STRONG LAW OF LARGE NUMBERS FOR ρ^*-MIXING SEQUENCES WITH DIFFERENT DISTRIBUTIONS

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Strong law of large numbers and complete convergence for ρ^*-mixing sequences with different distributions are investigated. The results obtained improve the relevant results by Utev and Peligrad (2003).

1. Introduction

Let nonempty sets $S,T \subset \mathcal{N}$, and define $\mathcal{F}_S = \sigma(X_k,\ k \in S)$, and the maximal correlation coefficient $\rho_n^* = \sup \operatorname{corr}(f,g)$ where the supremum is taken over all (S,T) with $\operatorname{dist}(S,T) \geq n$ and all $f \in L_2(\mathcal{F}_S)$, $g \in L_2(\mathcal{F}_T)$ and where $\operatorname{dist}(S,T) = \inf_{x\in S, y\in T} |x-y|$.

A sequence of random variables $\{X_n,\ n \geq 1\}$ on a probability space $\{\Omega, \mathcal{F}, P\}$ is called ρ^*-mixing if

$$\lim_{n\to\infty} \rho_n^* < 1. \tag{1.1}$$

As for ρ^*-mixing sequences of random variables, Bryc and Smoleński [1] established the moments inequality of partial sums. Peligrad [10] obtained a CLT and established an invariance principles. Peligrad [11] established the Rosenthal-type maximal inequality. Utev and Peligrad [16] obtained invariance principles of nonstationary sequences.

As for negatively associated (NA) random variables, Joag [6] gave the following definition.

Definition 1.1 (Joag [6]). A finite family of random variables $\{X_i,\ 1 \leq i \leq n\}$ is said to be negatively associated (NA) if for every pair of disjoint subsets T_1 and T_2 of $\{1,2,\ldots,n\}$,

$$\operatorname{Cov}\left(f_1(X_i,\ i \in T_1), f_2(X_j,\ j \in T_2)\right) \leq 0, \tag{1.2}$$

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whenever f_1 and f_2 are coordinatewise increasing and the covariance exists. An infinite family is negatively associated if every finite subfamily is negatively associated.

Recently, some authors focused on the problem of limiting behavior of partial sums of NA sequences. Su et al. [15] derived some moment inequalities of partial sums and a weak convergence for a strong stationary NA sequence. Lin [9] set up an invariance principal for NA sequences. Su and Qin [15] also studied some limiting results for NA sequences. More recently, Liang and Su [8], Liang [7] considered some complete convergence for weighted sums of NA sequences. Those results, especially some moment inequality by Huang and Xu [5], Shao [13], and Yang [17] undoubtedly propose important theory guide in further apply for the NA sequence.

The main purpose of this paper is to establish a strong law of large numbers and complete convergence for ρ^*-mixing sequences or NA sequences with different distributions that are investigated. The results obtained improve the relevant results by Utev and Peligrad [16].

2. Main results

Throughout this paper, C will represent a positive constant though its value may change from one appearance to the next, and $a_n = O(b_n)$ will mean $a_n \leq Cb_n$. And $a_n \ll b_n$ will mean $a_n = O(b_n)$.

In order to prove our results, we need the following lemma and the concept of complete convergence.

Definition 2.1 (Hsu and Robbins [4]). Let $\{X, X_n,\ n \geq 1\}$ be a sequence of random variables, if for any $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} P(|X_n - X| > \varepsilon) < \infty \tag{2.1}$$

holds, $\{X_n,\ n \geq 1\}$ is called completely converging to X.

As for complete convergence, let now $\{X, X_n,\ n \geq 1\}$ be a sequence of independent identically distributed random variables and denote $S_n = \sum_{i=1}^{n} X_i$. The Hsu-Robbins-Erdös law of large numbers (Hsu and Robbins [4], Erdös [3]) states that

$$\forall\, \varepsilon > 0, \quad \sum_{n=1}^{\infty} P(|S_n| > \varepsilon n) < \infty \tag{2.2}$$

is equivalent to $EX = 0$ and $EX^2 < \infty$.

This is a fundamental theorem in probability theory and has been intensively investigated by many authors in the past decades as we can see by Petrov [12], Chow and Teicher [2], and Stout [14]. There have been many extensions in various directions of the Hsu-Robbins-Erdös law of large numbers.

