

DYNAMICS OF A NONAUTONOMOUS SEMIRATIO-DEPENDENT PREDATOR-PREY SYSTEM WITH NONMONOTONIC FUNCTIONAL RESPONSES

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A nonautonomous semiratio-dependent predator-prey system with nonmonotonic functional responses is investigated. For general nonautonomous case, positive invariance, permanence, and globally asymptotic stability for the system are studied. For the periodic (almost periodic) case, sufficient conditions for existence, uniqueness, and stability of a positive periodic (almost periodic) solution are obtained.

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1. Introduction

The Lotka-Volterra system is a rudimentary model on mathematical ecology and has been studied extensively in [6, 8–16, 19, 24–26] and the references cited therein. But in the Lotka-Volterra model, the fact that there are upper limits to the rates of increase of both prey and predator is not recognized. In [17, 18], Leslie introduced a predator-prey model where the “carrying capacity” of the predator’s environment is proportional to the number of prey. Leslie stresses the above fact. In the case of continuous time, these considerations lead to the following autonomous model:

$$\begin{aligned}\frac{dH}{dt} &= (r_1 - b_1H)H - a_1PH, \\ \frac{dP}{dt} &= \left(r_2 - a_2\frac{P}{H}\right)P,\end{aligned}\tag{1.1}$$

which is known as the Leslie-Gower predator-prey model [20]. If we assume that the predator consumes the prey according to the functional response $f(H)$, then the system (1.1) formulates as the following:

$$\frac{dH}{dt} = (r_1 - b_1H)H - f(H)P, \quad \frac{dP}{dt} = \left(r_2 - a_2\frac{P}{H}\right)P,\tag{1.2}$$

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which is the so-called semiratio-dependent predator-prey system with the functional responses. Recently, Wang et al. [23] explored the dynamics of a class of the nonautonomous semiratio-dependent predator-prey systems with the functional responses

$$\begin{aligned}\frac{dH}{dt} &= (r_1(t) - b_1(t)H)H - f(t,H)P, \\ \frac{dP}{dt} &= \left(r_2(t) - a_2(t)\frac{P}{H} \right)P,\end{aligned}\tag{1.3}$$

where H and P stand for the population of the prey and the predator, respectively, $f(t,H)$ is the so-called predator functional response to prey, which describes the uptake of substrate by the microorganisms in microbial dynamics or chemical kinetics, and $\partial f(t,H)/\partial H > 0$ for $H > 0$, which implies that $f(t,H)$ is monotonic function with respect to H . However, there are experiments that indicate that nonmonotonic responses occur at the microbial level: when the nutrient concentration reaches a high level, an inhibitory effect on the specific growth rate may occur. This is often seen when microorganisms are used for waste decomposition or for water purification, see Bush and Cook [3]. The so-called Monod-Haldane function

$$f(H) = \frac{cH}{m^2 + bH + H^2}\tag{1.4}$$

has been proposed and used to model the inhibitory effect at high concentrations, see Andrews [1]. Collings [4] also used this response function to model mite predator-prey interactions and called it a Holling IV function. In experiments on the uptake of phenol by pure culture of *Pseudomonas putida* growing on phenol in continuous culture, Sokol and Howell [22] proposed a simplified Monod-Haldane function of the form

$$f(H) = \frac{cH}{m^2 + H^2}\tag{1.5}$$

and found that it fits their experimental data significantly better than the Monod-Haldane function and is simpler since it involves only two parameters. We would like to call this function a *simplified Monod-Haldane or Holling IV* response function.

So it is very interesting to study dynamics of a class of the so-called semiratio-dependent predator-prey systems with the nonmonotonic functional responses

$$\begin{aligned}\frac{dH(t)}{dt} &= \left(r_1(t) - \frac{a_1(t)}{m^2 + H^2(t)}P(t) - b_1(t)H(t) \right)H(t), \\ \frac{dP(t)}{dt} &= \left(r_2(t) - a_2(t)\frac{P(t)}{H(t)} \right)P(t).\end{aligned}\tag{1.6}$$

For the ecological sense of the system (1.6) we refer to [5, 21, 23, 27] and the references cited therein.

The plan of this paper is as follows: in Section 2, for general case, we will explore some basic problems for (1.6), such as positive invariance, permanence, and globally asymptotic stability for the system (1.6). In Section 3, for periodic case, sufficient conditions for

existence, uniqueness, and stability of a positive periodic solution are obtained. Section 4 is devoted to the case when all parameters are almost periodic, sufficient conditions for existence, uniqueness and stability of an almost periodic solution are also obtained. The methods used here will be comparison theorems, coincidence degree theory, and Liapunov function.

2. General nonautonomous case

In this section, we will address such properties as boundedness of solutions, permanence, and globally asymptotic stability of system (1.6). In the following discussion, we always assume that $r_i(t)$, $i = 1, 2$, $a_i(t)$, $i = 1, 2$, and $b_1(t)$ are all continuous and bounded above and below by positive constants.

Let $\mathbb{R}_+^2 = \{(H, P) \in \mathbb{R}^2 \mid H \geq 0, P \geq 0\}$. For a bounded continuous function $g(t)$ on \mathbb{R} , we use the following notations:

$$g^u = \sup_{t \in \mathbb{R}} g(t), \quad g^l = \inf_{t \in \mathbb{R}} g(t). \quad (2.1)$$

LEMMA 2.1. *Both the nonnegative and positive cones of \mathbb{R}^2 are positively invariant for system (1.6).*

Proof. Note that system (1.6) is equivalent to

$$\begin{aligned} H(t) &= H(t_0) \exp \left\{ \int_{t_0}^t \left(r_1(s) - \frac{a_1(s)}{m^2 + H^2(s)} P(s) - b_1(s) H(s) \right) ds \right\}, \\ P(t) &= P(t_0) \exp \left\{ \int_{t_0}^t \left(r_2(s) - a_2(s) \frac{P(s)}{H(s)} \right) ds \right\}. \end{aligned} \quad (2.2)$$

The assertion of the lemma follows immediately for all $t \geq t_0$. The proof is complete. \square

In the remainder of this paper, for biological reasons, we only consider solutions $(H(t), P(t))$ with $H(t_0) > 0$ and $P(t_0) > 0$.

Definition 2.2. The solution of system (1.6) is said to be ultimately bounded if there exists $B > 0$ such that for every solution $(H(t), P(t))$ of system (1.6), there exists $T > 0$ such that $\|(H(t), P(t))\| \leq B$, for all $t \geq t_0 + T$, where B is independent of particular solution while T may depend on the solution.

