# EXPLORING THE $q$-RIEMANN ZETA FUNCTION AND $q$-BERNOULLI POLYNOMIALS

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*Received 15 July 2004*

We study that the $q$-Bernoulli polynomials, which were constructed by Kim, are analytic continued to $\beta_s(z)$. A new formula for the $q$-Riemann zeta function $\zeta_q(s)$ due to Kim in terms of nested series of $\zeta_q(n)$ is derived. The new concept of dynamics of the zeros of analytic continued polynomials is introduced, and an interesting phenomenon of "scattering" of the zeros of $\beta_s(z)$ is observed. Following the idea of $q$-zeta function due to Kim, we are going to use "Mathematica" to explore a formula for $\zeta_q(n)$.

## 1. Introduction

Throughout this paper, $\mathbb{Z}$, $\mathbb{R}$, and $\mathbb{C}$ will denote the ring of integers, the field of real numbers, and the complex numbers, respectively.

When one talks of $q$-extension, $q$ is variously considered as an indeterminate, a complex number, or a $p$-adic number. In the complex number field, we will assume that $|q| < 1$ or $|q| > 1$. The $q$-symbol $[x]_q$ denotes $[x]_q = (1-q^x)/(1-q)$.

In this paper, we study that the $q$-Bernoulli polynomials due to Kim (see [2, 8]) are analytic continued to $\beta_s(z)$. By those results, we give a new formula for the $q$-Riemann zeta function due to Kim (cf. [4, 6, 8]) and investigate the new concept of dynamics of the zeros of analytic continued polynomials. Also, we observe an interesting phenomenon of "scattering" of the zeros of $\beta_s(z)$. Finally, we are going to use a software package called "Mathematica" to explore dynamics of the zeros from analytic continuation for $q$-zeta function due to Kim.

## 2. Generating $q$-Bernoulli polynomials and numbers

For $h \in \mathbb{Z}$, the $q$-Bernoulli polynomials due to Kim were defined as

$$\sum_{n=0}^{\infty} \frac{\beta_n(x,h \mid q)}{n!} t^n = -t \sum_{l=0}^{\infty} q^{l(h+1)+x} e^{[l+x]_q t} + (1-q)h \sum_{l=0}^{\infty} q^{lh} e^{[l+x]_q t}, \tag{2.1}$$

for $x, q \in \mathbb{C}$ (cf. [6, 8]).


Discrete Dynamics in Nature and Society 2005:2 (2005) 171–181
DOI: 10.1155/DDNS.2005.171

In the special case $x = 0$, $\beta_n(0,h \mid q) = \beta_n(h \mid q)$ are called $q$-Bernoulli numbers (cf. [1, 5, 7, 8]).

By (2.1), we easily see that

$$\beta_n(x,h \mid q) = \frac{1}{(1-q)^n}\sum_{j=0}^{n}\binom{n}{j}(-1)^j\frac{j+h}{[j+h]_q}q^{jx}, \quad \text{(cf. [2, 6])}, \tag{2.2}$$

where $\binom{n}{j}$ is a binomial coefficient.

In (2.1), it is easy to see that

$$q^h(q\beta(h \mid q)+1)^n - \beta_n(h \mid q) = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{if } n > 1, \end{cases} \tag{2.3}$$

with the usual convention of replacing $\beta^n(h \mid q)$ by $\beta_n(h \mid q)$.

By differentiating both sides with respect to $t$ in (2.1), we have

$$\beta_m(h \mid q) = -m\sum_{n=0}^{\infty}q^{hn}[n]_q^{m-1} - (q-1)(m+h)\sum_{n=0}^{\infty}q^{hn}[n]_q^m. \tag{2.4}$$

