see that

$$x(t) \equiv d \in \overline{\mathbb{P}}(v, d; \omega, l_2; \psi, b), \quad \psi(x) = d > b. \tag{3.16}$$

Consequently,

$$\left\{ x \in \overline{\mathbb{P}}(v, d; \omega, l_2; \psi, b) : \psi(x) > b \right\} \neq \emptyset. \tag{3.17}$$

Hence, for $x \in \overline{\overline{\mathbb{P}}}(v, d; \omega, l_2; \psi, b)$, there are

$$b \le x(t) \le d, \quad \left|x^{\Delta}(t)\right| \le l_2 \quad \forall t \in [0, \xi]_{\mathbb{T}}. \tag{3.18}$$

In view of assumption (H6), we have

$$f\left(t, x(t), x^{\Delta}(t)\right) > \varphi_p\left(\frac{b}{F}\right) \quad \forall t \in [0, \xi]_{\mathbb{T}}. \tag{3.19}$$

It follows that

$$
\begin{aligned}
\psi(\Psi x) &= \min_{t \in [0,\xi]_{\mathbb{T}}} (\Psi x)(t) \\
&= (\Psi x)(\xi) \\
&= \int_0^T \Theta(\beta + V(s)) \varphi_q\left(\int_0^s \lambda f\left(r, x(r), x^{\Delta}(r)\right) \nabla r\right) \Delta s \\
&\quad + \int_\xi^T \varphi_q\left(\int_0^s \lambda f\left(r, x(r), x^{\Delta}(r)\right) \nabla r\right) \Delta s \\
&\ge \int_\xi^T \varphi_q\left(\int_0^s \lambda f\left(r, x(r), x^{\Delta}(r)\right) \nabla r\right) \Delta s \\
&\ge \int_\xi^T \varphi_q\left(\int_0^\xi \lambda f\left(r, x(r), x^{\Delta}(r)\right) \nabla r\right) \Delta s \\
&> \int_\xi^T \varphi_q\left(\int_0^\xi \lambda \varphi_p\left(\frac{b}{F}\right) \nabla r\right) \Delta s \\
&= \lambda^{q-1} \xi^{q-1} (T - \xi) \frac{b}{F} \\
&= F \cdot \frac{b}{F} \\
&= b.
\end{aligned}
\tag{3.20}
$$

Therefore, $\psi(\Psi x) > b$ for $x \in \overline{\overline{\mathbb{P}}}(v, d; \omega, l_2; \psi, b)$. So condition (C1) in Lemma 2.10 is satisfied. Using a similar proof to Theorem 3.1, the other conditions in Lemma 2.10 are satisfied. By Lemma 2.10, BVPs (1.1)-(1.2) have at least three positive solutions. The proof is complete. $\square$

4. An Example

Example 4.1. Consider the following second-order Laplacian dynamic equations on time scales

$$\left(\varphi_{1.5}\left(x^{\Delta}(t)\right)\right)^{\nabla} + f\left(t, x(t), x^{\Delta}(t)\right) = 0, \quad t \in (0, 1)_{\mathbb{T}} \tag{4.1}$$

with integral boundary condition

$$x^{\Delta}(0) = 0, \quad 3x(1) - x(0) = \int_0^1 e^{s-1}x(s)\nabla s, \tag{4.2}$$

where

$$f(t, u, v) = \begin{cases} 10^{-5}(t + 5|v|) + 6|u| & \forall (t, u, v) \in ([0,1]_{\mathbb{T}} \times [0, 12] \times (-\infty, +\infty)), \\ 10^{-5}(t + 5|v|) + 72 & \forall (t, u, v) \in ([0,1]_{\mathbb{T}} \times (12, +\infty] \times (-\infty, +\infty)). \end{cases} \tag{4.3}$$

Then BVPs (4.1)-(4.2) have at least three positive solutions.

Proof. Take $\epsilon_0 = 0.25$, $\epsilon = 0.5$, $r_1 = l_1 = 0.009$, $r_2 = 30000$, $l_2 = 10000$, and $b = 4$. It follows that

$$\Theta = \frac{1}{\alpha - \beta - g_0} = \frac{1}{3 - 1 - \int_0^1 e^{s-1}\nabla s} \le \frac{1}{3 - 1 - 1} = 1,$$

$$\Theta = \frac{1}{\alpha - \beta - g_0} = \frac{1}{3 - 1 - \int_0^1 e^{s-1}\nabla s} \ge \frac{1}{3 - 1 - e^{-1}} = 0.5. \tag{4.4}$$

From (4.1)-(4.2), it is easy to obtain

$$V(t) = \int_0^t g(s)\nabla s = \int_0^t e^{s-1}\nabla s \le 1 \quad \forall t \in [0, 1]_{\mathbb{T}},$$

$$V(t) = \int_0^t g(s)\nabla s = \int_0^t e^{s-1}\nabla s \ge e^{-1} \ge 0.25 \quad \forall t \in [0, 1]_{\mathbb{T}},$$

$$K = \int_0^1 \Theta(1 + V(s))s^{3-1}\Delta s + \int_0^1 s^{3-1}\Delta s \le 3 = \overline{K}, \quad L = 1, \tag{4.5}$$

$$N = (0.5 - 0.25)^{3-1} \int_{0.5}^1 \Theta(1 + V(s))\Delta s \le 0.07 = \overline{N},$$

$$N = (0.5 - 0.25)^{3-1} \int_{0.5}^1 \Theta(1 + V(s))\Delta s > 0.01.$$

Hence, we have

$$\frac{b}{N} \le 400 < 10000 = \min\left\{\frac{r_2}{K}, \frac{l_2}{L}\right\}, \tag{4.6}$$

$$\alpha b - g_0 b - r_2\beta \le 12 - 30000 < 0.$$

Moreover, we have

(H3) for all $(t, u, v) \in [0, 1]_{\mathbb{T}} \times [0, 30000] \times [-10000, 10000]$,

$$f(t, u, v) < 80 < 100 = \min\left\{\varphi_{1.5}\left(\frac{r_2}{K}\right), \varphi_{1.5}\left(\frac{l_2}{L}\right)\right\} \leq \min\left\{\varphi_p\left(\frac{r_2}{K}\right), \varphi_p\left(\frac{l_2}{L}\right)\right\}; \qquad (4.7)$$

(H4) for all $(t, u, v) \in [0, 1]_{\mathbb{T}} \times [0, 0.009] \times [-0.009, 0.009]$,

$$f(t, u, v) \leq 0.05401045 < \min\left\{\varphi_{1.5}\left(\frac{r_1}{K}\right), \varphi_{1.5}\left(\frac{l_1}{L}\right)\right\} \leq \min\left\{\varphi_p\left(\frac{r_1}{K}\right), \varphi_p\left(\frac{l_1}{L}\right)\right\}; \qquad (4.8)$$

(H5) for all $(t, u, v) \in [0.25, 0.5]_{\mathbb{T}} \times [4, 12] \times [-10000, 10000]$,

$$f(t, u, v) \geq 6|u| \geq 24 > \varphi_{1.5}\left(\frac{b}{N}\right). \qquad (4.9)$$

Therefore, conditions (H3)–(H5) in Theorem 3.1 are satisfied. Further, it is easy to verify that the other conditions in Theorem 3.1 hold. By Theorem 3.1, BVPs (4.1)-(4.2) have at least three positive solutions. The proof is complete. □

Acknowledgment

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