Definition 2.3. System (1.6) is said to be permanent if there exist positive constants δ, Δ with $0 < \delta < \Delta$ such that

$$\begin{aligned} \min \left\{ \liminf_{t \rightarrow +\infty} H(t), \liminf_{t \rightarrow +\infty} P(t) \right\} &\geq \delta, \\ \max \left\{ \limsup_{t \rightarrow +\infty} H(t), \limsup_{t \rightarrow +\infty} P(t) \right\} &\leq \Delta, \end{aligned} \quad (2.3)$$

for all solutions of system (1.6) with positive initial values.

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THEOREM 2.4. *If*

$$r_1^l - \frac{a_1^u}{m^2} M_2^\epsilon > 0, \quad (2.4)$$

then the set Γ_ϵ defined by

$$\Gamma_\epsilon = \{(H(t), P(t)) \in \mathbb{R}^2 \mid m_1^\epsilon \leq H \leq M_1^\epsilon, m_2^\epsilon \leq P \leq M_2^\epsilon\} \quad (2.5)$$

is positively invariant with respect to system (1.6), where

$$\begin{aligned} M_1^\epsilon &= \frac{r_1^u}{b_1^l} + \epsilon, & M_2^\epsilon &= \frac{r_2^u M_1^\epsilon}{a_2^l}, \\ m_1^\epsilon &= \frac{r_1^l - (a_1^u/m^2) M_2^\epsilon}{b_1^u} - \epsilon, & m_2^\epsilon &= \frac{r_2^l m_1^\epsilon}{a_2^u}, \end{aligned} \quad (2.6)$$

and $\epsilon \geq 0$ is sufficiently small so that $m_1^\epsilon > 0$.

Proof. Let $(H(t), P(t))$ be the solution of system (1.6) through $(H(t_0), P(t_0))$ with

$$m_1^\epsilon \leq H(t_0) \leq M_1^\epsilon, \quad m_2^\epsilon \leq P(t_0) \leq M_2^\epsilon. \quad (2.7)$$

From the first equation in (1.6) and the positivity of the solutions of (1.6), it follows that

$$\begin{aligned} H'(t) &\leq H(t)(r_1(t) - b_1(t)H(t)) \leq H(t)(r_1^u - b_1^l H(t)) \\ &\leq b_1^l H(t) \left(\frac{r_1^u}{b_1^l} + \epsilon - H(t) \right) = b_1^l H(t)(M_1^\epsilon - H(t)), \quad t \geq t_0. \end{aligned} \quad (2.8)$$

A standard comparison argument shows that

$$0 < H(t_0) \leq M_1^\epsilon \implies H(t) \leq M_1^\epsilon, \quad t \geq t_0, \quad (2.9)$$

which together with the second equation in (1.6) produces

$$\begin{aligned} P'(t) &\leq P(t) \left(r_2^u - a_2^l \frac{P(t)}{M_1^\epsilon} \right) = \frac{a_2^l}{M_1^\epsilon} P(t) \left(\frac{r_2^u M_1^\epsilon}{a_2^l} - P(t) \right) \\ &= \frac{a_2^l}{M_1^\epsilon} P(t) (M_2^\epsilon - P(t)), \quad t \geq t_0, \end{aligned} \quad (2.10)$$

and hence

$$0 < P(t_0) \leq M_2^\epsilon \implies P(t) \leq M_2^\epsilon, \quad t \geq t_0. \quad (2.11)$$

Similarly, the first equation of system (1.6) also yields

$$\begin{aligned} H'(t) &\geq H(t)(r_1^l - a_1^u P(t) - b_1^u H(t)) \geq H(t) \left(r_1^l - \frac{a_1^u}{m_2^\epsilon} M_2^\epsilon - H(t) \right) \\ &\geq b_1^u H(t) \left(\frac{r_1^l - (a_1^u/m_2^\epsilon) M_2^\epsilon}{b_1^u} - \epsilon - H(t) \right) = b_1^u H(t) (m_1^\epsilon - H(t)), \quad t \geq t_0, \end{aligned} \quad (2.12)$$

and therefore,

$$H(t_0) \geq m_1^\epsilon \implies H(t) \geq m_1^\epsilon, \quad t \geq t_0. \quad (2.13)$$

Moreover, by the second equation of system (1.6), we have

$$\begin{aligned} P'(t) &\geq P(t) \left(r_2^l - a_2^u \frac{P(t)}{m_1^\epsilon} \right) = \frac{a_2^u}{m_1^\epsilon} P(t) \left(\frac{r_2^l m_1^\epsilon}{a_2^u} - P(t) \right) \\ &= \frac{a_2^u}{m_1^\epsilon} P(t) (m_2^\epsilon - P(t)), \quad t \geq t_0, \end{aligned} \quad (2.14)$$

which implies

$$P(t_0) \geq m_2^\epsilon \implies P(t) \geq m_2^\epsilon, \quad t \geq t_0. \quad (2.15)$$

Thus, Γ_ϵ is positive invariant for system (1.6), and the proof is complete. \square

LEMMA 2.5. *Let $(H(t), P(t))$ be a solution of system (1.6) with $H(t_0) > 0$ and $P(t_0) > 0$. Then,*

$$\limsup_{t \rightarrow \infty} H(t) \leq M_1^0. \quad (2.16)$$

Moreover, assume that (2.4) holds, then

$$\liminf_{t \rightarrow \infty} H(t) \geq m_1^0. \quad (2.17)$$

Proof. Noting that (2.8) and (2.12) are valid, the conclusion follows from a standard comparison arguments directly.

For the predator population, we can also have some estimates.

LEMMA 2.6. *Assume that (2.4) holds, then*

$$\liminf_{t \rightarrow \infty} P(t) \geq m_2^0, \quad \limsup_{t \rightarrow \infty} P(t) \leq M_2^0. \quad (2.18)$$

Proof. Since $\limsup_{t \rightarrow \infty} H(t) \leq M_1^0$, for any sufficient small $\epsilon > 0$, there is some $t_1 > t_0$ such that for $t \geq t_1$,

$$H(t) \leq M_1^0 + \epsilon. \quad (2.19)$$

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Then, from the second equation of system (1.6), it follows that

$$\begin{aligned} P'(t) &\leq P(t) \left(r_2^u - a_2^l \frac{P(t)}{M_1^\epsilon} \right) = \frac{a_2^l}{M_1^\epsilon} P(t) \left(\frac{r_2^u M_1^\epsilon}{a_2^l} - P(t) \right) \\ &= \frac{a_2^l}{M_1^\epsilon} P(t) (M_2^\epsilon - P(t)), \quad t \geq t_1. \end{aligned} \tag{2.20}$$

Hence, by using the comparison theorem of ordinary differential equations and the arbitrariness of ϵ , we have

$$\limsup_{t \rightarrow \infty} P(t) \leq M_2^0. \tag{2.21}$$

By a similar argument, we can easily show that

$$\liminf_{t \rightarrow \infty} P(t) \geq m_2^0. \tag{2.22}$$

The proof is complete. □

Lemmas 2.5 and 2.6 immediately lead to the following.