Expanding (2.1) as a series and matching the coefficients on both sides give

$$\begin{aligned} &\beta_0(2 \mid q) = \frac{2}{[2]_q}, \qquad \beta_1(2 \mid q) = \frac{2q+1}{[2]_q[3]_q}, \qquad \beta_2(2 \mid q) = \frac{2q^2}{[3]_q[4]_q}, \\ &\beta_3(2 \mid q) = -\frac{q^2(q-1)(2[3]_q+q)}{[3]_q[4]_q[5]_q}, \ldots, \qquad \beta_0(h \mid q) = \frac{h}{[h]_q}, \\ &\beta_1(h \mid q) = -\frac{(1+q+\cdots+q^{h-1})+q(1+q+\cdots+q^{h-2})+\cdots+q^{h-1}}{[h]_q[h+1]_q}, \ldots. \end{aligned} \tag{2.5}$$

By (2.1), the $q$-Bernoulli polynomials can be written as

$$\beta_m(x,h \mid q) = \sum_{j=0}^{m}\binom{m}{j}[x]_q^{n-j}q^{jx}\beta_j(h \mid q). \tag{2.6}$$

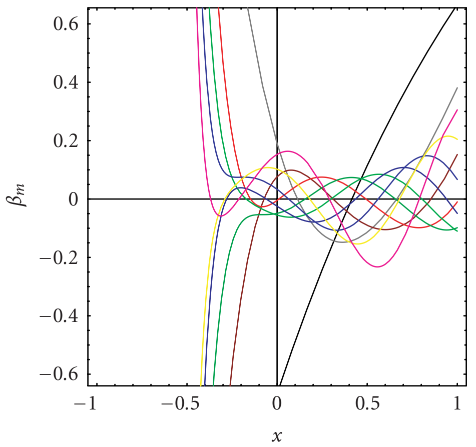


Figure 3.1. The curve of $\beta_m(x,1 \mid 1/2)$, $1 \le m \le 10$, $-1 \le x \le 1$.

In the case $h=0$, $\beta_m(x,0 \mid q)$ will be symbolically written as $\beta_{m,q}(x)$. Let $G_q(x,t)$ be the generating function of $q$-Bernoulli polynomials as follows:

$$G_q(x,t) = \sum_{n=0}^{\infty} \beta_{n,q}(x)\frac{t^n}{n!}. \tag{2.7}$$

Then we easily see that

$$G_q(x,t) = \frac{q-1}{\log q} e^{t/(1-q)} - t\sum_{n=0}^{\infty} q^{h+x} e^{[n+x]_q t}, \quad |t|<1, \ (\text{cf. } [2,3,4,6]). \tag{2.8}$$

For $x=0$, $\beta_{n,q} = \beta_{n,q}(0)$ will be called $q$-Bernoulli numbers.

By (2.8), we easily see that

$$\beta_{m,q}(n) - \beta_{m,q} = m\sum_{l=0}^{n-1} q^l [l]_q^{m-1}. \tag{2.9}$$

Thus, we have

$$\sum_{l=0}^{n-1} q^l [l]_q^{m-1} = \frac{1}{m}\sum_{l=0}^{m-1}\binom{m}{l} q^{nl}\beta_{l,q}[n]_q^{m-l} + \frac{1}{m}(1-q^{mn})\beta_{m,q}. \tag{2.10}$$

## 3. Beautiful shape of $q$-Bernoulli polynomials

In this section, we display the shapes of the $q$-Bernoulli polynomials $\beta_m(x,1|1/2)$. For $m=1,2,\ldots,10$, we can draw a plot of $\beta_m(x,1|1/2)$, respectively. This shows the ten plots combined into one. For $m=1,\ldots,10,q$, Figure 3.1 displays the shapes of the $q$-Bernoulli