Lemma 2.2 (Utev and Peligrad [16]). *Let $\{X_i,\ i \geq 1\}$ be a ρ^*-mixing sequence of random variables, $EX_i = 0$, $E|X_i|^p < \infty$ for some $p \geq 2$ and for every $i \geq 1$. Then there exists $C = C(p)$,* *such that*

$$E\max_{1\le k\le n}\left|\sum_{i=1}^{k}X_i\right|^p\le C\left\{\sum_{i=1}^{n}E|X_i|^p+\left(\sum_{i=1}^{n}EX_i^2\right)^{p/2}\right\}. \tag{2.3}$$

Lemma 2.3 (Shao [13]). *Let $\{X_i,\ i\ge 1\}$ be a sequence of NA random variables, $EX_i=0$, $E|X_i|^p<\infty$ for some $p\ge 2$ and for every $i\ge 1$. Then there exists $C=C(p)$, such that*

$$E\max_{1\le k\le n}\left|\sum_{i=1}^{k}X_i\right|^p\le C\left\{\sum_{i=1}^{n}E|X_i|^p+\left(\sum_{i=1}^{n}EX_i^2\right)^{p/2}\right\}. \tag{2.4}$$

Now we state the main result of this paper.

Theorem 2.4. *Let $\{X,X_i,\ i\ge 1\}$ be ρ^*-mixing sequence with $E|X|^p<\infty$, $0<p<2$. Let $S_n=\sum_{i=1}^{n}X_i$, $P(|X_i|>x)\ll P(|X|>x)$, for all $x>0$, $i\ge 1$. When $1\le p<2$, let $EX=0$. Then,*

$$\forall\,\varepsilon>0,\quad \sum_{n=1}^{\infty}n^{-1}P\left(\max_{1\le j\le n}|S_j|>\varepsilon n^{1/p}\right)<\infty. \tag{2.5}$$

Proof of Theorem 2.4. For any $i\ge 1$, let $X_i^{(n)}=X_iI(|X_i|\le n^{1/p})$, $S_j^{(n)}=\sum_{i=1}^{j}(X_i^{(n)}-EX_i^{(n)})$. for all $\varepsilon>0$, then

$$\begin{aligned}
P&\left(\max_{1\le j\le n}|S_j|>\varepsilon n^{1/p}\right)\\
&\le P\left(\max_{1\le j\le n}|X_j|>n^{1/p}\right)+P\left(\max_{1\le j\le n}\left|S_j^{(n)}+\sum_{i=1}^{j}EX_i^{(n)}\right|>\varepsilon n^{1/p}\right)\\
&\le P\left(\max_{1\le j\le n}|X_j|>n^{1/p}\right)+P\left(\max_{1\le j\le n}|S_j^{(n)}|>\varepsilon n^{1/p}-\max_{1\le j\le n}\left|\sum_{i=1}^{j}EX_i^{(n)}\right|\right).
\end{aligned} \tag{2.6}$$

When n large enough, first we show that

$$n^{-1/p}\max_{1\le j\le n}\left|\sum_{i=1}^{j}EX_i^{(n)}\right|\longrightarrow 0. \tag{2.7}$$

In fact

(i) if $p<1$, then

$$\begin{aligned}
n^{-1/p}\max_{1\le j\le n}\left|\sum_{i=1}^{j}EX_i^{(n)}\right|&\le n^{-1/p}\sum_{i=1}^{n}E|X_i|I(|X_i|\le n^{1/p})\\
&\le n^{1-1/p}E|X|I(|X|\le n^{1/p})\\
&=n^{1-1/p}\sum_{k=1}^{n}E|X|I(k-1<|X|^p\le k),
\end{aligned} \tag{2.8}$$

because

$$\begin{aligned}\sum_{k=1}^{\infty} k^{1-1/p} E|X| I(k-1<|X|^p \le k) &\le \sum_{k=1}^{\infty} E|X|^p I(k-1<|X|^p \le k) \\ &\le \sum_{k=1}^{\infty} E|X|^p I(k-1<|X|^p \le k) = E|X|^p < \infty.\end{aligned} \tag{2.9}$$

By Kronecker lemma, we get $n^{1-1/p}\sum_{k=1}^{n} E|X|I(k-1<|X|^p \le k) \to 0$, $n \to \infty$, so

$$n^{-1/p} \max_{1\le j\le n} \left| \sum_{i=1}^{j} EX_i^{(n)} \right| \longrightarrow 0, \quad n \longrightarrow \infty; \tag{2.10}$$

(ii) if $1 \le p < 2$, by $EX = 0$, then

$$\begin{aligned} n^{-1/p} \max_{1\le j\le n} \left| \sum_{i=1}^{j} EX_i^{(n)} \right| &\le n^{-1/p} \sum_{i=1}^{n} E|X_i| I(|X_i| > n^{1/p}) \\ &\le E|X|^p I(|X| > n^{1/p}) \longrightarrow 0. \end{aligned} \tag{2.11}$$

Equations (2.10) and (2.11) imply (2.7).

From (2.6) and (2.7) it follows that for n large enough, we have $P(\max_{1\le j\le n}|S_j| > \varepsilon n^{1/p}) \le \sum_{j=1}^{n} P(|X_j| > n^{1/p}) + P(\max_{1\le j\le n}|S_j^{(n)}| > \varepsilon/2n^{1/p})$.