THEOREM 2.7. *Assume that (2.4) holds, then system (1.6) is permanent.*

From the proofs of Lemmas 2.5 and 2.6, we can easily obtain the following ultimate boundedness of Γ_ϵ with $\epsilon > 0$ sufficiently small.

THEOREM 2.8. *Assume that (2.4) holds, then the set Γ_ϵ with $\epsilon > 0$ defined by (2.5) is an ultimately bounded region of system (1.6).*

Definition 2.9. A bounded nonnegative solution $(H^*(t), P^*(t))$ of (1.6) is said to be globally asymptotically stable (or globally attractive) if for any other solution $(H(t), P(t))$ of (1.6) with positive initial values the following holds:

$$\lim_{t \rightarrow +\infty} (|H(t) - H^*(t)| + |P(t) - P^*(t)|) = 0. \tag{2.23}$$

Remark 2.10. In general, if the above property holds for any two solutions with positive initial values, then we say system (1.6) is globally asymptotically stable. One can easily show that if system (1.6) has a bounded positive solution which is globally asymptotically stable, then system (1.6) is globally asymptotically stable, and vice versa.

The following lemma is from Barbalat [2], and will be employed in establishing the globally asymptotic stability of system (1.6).

LEMMA 2.11. *let h be a real number and let f be a nonnegative function defined on $[h, +\infty)$ such that f is integrable on $[h, +\infty)$ and is uniformly continuous on $[h, +\infty)$, then $\lim_{t \rightarrow +\infty} f(t) = 0$.*

THEOREM 2.12. Let $(H^*(t), P^*(t))$ be a bounded positive solution of system (1.6). Assume that (2.4) holds, and

$$\begin{aligned} b_1^l(t) + \frac{2m_2^\epsilon m_1^\epsilon}{(m^2 + (M_1^\epsilon)^2)^2} - \frac{M_2^\epsilon}{(m_1^\epsilon)^2} &> 0, \\ \frac{1}{M_1^\epsilon} - a_1^u(t) - \frac{1}{(m^2 + (m_1^\epsilon)^2)} &> 0, \end{aligned} \quad (2.24)$$

where $M_i^\epsilon, m_i^\epsilon, i = 1, 2$, are defined in (2.6). Then $(H^*(t), P^*(t))$ is globally asymptotically stable.

Proof. Let $(H(t), P(t))$ be any solution of system (1.6) with a positive initial value. Since Γ_ϵ is an ultimately bounded region of system (1.6), there exists a $T_1 > 0$, such that $(H(t), P(t)) \in \Gamma_\epsilon$ and $(H^*(t), P^*(t)) \in \Gamma_\epsilon$ for all $t \geq t_0 + T_1$.

Consider a Lyapunov function defined by

$$V(t) = |\ln \{H(t)\} - \ln \{H^*(t)\}| + |\ln \{P(t)\} - \ln \{P^*(t)\}|, \quad t \geq t_0. \quad (2.25)$$

Calculating the right derivative of $V(t)$ along the solution of system (1.6), we derive for $t \geq t_0 + T_1$ that

$$\begin{aligned} D^+ V(t) &= \operatorname{sgn}(H(t) - H^*(t)) \\ &\quad \times \left(-a_1(t) \left(\frac{P(t)}{m^2 + H^2(t)} - \frac{P^*(t)}{m^2 + (H^*(t))^2} \right) - b_1(t)(H(t) - H^*(t)) \right) \\ &\quad + \operatorname{sgn}(P(t) - P^*(t)) \left(-a_2(t) \left(\frac{P(t)}{H(t)} - \frac{P^*(t)}{H^*(t)} \right) \right) \\ &\leq - \left(b_1^l + \frac{P^*(t)(H(t) + H^*(t))}{(m^2 + H^2(t))(m^2 + (H^*(t))^2)} - \frac{P(t)}{H(t)H^*(t)} \right) |H(t) - H^*(t)| \\ &\quad - \left(\frac{1}{H^*(t)} - a_1^u - \frac{1}{(m^2 + H^2(t))} \right) |P(t) - P^*(t)| \\ &\leq - \left(b_1^l + \frac{2m_2^\epsilon m_1^\epsilon}{(m^2 + (M_1^\epsilon)^2)^2} - \frac{M_2^\epsilon}{(m_1^\epsilon)^2} \right) |H(t) - H^*(t)| \\ &\quad - \left(\frac{1}{M_1^\epsilon} - a_1^u - \frac{1}{(m^2 + (m_1^\epsilon)^2)} \right) |P(t) - P^*(t)|. \end{aligned} \quad (2.26)$$

From (2.24) it follows that there exists a positive constant $\mu > 0$ such that

$$D^+ V(t) \leq -\mu [|H(t) - H^*(t)| + |P(t) - P^*(t)|], \quad t \geq t_0 + T_1. \quad (2.27)$$

Integrating on both sides of (2.27) from $t_0 + T_1$ to t produces

$$\begin{aligned} V(t) + \mu \int_{t_0+T_1}^t [|H(s) - H^*(s)| + |P(s) - P^*(s)|] ds \\ \leq V(t_0 + T_1) < +\infty, \quad t \geq t_0 + T_1. \end{aligned} \quad (2.28)$$

Then

$$\int_{t_0+T_1}^t [|H(s) - H^*(s)| + |P(s) - P^*(s)|] ds \leq \mu^{-1} V(t_0 + T_1) < +\infty, \quad t \geq t_0 + T_1, \quad (2.29)$$

and hence, $|H(t) - H^*(t)| + |P(t) - P^*(t)| \in L^1([t_0 + T_1, +\infty))$.

The boundedness of $H^*(t)$ and $P^*(t)$ and the ultimate boundedness of $H(t)$ and $P(t)$ imply that $H(t)$, $P(t)$, $H^*(t)$, and $P^*(t)$ all have a bounded derivative for $t \geq t_0 + T_1$. Then, it follows that $|H(t) - H^*(t)| + |P(t) - P^*(t)|$ is uniformly continuous on $[t_0 + T_1, +\infty)$. By Lemma 2.11, we have

$$\lim_{t \rightarrow +\infty} (|H(t) - H^*(t)| + |P(t) - P^*(t)|) = 0. \quad (2.30)$$

The proof is complete. \square

3. Periodic case

Throughout this section, we will assume that $r_i(t)$, $a_i(t)$, $i = 1, 2$, $b_1(t)$, are positive ω -period functions, that is, system (1.6) is ω -period system and will study the existence and stability of a positive periodic solution of (1.6).