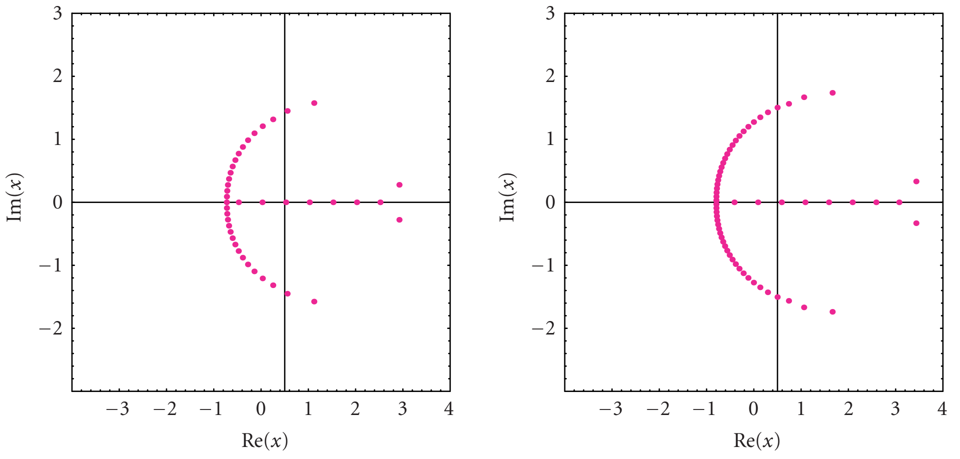


Figure 3.2. Zeros of $q$-Bernoulli polynomials $\beta_m(x,1 \mid 1/2)$, $m = 40,60$, and $x \in \mathbb{C}$.

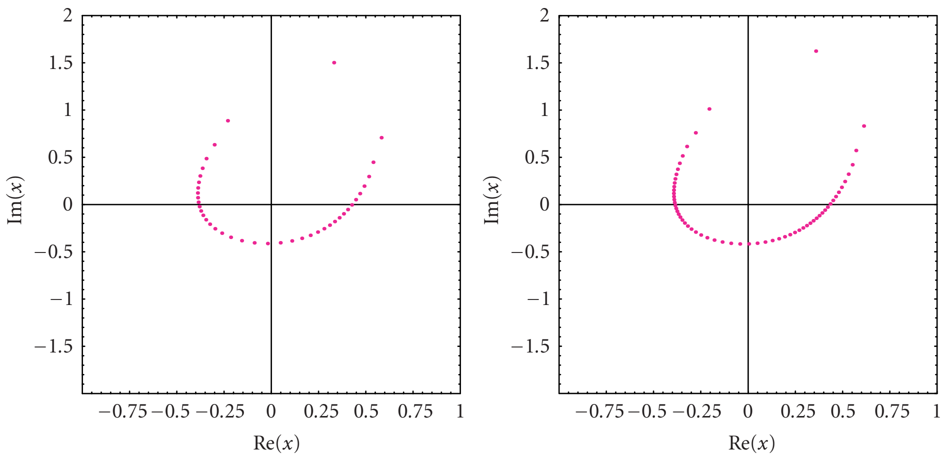


Figure 3.3. Zeros of $q$-Bernoulli polynomials $\beta_m(x,1 \mid -1/2)$, $m = 40,60$, and $x \in \mathbb{C}$.

polynomials $\beta_m(x,1|1/2)$. We plot the zeros of $\beta_m(x,1|1/2)$, $m = 40$, $m = 60$, and $x \in \mathbb{C}$ (Figure 3.2). We plot the zeros of $\beta_m(x,1|-1/2)$, $m = 40$, $m = 60$, and $x \in \mathbb{C}$ (Figure 3.3). We plot the zeros of $\beta_m(x,1|11/10)$, $m = 40$, $m = 60$, and $x \in \mathbb{C}$ (Figure 3.4). We plot the zeros of $\beta_m(x,1|-11/10)$, $m = 40$, $m = 60$, and $x \in \mathbb{C}$ (Figure 3.5). Stacks of zeros of $\beta_n(x,1|1/2)$, $1 \le n \le 60$, from a 3D structure are presented in Figure 3.6. The curve $\beta(s)$ runs through the points $\beta_{-n}(n|1/2)$ (Figure 3.7). We draw the curve of $\beta_{-n}(n|q)$ and $\lim_{n\to\infty} = n\zeta_q(n+1)$, $q = 3/10, 5/10, 7/10, 9/10, 99/100, 999/1000$ (Figures 3.8, 3.9, and 3.10).