Hence we need only to prove that

$$\begin{aligned} I &=: \sum_{n=1}^{\infty} n^{-1} \sum_{j=1}^{n} P(|X_j| > n^{1/p}) < \infty, \\ II &=: \sum_{n=1}^{\infty} n^{-1} P\left(\max_{1\le j\le n} |S_j^{(n)}| > \frac{\varepsilon}{2} n^{1/p} \right) < \infty. \end{aligned} \tag{2.12}$$

By $E|X|^p < \infty$, then

$$I \le C \sum_{n=1}^{\infty} P(|X| > n^{1/p}) \ll E|X|^p < \infty. \tag{2.13}$$

By Lemma 2.2, it follows that

$$\begin{aligned} II &\ll \sum_{n=1}^{\infty} n^{-1-\alpha q} E \max_{1\le j\le n} |S_j^{(n)}|^q \\ &\ll \sum_{n=1}^{\infty} n^{-1-\alpha q} \left\{ \sum_{j=1}^{n} E|X_j^{(n)}|^q + \left(\sum_{j=1}^{n} E|X_j^{(n)}|^2 \right)^{q/2} \right\} \\ &=: II_1 + II_2. \end{aligned} \tag{2.14}$$

Let $q = 2$, we have

$$\begin{aligned}
II_1 &\le C\sum_{n=1}^{\infty} n^{-\alpha q}E|X|^{q}I(|X| \le n^{1/p}) \\
&= \sum_{n=1}^{\infty} n^{-\alpha q}\sum_{k=1}^{n} E|X|^{q}I(k-1 < |X|^{p} \le k) \\
&= \sum_{k=1}^{\infty}\sum_{n=k}^{\infty} n^{-\alpha q}E|X|^{q}I(k-1 < |X|^{p} \le k) \\
&\ll \sum_{k=1}^{\infty} kP(k-1 < |X|^{p} \le k) \ll E|X|^{p} < \infty.
\end{aligned} \tag{2.15}$$

Let $q = 2$, then $II_2 = II_1 < \infty$. So $II < \infty$. Now we complete the proof of Theorem 2.4. □

Corollary 2.5. *Under the conditions of Theorem 2.4,*

$$\lim_{n\to\infty} \frac{S_n}{n^{1/p}} = 0 \quad a.s. \tag{2.16}$$

Proof of Corollary 2.5. For all $\varepsilon > 0$, by (2.5), we have

$$\sum_{n=1}^{\infty} n^{-1}P\Big(\max_{1\le j\le n} |S_j| > \varepsilon n^{1/p}\Big) < \infty. \tag{2.17}$$

Then we have

$$\sum_{k=0}^{\infty}\sum_{n=2^k}^{2^{k+1}-1} (2^{k+1}-1)^{-1}P\Big(\max_{1\le j\le 2^k} |S_j| > \varepsilon n^{1/p}\Big) < \infty. \tag{2.18}$$

So

$$\sum_{k=0}^{\infty} P\Big(\max_{1\le j\le 2^k} |S_j| > \varepsilon 2^{(k+1)/p}\Big) < \infty. \tag{2.19}$$

By the Borel-Cantelli lemma, we have

$$\max_{1\le j\le 2^k} \frac{|S_j|}{2^{k/p}} \longrightarrow 0 \quad \text{a.s.} \tag{2.20}$$

For all positive integers n, there existes a nonnegative integer k_0, such that $2^{k_0} \le n < 2^{k_0+1}$.

Thus

$$\frac{|S_n|}{n^{1/p}} \le \max_{1\le j\le 2^{k_0+1}} \frac{|S_j|}{2^{k_0/p}} \longrightarrow 0 \quad \text{a.s.} \tag{2.21}$$

Thus we have

$$\lim_{n\to\infty}\frac{S_n}{n^{1/p}}=0 \quad \text{a.s.} \tag{2.22}$$

Now we complete the proof of Corollary 2.5. □

THEOREM 2.6. *Let $\{X,X_i,\ i\geq 1\}$ be NA sequence with $E|X|^p<\infty$, $0<p<2$. Let $S_n=\sum_{i=1}^n X_i$, $P(|X_i|>x)\ll P(|X|>x)$, for all $x>0$, $i\geq 1$. When $1\leq p<2$, let $EX=0$. Then,*

$$\forall\,\varepsilon>0,\quad \sum_{n=1}^{\infty}n^{-1}P\Big(\max_{1\leq j\leq n}\left|S_j\right|>\varepsilon n^{1/p}\Big)<\infty. \tag{2.23}$$

Proof of Thereom 2.6. Using Lemma 2.3 instead of Lemma 2.2, the proof of Theorem 2.6 is similar to the proof of Theorem 2.4. □

COROLLARY 2.7. *Under the conditions of Theorem 2.6,*

$$\lim_{n\to\infty}\frac{S_n}{n^{1/p}}=0 \quad \textit{a.s.} \tag{2.24}$$

Proof of Corollary 2.7. The proof of Corollary 2.7 is similar to the proof of Corollary 2.5. □

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