LEMMA 3.1 (Brouwer fixed point theorem). *Let σ be a continuous operator that maps a closed, bounded, convex subset $\overline{\Omega} \subset \mathbb{R}^n$ into itself. Then $\overline{\Omega}$ contains at least one fixed point of the operator σ , that is, there exists an $x^* \in \overline{\Omega}$ such that $\sigma(x^*) = x^*$.*

THEOREM 3.2. *Assume that (2.4) holds, then system (1.6) has at least one positive periodic solution of period ω , say $(\overline{H}(t), \overline{P}(t))$ which lies in Γ_ϵ .*

Proof. Define a shift operator, which is also known as a Poincaré mapping $\sigma : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ by

$$\sigma((H_0, P_0)) = (H(\omega, t_0, (H_0, P_0)), P(\omega, t_0, (H_0, P_0))), \quad (H_0, P_0) \in \mathbb{R}^2, \quad (3.1)$$

where $(H(\omega, t_0, (H_0, P_0)), P(\omega, t_0, (H_0, P_0)))$ denotes the solution of system (1.6) through the point $(t_0, (H_0, P_0))$. Theorem 2.4 tells us that Γ_ϵ is a positive invariant with respect to system (1.6), and hence, the operator σ defined above maps Γ_ϵ into itself, that is, $\sigma(\Gamma_\epsilon) \subset \Gamma_\epsilon$. Since the solution of system (1.6) is continuous with respect to the initial value, the operator σ is continuous. It is easy to show that Γ_ϵ is a bounded, closed, convex set in \mathbb{R}^2 . By Lemma 3.1, σ has at least one fixed point in Γ_ϵ , that is, there exists an $(\overline{H}(t), \overline{P}(t)) \in \Gamma_\epsilon$ such that

$$(\overline{H}(t), \overline{P}(t)) = (H(\omega, t_0, (\overline{H}, \overline{P})), P(\omega, t_0, (\overline{H}, \overline{P}))). \quad (3.2)$$

Therefore, there exists at least one positive periodic solution, say $(\overline{H}(t), \overline{P}(t))$, and the invariance of Γ_ϵ assures that $(\overline{H}(t), \overline{P}(t)) \in \Gamma_\epsilon$. The proof is complete. \square

Next, we will employ an alternative approach, that is, a continuation theorem in coincidence degree theorem, to establish some different criteria for the same problem. To this end, we need some preparation as follows.

Let X and Z be two Banach spaces. Consider an operator equation

$$Lx = \lambda Nx, \quad \lambda \in (0, 1), \tag{3.3}$$

where $L : \text{Dom}L \cap X \rightarrow Z$ is a linear operator and λ is a parameter. Let P and Q denote two projectors such that

$$P : X \cap \text{Dom}L \rightarrow \text{Ker}L, \quad Q : Z \rightarrow Z/\text{Im}L. \tag{3.4}$$

In the sequel, we will use the following result of Gaines and Mawhin [7, page 40].

LEMMA 3.3. *Let X and Z be two Banach spaces and let L be a Fredholm mapping of index zero. Assume that $N : \overline{\Omega} \rightarrow Z$ is L -compact on $\overline{\Omega}$ with Ω open bounded in X . Furthermore assume that*

(a) *for each $\lambda \in (0, 1)$, $x \in \partial\Omega \cap \text{Dom}L$,*

$$Lx \neq \lambda Nx; \tag{3.5}$$

(b) *for each $x \in \partial\Omega \cap \text{Ker}L$,*

$$QNx \neq 0, \quad \text{deg}\{QNx, \Omega \cap \text{Ker}L, 0\} \neq 0. \tag{3.6}$$

Then the equation $Lx = Nx$ has at least one solution in $\overline{\Omega} \cap \text{Dom}L$.

Recall that a linear mapping $L : \text{Dom}L \cap X \rightarrow Z$ with $\text{Ker}L = L^{-1}(0)$ and $\text{Im}L = L(\text{Dom}L)$ will be called a Fredholm mapping if the following two conditions hold:

- (i) $\text{Ker}L$ has a finite dimension;
- (ii) $\text{Im}L$ is closed and has a finite codimension.

Recall also that the codimension of $\text{Im}L$ is the dimension of $Z/\text{Im}L$, that is, the dimension of the cokernel $\text{coker}L$ of L .

When L is a Fredholm mapping, its index is the integer $\text{Ind}L = \dim \text{ker}L - \text{codim} \text{Im}L$.

We will say that a mapping N is L -compact on Ω if the mapping $QN : \overline{\Omega} \rightarrow Z$ is continuous, $QN(\overline{\Omega})$ is bounded, and $K_p(I - Q)N : \overline{\Omega} \rightarrow X$ is compact, that is, it is continuous and $K_p(I - Q)N(\overline{\Omega})$ is relatively compact, where $K_p : \text{Im}L \rightarrow \text{Dom}L \cap \text{Ker}P$ is a inverse of the restriction L_p of L to $\text{Dom}L \cap \text{Ker}P$, so that $LK_p = I$ and $K_pL = I - P$.

For convenience, we will introduce the notation

$$\bar{u} = \frac{1}{\omega} \int_0^\omega u(t)dt, \tag{3.7}$$

where u is a periodic continuous function with period ω .

THEOREM 3.4. *If the system of algebraic equations*

$$\begin{aligned} \bar{r}_1 - \frac{\bar{a}_1}{m^2 + v_1^2} v_2 - \bar{b}_1 v_1 &= 0, \\ \bar{r}_2 - \bar{a}_2 \frac{v_2}{v_1} &= 0 \end{aligned} \tag{3.8}$$

has a unique solution $(v_1^*, v_2^*)^T \in \text{int } \mathbb{R}_+^2$ with $v_i^* > 0$, $i = 1, 2$, then system (1.6) has at least one positive ω -periodic solution.