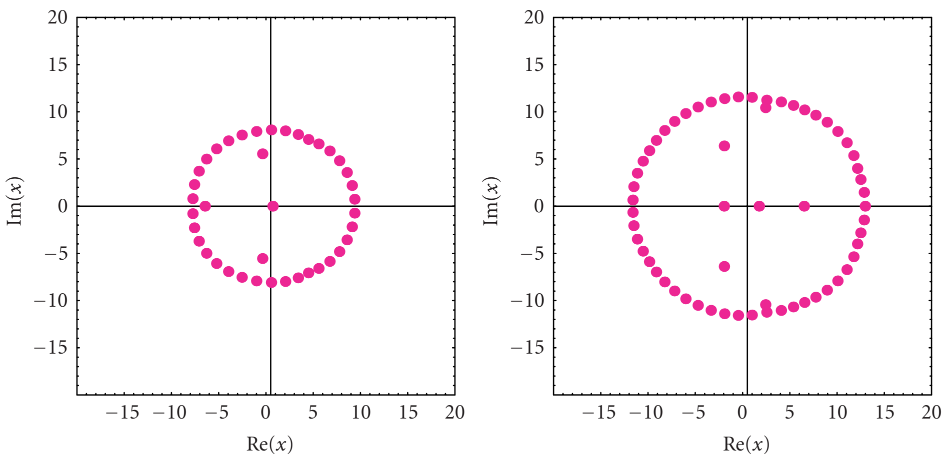


Figure 3.4. Zeros of $q$-Bernoulli polynomials $\beta_m(x,1 \mid 11/10)$, $m = 40, 60$, and $x \in \mathbb{C}$.

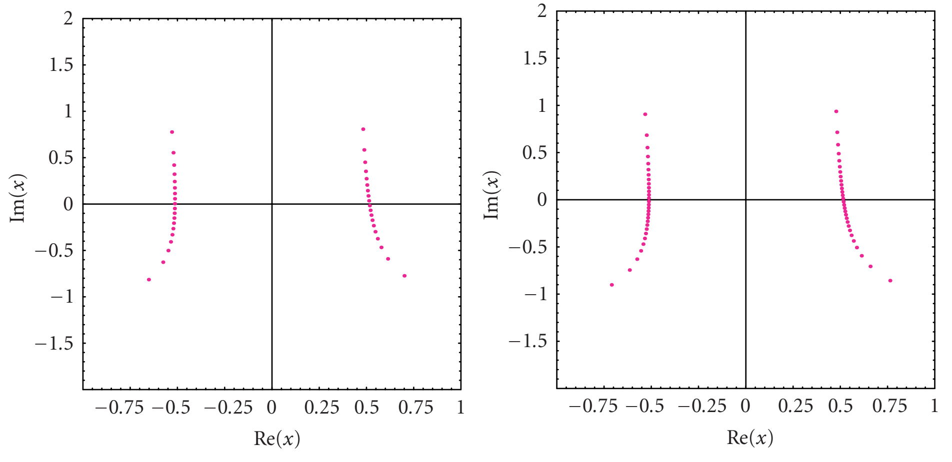


Figure 3.5. Zeros of $q$-Bernoulli polynomials $\beta_m(x,1 \mid -11/10)$, $m = 40, 60$, and $x \in \mathbb{C}$.

## 4. $q$-Riemann zeta function

We display the plot of $\beta_q(s)$, $0.1 \le s \le 0.9$, $1.1 \le q \le 2$ (Figure 4.1). We display the plot of $\beta_q(s)$, $1.03 \le s \le 2$, $0.1 \le q \le 2$ (Figure 4.2). We draw the curve of $\zeta_q(n)$, $q = 7/10$, $9/10$ (Figure 4.3). We draw the curve of $\beta_{-q}(s,w)$, $2 \le s \le 3$, $-0.5 \le w \le 0.5$, $q = 11/10$ (Figure 4.4).