Proof. Since

$$\begin{aligned} H(t) &= H(0) \exp \left\{ \int_0^t \left[r_1(s) - \frac{a_1(s)}{m^2 + H^2(s)} P(s) - b_1(s) H(s) \right] ds \right\}, \\ P(t) &= P(0) \exp \left\{ \int_0^t \left[r_2(s) - a_2(s) \frac{P(s)}{H(s)} \right] ds \right\}, \end{aligned} \tag{3.9}$$

the solution of system (1.6) remains positive for $t \geq 0$, we can let

$$H(t) = \exp \{x_1(t)\}, \quad P(t) = \exp \{x_2(t)\}, \tag{3.10}$$

and derive that

$$\begin{aligned} \frac{dx_1(t)}{dt} &= r_1(t) - \frac{a_1(t)}{m^2 + \exp [2x_1(t)]} \exp [x_2(t)] - b_1(t) \exp [x_1(t)], \\ \frac{dx_2(t)}{dt} &= r_2(t) - a_2(t) \frac{\exp \{x_2(t)\}}{\exp \{x_1(t)\}}. \end{aligned} \tag{3.11}$$

In order to use Lemma 2.1 to system (1.6), we take

$$X = Z = \{x(t) = (x_1(t), x_2(t))^T \in C(\mathbb{R}, \mathbb{R}^2) : x(t + \omega) = x(t)\} \tag{3.12}$$

and denote

$$\|x\| = \left\| (x_1(t), x_2(t))^T \right\| = \max_{t \in [0, \omega]} |x_1(t)| + \max_{t \in [0, \omega]} |x_2(t)|. \tag{3.13}$$

Then X and Z are Banach spaces when they are endowed with the norms $\|\cdot\|$.

Set

$$Nx = \begin{bmatrix} r_1(t) - \frac{a_1(t)}{m^2 + \exp [2x_1(t)]} \exp [x_2(t)] - b_1(t) \exp [x_1(t)] \\ r_2(t) - a_2(t) \frac{\exp \{x_2(t)\}}{\exp [x_1(t)]} \end{bmatrix}, \tag{3.14}$$

$$Lx = x', \quad Px = \frac{1}{\omega} \int_0^\omega x(t) dt, \quad x \in X, \quad Qz = \frac{1}{\omega} \int_0^\omega z(t) dt, \quad z \in Z.$$

Evidently, $\text{Ker } L = \{x \mid x \in X, x = \mathbb{R}^2\}$, $\text{Im } L = \{z \mid z \in Z, \int_0^\omega z(t) dt = 0\}$ is closed in Z and $\dim \text{Ker } L = \text{codim Im } L = 2$. Hence, L is a Fredholm mapping of index zero. Furthermore,

the generalized inverse (to L) $K_p : \text{Im } L \rightarrow \text{Ker } P \cap \text{dom } L$ has the form

$$K_p(z) = \int_0^t z(s) ds - \frac{1}{\omega} \int_0^\omega \int_0^t z(s) ds dt. \quad (3.15)$$

Thus

$$QNx = \begin{bmatrix} \frac{1}{\omega} \int_0^\omega \left[r_1(t) - \frac{a_1(t)}{m^2 + \exp[2x_1(t)]} \exp\{x_2(t)\} - b_1(t) \exp\{x_1(t)\} \right] dt \\ \frac{1}{\omega} \int_0^\omega \left[r_2(t) - a_2(t) \frac{\exp\{x_2(t)\}}{\exp\{x_1(t)\}} \right] dt \end{bmatrix},$$

$$K_p(I - Q)N = \begin{bmatrix} \int_0^t \left[r_1(s) - \frac{a_1(s)}{m^2 + \exp[2x_1(s)]} \exp\{x_2(s)\} - b_1(s) \exp\{x_1(s)\} \right] ds \\ \int_0^t \left[r_2(s) - a_2(s) \frac{\exp\{x_2(s)\}}{\exp\{x_1(s)\}} \right] ds \end{bmatrix}$$

$$- \begin{bmatrix} \frac{1}{\omega} \int_0^\omega \int_0^t \left[r_1(s) - \frac{a_1(s)}{m^2 + \exp[2x_1(s)]} \exp\{x_2(s)\} - b_1(s) \exp\{x_1(s)\} \right] ds dt \\ \frac{1}{\omega} \int_0^\omega \int_0^t \left[r_2(s) - a_2(s) \frac{\exp\{x_2(s)\}}{\exp\{x_1(s)\}} \right] ds dt \end{bmatrix}$$

$$- \begin{bmatrix} \left(\frac{t}{\omega} - \frac{1}{2} \right) \int_0^\omega \left[r_1(t) - \frac{a_1(t)}{m^2 + \exp[2x_1(t)]} \exp\{x_2(t)\} - b_1(t) \exp\{x_1(t)\} \right] dt \\ \left(\frac{t}{\omega} - \frac{1}{2} \right) \int_0^\omega \left[r_2(t) - a_2(t) \frac{\exp\{x_2(t)\}}{\exp\{x_1(t)\}} \right] dt \end{bmatrix}. \quad (3.16)$$

Clearly, QN and $K_p(I - Q)N$ are continuous and, moreover, $QN(\overline{\Omega})$, $K_p(I - Q)N(\overline{\Omega})$ are relatively compact for any open bounded set $\Omega \subset X$. Hence, N is L -compact on $\overline{\Omega}$, here Ω is any open bounded set in X .

Now we reach the position to search for an appropriate open bounded subset Ω for the application of Lemma 2.1. Corresponding to equation $Lx = \lambda Nx$, $\lambda \in (0, 1)$, we have

$$x'_1(t) = \lambda \left[r_1(t) - \frac{a_1(t)}{m^2 + \exp[2x_1(t)]} \exp\{x_2(t)\} - b_1(t) \exp\{x_1(t)\} \right],$$

$$x'_2(t) = \lambda \left[r_2(t) - a_2(t) \frac{\exp\{x_2(t)\}}{\exp\{x_1(t)\}} \right]. \quad (3.17)$$

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Suppose that $x(t) = (x_1, x_2) \in X$ is a solution of system (3.17) for a certain $\lambda \in (0, 1)$. By integrating (3.17) over the interval $[0, \omega]$, we obtain

$$\int_0^\omega \left[r_1(t) - \frac{a_1(t)}{m^2 + \exp[2x_1(t)]} \exp\{x_2(t)\} - b_1(t) \exp\{x_1(t)\} \right] dt = 0, \quad (3.18)$$

$$\int_0^\omega \left[r_2(t) - a_2(t) \frac{\exp\{x_2(t)\}}{\exp\{x_1(t)\}} \right] dt = 0.$$

Hence

$$\int_0^\omega \left[\frac{a_1(t)}{m^2 + \exp[2x_1(t)]} \exp\{x_2(t)\} + b_1(t) \exp\{x_1(t)\} \right] dt = \bar{r}_1 \omega, \quad (3.19)$$

$$\int_0^\omega \left[a_2(t) \frac{\exp\{x_2(t)\}}{\exp\{x_1(t)\}} \right] dt = \bar{r}_2 \omega. \quad (3.20)$$

From (3.17), (3.19), and (3.20), we obtain

$$\int_0^\omega |x'_1(t)| dt < \int_0^\omega \left[\frac{a_1(t)}{m^2 + \exp[2x_1(t)]} \exp\{x_2(t)\} + b_1(t) \exp\{x_1(t)\} \right] dt + \int_0^\omega |r_1(t)| dt = 2\bar{r}_1 \omega, \quad (3.21)$$

$$\int_0^\omega |x'_2(t)| dt < \int_0^\omega \left[a_2(t) \frac{\exp\{x_2(t)\}}{\exp\{x_1(t)\}} \right] dt + \bar{r}_2 \omega = 2\bar{r}_2 \omega.$$

Note that $(x_1(t), x_2(t))^T \in X$, then there exists $\xi_i, \eta_i \in [0, \omega]$, $i = 1, 2$ such that

$$x_i(\xi_i) = \min_{t \in [0, \omega]} x_i(t), \quad x_i(\eta_i) = \max_{t \in [0, \omega]} x_i(t), \quad i = 1, 2. \quad (3.22)$$

By (3.19) and (3.22), we obtain

$$\bar{r}_1 \omega \geq \bar{b}_1 \omega \exp\{x_1(\xi_1)\}, \quad x_1(\xi_1) \leq \ln \left\{ \frac{\bar{r}_1}{\bar{b}_1} \right\}, \quad (3.23)$$

$$x_1(t) \leq x_1(\xi_1) + \int_0^\omega |x'_1(t)| dt < \ln \left\{ \frac{\bar{r}_1}{\bar{b}_1} \right\} + 2\bar{r}_1 \omega. \quad (3.24)$$

In addition, from (3.22) and system (3.17), we obtain

$$r_2(\xi_2) - a_2(\xi_2) \frac{\exp\{x_2(\xi_2)\}}{\exp\{x_1(\xi_2)\}} = 0,$$

$$r_2(\xi_2) - a_2(\xi_2) \frac{1}{\exp\{x_1(\eta_1)\}} \geq 0, \quad (3.25)$$

$$x_1(\eta_1) \geq \ln \frac{a_2(\xi_2)}{r_2(\xi_2)} \geq \min_{t \in [0, \infty]} \left\{ \ln \left(\frac{a_2(t)}{r_2(t)} \right) \right\}.$$

Then

$$x_1(t) \geq x_1(\eta_1) - \int_0^\omega |x'_1(t)| dt \geq \min_{t \in [0, \infty)} \left\{ \ln \left(\frac{a_2(t)}{r_2(t)} \right) \right\} - 2\bar{r}_1\omega. \quad (3.26)$$

It follows from (3.24) and (3.26) that

$$\max_{t \in [0, \omega]} |x_1(t)| \leq \max \left\{ \left| \ln \left\{ \frac{\bar{r}_1}{b_1} \right\} + 2\bar{r}_1\omega \right|, \left| \min_{t \in [0, \infty)} \left\{ \ln \left(\frac{a_2(t)}{r_2(t)} \right) \right\} - 2\bar{r}_1\omega \right| \right\} := M_1. \quad (3.27)$$

In view of (3.20) and (3.22), we have

$$\bar{r}_2\omega \leq \bar{a}_2\omega \exp \{x_2(\eta_2)\}, \quad (3.28)$$

that is

$$x_2(\eta_2) \geq \ln \left\{ \frac{\bar{r}_2}{\bar{a}_2} \right\}. \quad (3.29)$$

Then

$$x_2(t) \geq x_2(\eta_2) - \int_0^\omega |x'_2(t)| dt < \ln \left\{ \frac{\bar{r}_2}{\bar{a}_2} \right\} - 2\bar{r}_2\omega. \quad (3.30)$$

By virtue of (3.19), (3.24), and (3.22), we obtain that

$$\bar{r}_1\omega \geq \frac{\bar{a}_1}{m^2 + \exp[2M_1]} \omega \exp \{x_2(\xi_2)\}, \quad (3.31)$$

and so

$$x_2(\xi_2) \leq \ln \left\{ \frac{\bar{r}_1(m^2 + \exp[2M_1])}{\bar{a}_1} \right\}. \quad (3.32)$$

Then

$$x_2(t) \leq x_2(\xi_2) + \int_0^\omega |x'_2(t)| dt < \ln \left\{ \frac{\bar{r}_1(m^2 + \exp[2M_1])}{\bar{a}_1} \right\} + 2\bar{r}_2\omega. \quad (3.33)$$

It follows from (3.30) and (3.33) that

$$\max_{t \in [0, \omega]} |x_2(t)| \leq \max \left\{ \left| \ln \left\{ \frac{\bar{r}_2}{\bar{a}_2} \right\} - 2\bar{r}_2\omega \right|, \left| \frac{\bar{r}_1(m^2 + \exp[2M_1])}{\bar{a}_1} + 2\bar{r}_2\omega \right| \right\} := M_2. \quad (3.34)$$

Clearly, M_i , $i = 1, 2$, are independent of λ . Under the assumption in Theorem 3.4, it is easy to show that the system of algebraic equations

$$\bar{r}_1 - \frac{\bar{a}_1}{m^2 + v_1^2} v_2 - \bar{b}_1 v_1 = 0, \quad \bar{r}_2 - \bar{a}_2 \frac{v_2}{v_1} = 0 \quad (3.35)$$

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has a unique solution $(v_1^*, v_2^*)^T \in \text{int } \mathbb{R}_+^2$ with $v_i^* > 0, i = 1, 2$. Denote $M = M_1 + M_2 + M_3$, where $M_3 > 0$ is taken sufficiently large such that

$$|(\ln \{v_1^*\}, \ln \{v_2^*\})| = |\ln \{v_1^*\}| + |\ln \{v_2^*\}| < M_3, \quad (3.36)$$

and define

$$\Omega = \{x(t) \in X : \|x\| < M\}. \quad (3.37)$$

It is clear that Ω satisfies the condition (a) of Lemma 2.1. When

$$x = (x_1, x_2)^T \in \partial\Omega \cap \text{Ker } L = \partial\Omega \cap \mathbb{R}^2, \quad (3.38)$$

x is a constant vector in \mathbb{R}^2 with $\|x\| = M$. Then

$$QNx = \begin{bmatrix} \bar{r}_1 - \frac{\bar{a}_1}{m^2 + v_1^2} \exp \{x_2\} - \bar{b}_1 \exp \{x_1\} \\ \bar{r}_2 - \bar{a}_2 \frac{\exp \{x_2\}}{\exp \{x_1\}} \end{bmatrix} \neq 0. \quad (3.39)$$