The $q$-Riemann zeta function due to Kim was defined as

$$\zeta_q^{(h)}(s) = \frac{1-s+h}{1-s}(q-1)\sum_{n=1}^{\infty}\frac{q^{nh}}{[n]_q^{s-1}} + \sum_{n=1}^{\infty}\frac{q^{nh}}{[n]_q^{s}}, \quad \text{for } s,h \in \mathbb{C}, \text{ (cf. [6, 8]).} \tag{4.1}$$

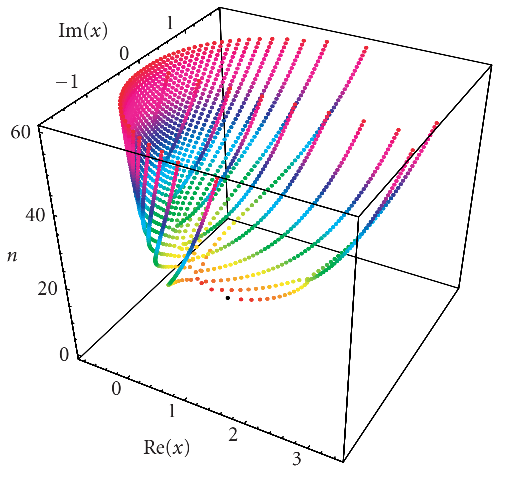


Figure 3.6. Stacks of zeros of $q$-Bernoulli polynomials $\beta_n(x,1 \mid 1/2)$, $1 \le n \le 60$, from a 3D structure.

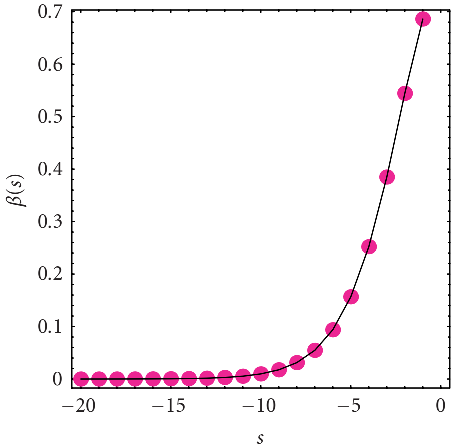


Figure 3.7. The curve $\beta(s)$ runs through the points $\beta_{-n}(n \mid 1/2)$.

For $k \in \mathbb{N}$, $h \in \mathbb{Z}$, it was known that

$$\zeta_q^{(h)}(1-k) = -\frac{\beta_k(h \mid q)}{k}, \quad \text{(cf. [6, 8])}. \tag{4.2}$$

In the special case $h = s-1$, $\zeta_q^{(s-1)}(s)$ will be written as $\zeta_q(s)$. For $s \in \mathbb{C}$, we note that

$$\zeta_q(s) = \sum_{n=1}^{\infty} \frac{q^{n(s-1)}}{[n]_q^s}, \quad \text{(cf. [6, 8])}. \tag{4.3}$$