Furthermore, in view of assumption in Theorem 3.4, it can be easily seen that

$$\deg \{QNx, \Omega \cap \text{Ker } L, 0\} \neq 0. \quad (3.40)$$

By now we know that Ω verifies all the requirements of Lemma 3.3 and then system (3.11) has at least one ω -periodic solution. By the medium of (3.10), we derive that system (1.6) has at least one positive ω -periodic solution. The proof is complete. \square

4. Almost periodic case

The assumption of almost periodicity of system (1.6) is a way of incorporating the time dependent variability of the environment, especially when the various components of the environment are periodic with not necessary commensurate period. Mathematically, system (1.6) will denote a generation of an autonomous and periodic system. Therefore, throughout this section, in addition to the assumptions in Section 2, we further assume that $r_i(t), a_i(t), i = 1, 2, b_1(t)$ are almost periodic. Thus, all the theorems in Section 2 remain valid.

Let

$$H(t) = \exp(\tilde{H}(t)), \quad P(t) = \exp(\tilde{P}(t)), \quad (4.1)$$

then system (1.6) becomes

$$\begin{aligned} \frac{d\tilde{H}(t)}{dt} &= r_1(t) - \frac{a_1(t)}{m^2 + \exp 2(\tilde{H}(t))} \exp(\tilde{P}(t)) - b_1(t) \exp(\tilde{H}(t)), \\ \frac{d\tilde{P}(t)}{dt} &= r_2(t) - a_2(t) \frac{\exp\{\tilde{P}(t)\}}{\exp\{\tilde{H}(t)\}}. \end{aligned} \quad (4.2)$$

By Theorem 2.8, it is not difficult to prove the following theorem.

THEOREM 4.1. *Assume that (2.4) holds, then the set Γ_ϵ defined by*

$$\Gamma_\epsilon^* = \{(H(t), P(t)) \in \mathbb{R}^2 \mid \ln(m_1^\epsilon) \leq H \leq \ln(M_1^\epsilon), \ln(m_2^\epsilon) \leq P \leq \ln(M_2^\epsilon)\} \quad (4.3)$$

is positively invariant and ultimately bounded region of system (4.2), where $M_i^\epsilon, m_i^\epsilon, i = 1, 2$, are defined in (2.6).

In order to prove the main result of this section, we will first introduce a useful lemma. Consider the ordinary differential equation

$$x' = f(t, x), \quad f(t, x) \in C(R \times D, \mathbb{R}^n), \quad (4.4)$$

where D is an open set in \mathbb{R}^n , $f(t, x)$ is almost periodic in t uniformly with respect to $x \in D$.

To discuss the existence of an almost periodic solution of (4.4), we consider the product system of (4.4)

$$\begin{aligned} x' &= f(t, x), \\ y' &= f(t, y). \end{aligned} \quad (4.5)$$

LEMMA 4.2 [28, Theorem 19.1]. *Suppose that there exists a Lyapunov function $V(t, x, y)$ defined on $[0, +\infty) \times D \times D$ which satisfies the following conditions:*

- (i) $\alpha(\|x - y\|) \leq V(t, x, y) \leq \beta(\|x - y\|)$, where $\alpha(y)$ and $\beta(y)$ are continuous, increasing and positive definite;
- (ii) $|V(t, x_1, y_1) - V(t, x_2, y_2)| \leq K\{\|x_1 - x_2\| + \|y_1 - y_2\|\}$, where $K > 0$ is a constant;
- (iii) $V'_{(4.3)}(t, x, y) \leq -\mu V(|x - y|)$, where $\mu > 0$ is a constant.

Moreover, suppose that system (4.4) has a solution that remains in a compact set $S \subset D$ for all $t \geq t_0 \geq 0$. Then system (4.4) has a unique almost periodic solution in S , which is uniformly asymptotically stable in D .

THEOREM 4.3. *Assume that (2.4) holds, and*

$$\begin{aligned} b_1^l(t) + \frac{2m_2^\epsilon m_1^\epsilon}{(m^2 + (M_1^\epsilon)^2)^2} - \frac{M_2^\epsilon}{(m_1^\epsilon)^2} &> 0, \\ \frac{1}{M_1^\epsilon} - a_1^u(t) - \frac{1}{(m^2 + (m_1^\epsilon)^2)} &> 0, \end{aligned} \quad (4.6)$$

where $M_i^\epsilon, m_i^\epsilon, i = 1, 2$, are defined in (2.6). Then system (1.6) has an unique positive almost periodic solution, which is globally asymptotically stable, especially uniformly globally asymptotically stable in Γ_ϵ^ .*

Consider the product system of (4.2)

$$\begin{aligned}
\frac{d\tilde{H}_1(t)}{dt} &= r_1(t) - \frac{a_1(t)}{m^2 + \exp(2\tilde{H}_1(t))} \exp(\tilde{P}_1(t)) - b_1(t) \exp(\tilde{H}_1(t)), \\
\frac{d\tilde{P}_1(t)}{dt} &= r_2(t) - a_2(t) \frac{\exp\{\tilde{P}_1(t)\}}{\exp\{\tilde{H}_1(t)\}}, \\
\frac{d\tilde{H}_2(t)}{dt} &= r_1(t) - \frac{a_1(t)}{m^2 + \exp(2\tilde{H}_2(t))} \exp(\tilde{P}_2(t)) - b_1(t) \exp(\tilde{H}_2(t)), \\
\frac{d\tilde{P}_2(t)}{dt} &= r_2(t) - a_2(t) \frac{\exp\{\tilde{P}_2(t)\}}{\exp\{\tilde{H}_2(t)\}}.
\end{aligned} \tag{4.7}$$

Now we define a Lyapunov function on $[0, +\infty) \times \Gamma_\epsilon^* \times \Gamma_\epsilon^*$ as follows:

$$V(t, \tilde{H}_1, \tilde{P}_1, \tilde{H}_2, \tilde{P}_2) = |\tilde{H}_1(t) - \tilde{H}_2(t)| + |\tilde{P}_1(t) - \tilde{P}_2(t)|. \tag{4.8}$$