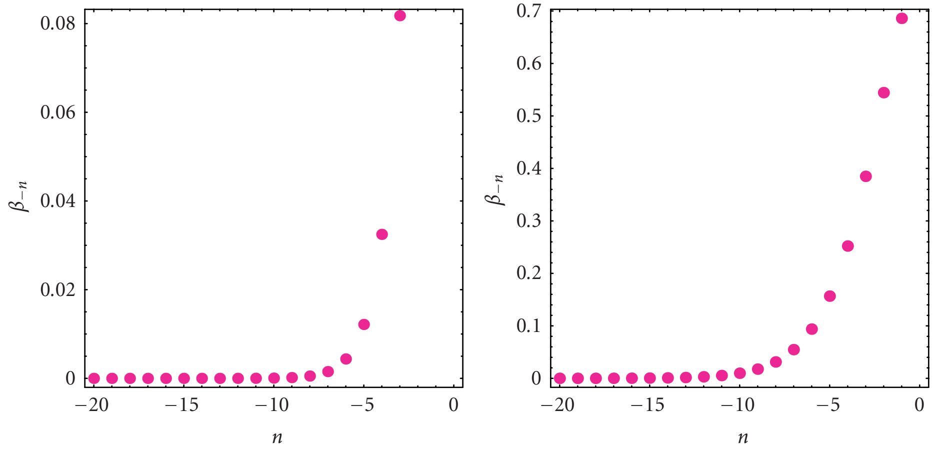


Figure 3.8. The curve of $\beta_{-n}(n \mid q)$ and $\lim_{n\to\infty} \beta_{-n} = n\zeta_q(n+1) = 0$, $q = 3/10, 5/10$
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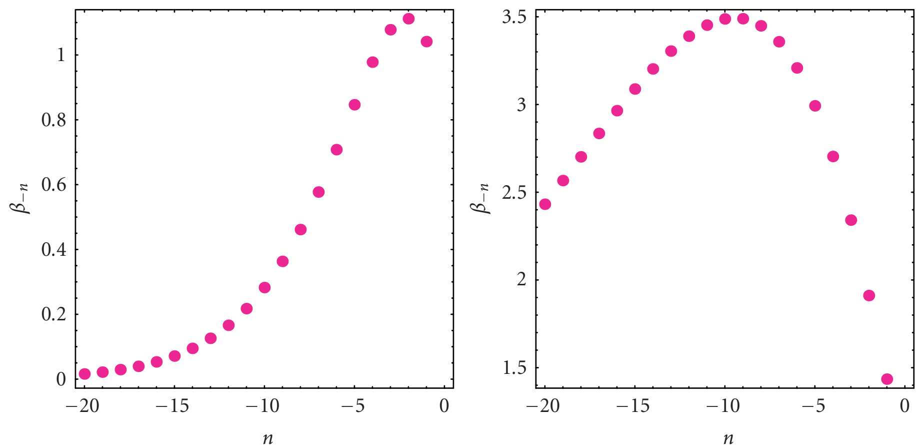


Figure 3.9. The curve of $\beta_{-n}(n \mid q)$ and $\lim_{n\to\infty} \beta_{-n} = n\zeta_q(n+1) = 0$, $q = 7/10, 9/10$
.

By (4.1), (4.2), and (4.3), we easily see that

$$\zeta_q(1-k) = -\frac{\beta_k(-k \mid q)}{k}, \quad \text{for } k \in \mathbb{N}, \text{ (cf. [3, 4, 6])}. \tag{4.4}$$

From the above analytic continuation of $q$-Bernoulli numbers, we consider

$$\begin{gathered} \beta_n = \beta_n(-n \mid q) \longmapsto \beta(s), \\ \zeta_q(-n) = -\frac{\beta_{n+1}(-n+1 \mid q)}{n+1} \longmapsto \zeta_q(-s) = -\frac{\beta(s+1)}{s+1} \Longrightarrow \zeta_q(1-s) = -\frac{\zeta(s)}{s}. \end{gathered} \tag{4.5}$$

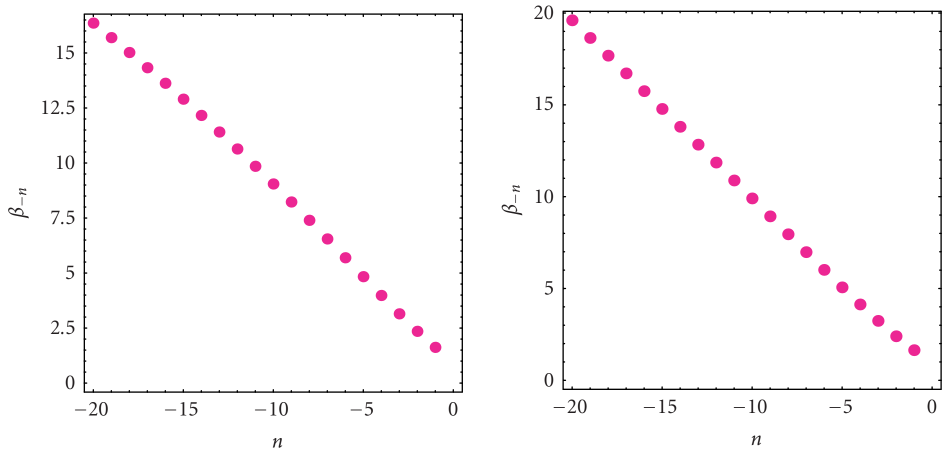


Figure 3.10. The curve of $\beta_{-n}(n \mid q)$, $q = 99/100, 999/1000$.