Then, condition (i) in Lemma 4.2 is satisfied for $\alpha(\gamma) = \beta(\gamma) = \gamma$ for $\gamma \geq 0$. In addition,

$$\begin{aligned}
&|V(t, \tilde{H}_1, \tilde{P}_1, \tilde{H}_2, \tilde{P}_2) - V(t, \tilde{H}_3, \tilde{P}_3, \tilde{H}_4, \tilde{P}_4)| \\
&= |(|\tilde{H}_1(t) - \tilde{H}_2(t)| + |\tilde{P}_1(t) - \tilde{P}_2(t)|) - (|\tilde{H}_3(t) - \tilde{H}_4(t)| + |\tilde{P}_3(t) - \tilde{P}_4(t)|)| \\
&\leq |\tilde{H}_1(t) - \tilde{H}_3(t)| + |\tilde{P}_1(t) - \tilde{P}_3(t)| + |\tilde{H}_2(t) - \tilde{H}_4(t)| + |\tilde{P}_2(t) - \tilde{P}_4(t)| \\
&\leq \|(\tilde{H}_1(t), \tilde{P}_1(t)) - (\tilde{H}_3(t), \tilde{P}_3(t))\| + \|(\tilde{H}_2(t), \tilde{P}_2(t)) - (\tilde{H}_4(t), \tilde{P}_4(t))\|,
\end{aligned} \tag{4.9}$$

which shows that the condition (ii) of Lemma 4.2 is also satisfied.

Let $(\tilde{H}_i(t), \tilde{P}_i(t))$, $i = 1, 2$, be any two solutions of (4.2) defined on $[0, +\infty) \times \Gamma_\epsilon^* \times \Gamma_\epsilon^*$.

Calculating the right derivative $D^+V(t)$ of $V(t)$ along the solutions of (4.2), we have

$$\begin{aligned}
D^+V(t) &= \operatorname{sgn}(\tilde{H}_1(t) - \tilde{H}_2(t)) \left(-\frac{a_1(t)}{m^2 + \exp(2\tilde{H}_1(t))} (\exp(\tilde{P}_1(t)) - \exp(\tilde{P}_2(t))) \right) \\
&\quad - b_1(t) (\exp(\tilde{H}_1(t)) - \exp(\tilde{H}_2(t))) \\
&\quad + \operatorname{sgn}(\tilde{P}_1(t) - \tilde{P}_2(t)) \left(-a_2(t) \left(\frac{\exp(\tilde{P}_1(t))}{\exp(\tilde{H}_1(t))} - \frac{\exp(\tilde{P}_2(t))}{\exp(\tilde{H}_2(t))} \right) \right)
\end{aligned}$$

$$\begin{aligned}
&\leq - \left(b_1^l + \frac{\exp(\tilde{P}_2(t)) (\exp(\tilde{H}_1(t)) + \exp(\tilde{H}_2(t)))}{(m^2 + \exp(2\tilde{H}_1(t))) (m^2 + \exp(2\tilde{H}_2(t)))} - \frac{P(t)}{H(t)H^*(t)} \right. \\
&\quad \left. - \frac{\exp(\tilde{P}_1(t))}{\exp(\tilde{H}_1(t)) \exp(\tilde{H}_2(t))} \right) \times |\exp(\tilde{H}_1(t)) - \exp(\tilde{H}_2(t))| \\
&\quad - \left(\frac{1}{\exp(\tilde{H}_2(t))} - a_1^u - \frac{1}{(m^2 + \exp(2\tilde{H}_1(t)))} \right) \times |\exp(\tilde{P}_1(t)) - \exp(\tilde{P}_2(t))| \\
&\leq - \left(b_1^l + \frac{2m_2^\epsilon m_1^\epsilon}{(m^2 + (M_1^\epsilon)^2)^2} - \frac{M_2^\epsilon}{(m_1^\epsilon)^2} \right) |\exp(\tilde{H}_1(t)) - \exp(\tilde{H}_2(t))| \\
&\quad - \left(\frac{1}{M_1^\epsilon} - a_1^u - \frac{1}{(m^2 + (m_1^\epsilon)^2)} \right) |\exp(\tilde{P}_1(t)) - \exp(\tilde{P}_2(t))|. \tag{4.10}
\end{aligned}$$

Note that

$$\begin{aligned}
\exp(\tilde{H}_1(t)) - \exp(\tilde{H}_2(t)) &= \exp(\xi(t)) (\tilde{H}_1(t) - \tilde{H}_2(t)), \\
\exp(\tilde{P}_1(t)) - \exp(\tilde{P}_2(t)) &= \exp(\eta(t)) (\tilde{P}_1(t) - \tilde{P}_2(t)), \tag{4.11}
\end{aligned}$$

where $\xi(t)$ lies between $\tilde{H}_1(t)$ and $\tilde{H}_2(t)$ while $\eta(t)$ lies between $\tilde{P}_1(t)$ and $\tilde{P}_2(t)$. Then we have

$$\begin{aligned}
D^+V(t) &\leq - \left(b_1(t) - \frac{M_2^\epsilon}{(m_1^\epsilon)^2} \right) m_1^\epsilon |\tilde{H}_1(t) - \tilde{H}_2(t)| \\
&\quad - \left(\frac{1}{M_1^\epsilon} - a_1(t) \right) m_2^\epsilon |\tilde{P}_1(t) - \tilde{P}_2(t)| \tag{4.12} \\
&\leq -\mu (|\tilde{H}_1(t) - \tilde{H}_2(t)| + |\tilde{P}_1(t) - \tilde{P}_2(t)|) \\
&= -\mu \|(\tilde{H}_1(t), \tilde{P}_1(t)) - (\tilde{H}_2(t), \tilde{P}_2(t))\|,
\end{aligned}$$

where

$$\mu = \min \left\{ \left\{ b_1^l + \frac{2m_2^\epsilon m_1^\epsilon}{(m^2 + (M_1^\epsilon)^2)^2} - \frac{M_2^\epsilon}{(m_1^\epsilon)^2} \right\}, \left\{ \frac{1}{M_1^\epsilon} - a_1^u - \frac{1}{(m^2 + (m_1^\epsilon)^2)} \right\} \right\} > 0. \tag{4.13}$$

Hence, the condition (iii) of Lemma 4.2 is verified as well. Therefore, by Theorem 4.1 and Lemma 4.2, it follows that system (4.2) has a unique almost periodic solution in Γ_ϵ^* , say $(\tilde{H}^*(t), \tilde{P}^*(t))$, which is uniformly asymptotically stable in Γ_ϵ^* . Hence, system (4.2) has a

unique positive almost periodic solution $(H^*(t), P^*(t))$ in Γ_ϵ^* , which is uniformly asymptotically stable in Γ_ϵ^* . By Theorem 2.12, we have that $(H^*(t), P^*(t))$ is globally asymptotically stable. The proof is complete. \square

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