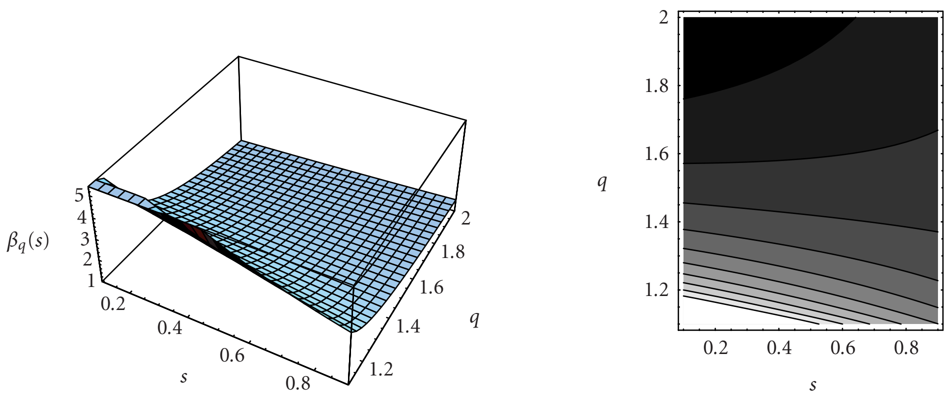


Figure 4.1. The plot of $\beta_q(s)$, $0.1 \le s \le 0.9$, $1.1 \le q \le 2$.

From relation (4.5), we can define the other analytic continued half of $q$-Bernoulli numbers,

$$\begin{gathered} \beta(s) = -s\zeta_q(1-s), \quad \beta(-s) = s\zeta_q(1+s) \\ \Longrightarrow \beta_{-n} = \beta_{-n}(n \mid q) = \beta(-n) = n\zeta_q(n+1), \quad n \in \mathbb{N}. \end{gathered} \tag{4.6}$$

The curve $\beta(s)$ runs through the points $\beta_{-n}$ and $\lim_{n\to\infty} \beta_{-n} = n\zeta_q(n+1) = 0$. However, the curve $\beta_{-n}(n \mid q)$ grows $\sim n$ asymptotically as $q \to 1$, $(-n) \to -\infty$.

$$\zeta_q(m) = \sum_{n=1}^{\infty} \frac{q^{n(m-1)}}{[n]_q^m} \Longrightarrow \lim_{m\to\infty} \zeta_q(m) = 0. \tag{4.7}$$

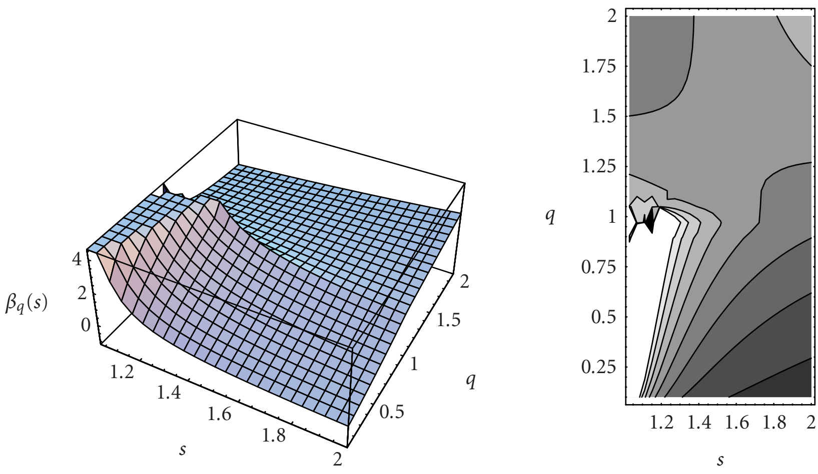


Figure 4.2. The plot of $\beta_q(s)$, $1.03 \leq s \leq 2$, $0.1 \leq q \leq 2$.

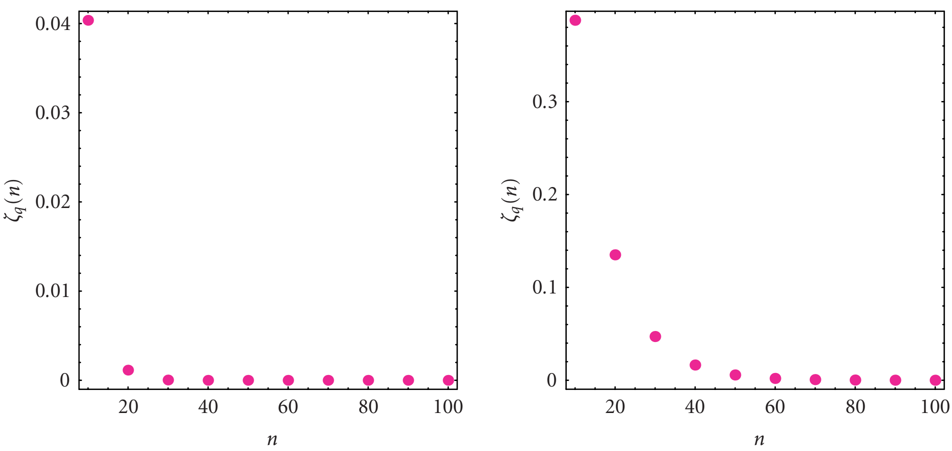


Figure 4.3. The curve of $\zeta_q(n)$, $q = 7/10, 9/10$.

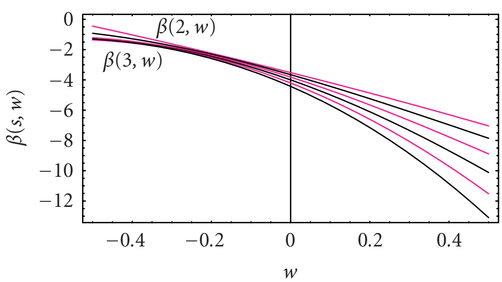


Figure 4.4. The curve of $\beta(s, w)$, $2 \leq s \leq 3$, $-0.5 \leq w \leq 0.5$, $q = 11/10$.

## 5. Analytic continuation of $q$-Bernoulli polynomials

For consistency with the redefinition of $\beta_n = \beta(n)$ in (4.5) and (4.6),

$$\beta_n(x) = \beta_n(x, -n \mid q) = \sum_{k=0}^{n} \binom{n}{k} \beta_k q^{kx} [x]_q^{n-k}. \tag{5.1}$$

The analytic continuation can be then obtained as

$$\begin{gathered}
n \longmapsto s \in \mathbb{R}, \qquad x \longmapsto w \in \mathbb{C}, \\
\beta_k \longmapsto \beta(k+s-[s] \mid q) = -(k+(s-[s]))\zeta_q(1-(k+(s-[s]))), \\
\binom{n}{k} \longmapsto \frac{\Gamma(1+s)}{\Gamma(1+k+(s-[s]))\Gamma(1+[s]-k)} \\
\Longrightarrow \beta_n(s) \longmapsto \beta(s, w \mid q) = \sum_{k=-1}^{[s]} \frac{\Gamma(1+s)\beta(k+s-[s])q^{(k+s-[s])w}[w]_q^{[s]-k}}{\Gamma(1+k+(s-[s]))\Gamma(1+[s]-k)} \\
= \sum_{k=0}^{[s]+1} \frac{\Gamma(1+s)\beta((k-1)+s-[s])q^{((k-1)+s-[s])w}[w]_q^{[s]+1-k}}{\Gamma(k+(s-[s]))\Gamma(2+[s]-k)},
\end{gathered} \tag{5.2}$$

where $[s]$ gives the integer part of $s$, and so $s-[s]$ gives the fractional part.

Deformation of the curve $\beta(2, w)$ into the curve $\beta(3, w)$ via the real analytic continuation $\beta(s, w)$, $2 \le s \le 3$, $-0.5 \le w \le 0.5$.

## Acknowledgments

The fourth author was supported by Kyungpook National University Research team Fund, 2003. The first author would like to dedicate this paper to the memory of Katsumi Shiratani.

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