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Research Article

A 3D Smale Horseshoe in a Hyperchaotic Discrete-Time System

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This paper presents a three-dimensional topological horseshoe in the hyperchaotic generalized Hénon map. It looks like a planar Smale horseshoe with an additional vertical expansion, so we call it 3D Smale horseshoe. In this way, a computer assisted verification of existence of hyperchaos is provided by means of interval analysis.

1. Introduction

The most significant result on chaotic dynamics is perhaps topological horseshoe theory [1]. Since horseshoes provide the "backbone" of chaotic attractors, they are essential to both the mathematical and physical analysis of a chaotic system.

The first and fundamental achievement on topological horseshoes is the theory about Smale horseshoes. A Smale horseshoe map is a sequence of basic topological operations consist of stretching (which gives sensitivity to initial conditions) and folding (which gives the attraction). Since trajectories in phase space cannot cross, the repeated stretching and folding operations result in an object of great topological complexity. The theory concerns on a set Q (usually diffeomorphic to a rectangle) in a two-dimensional manifold M and a diffeomorphism $\pi : Q \to M$. By using only hypotheses on the first iterate of π on Q, Smale concludes that there is a compact invariant set Q_I in Q which is homeomorphic to a shift on 2 symbols [2].

In this paper, we will show a picture of a 3D Smale horseshoe for a hyperchaotic discrete-time system. This new horseshoe looks like a planar Smale horseshoe with an additional vertical expansion. It concerns on a cuboid c and a diffeomorphism $H : c \to R^3$, here H is the hyperchaotic generalized Hénon map which is useful in many applications of discrete hyperchaos [3–5]. By means of interval analysis, we show that there exists a closed invariant set $c_I \subset c$ for which $H^4 \mid c_I$ is semiconjugate to a shift map on 2 symbols. Numerical studies suggest that $H^4 \mid c_I$ has two-directional expansions. In this way, a computer-assisted verification of hyperchaos is carried out by virtue of topological horseshoe theory.

2. A result on topological horseshoe theory

Let X be a metric space, Q is a compact subset of X, and $f : Q \to X$ is map satisfying the assumption that there exist m mutually disjoint compact subsets $Q_1, Q_2, \dots, Q_m$ of Q, the restriction of f to each Q_i, that is, $f \mid Q_i$ is continuous.

Definition 2.1. Let γ be a compact subset of Q, such that for each $1 \le i \le m$, $\gamma_i = \gamma \cap Q_i$ is nonempty and compact, then γ is called a connection with respect to $Q_1, Q_2, \dots, Q_m$. Let F be a family of connections γs with respect to $Q_1, Q_2, \dots, Q_m$ satisfying property: $\gamma \in F \Rightarrow f(\gamma_i) \in F$. Then F is said to be an f-connected family with respect to $Q_1, Q_2, \dots, Q_m$.

THEOREM 2.2. *Suppose that there exists an f-connected family with respect to $Q_1, Q_2, \dots, Q_m$. Then there exists a compact invariant set $K \subset Q$, such that $f \mid K$ is semiconjugate to m-shift dynamics, then* $\operatorname{ent}(f) \ge \log m$, *where* $\operatorname{ent}(f)$ *denotes the entropy of the map f.*

In addition, for every positive integer k, $\operatorname{ent}(f^k) = (1/k)\operatorname{ent}(f)$.

For details about the proof of this theorem, see [6], and for details of symbolic dynamics and horseshoe theory, see [7].

3. The 3D Smale horseshoe

The hyperchaotic generalized Hénon map $\mathbf{x}(k+1) = H(\mathbf{x}(k))$ can be described by a third order difference equation [3]:

$$\begin{aligned} x_1(k+1) &= 1.76 - x_2^2(k) - 0.1x_3(k), \\ x_2(k+1) &= x_1(k), \\ x_3(k+1) &= x_2(k). \end{aligned} \tag{3.1}$$

Its attractor is shown in Figure 3.1.

Now, we discuss that there exists a horseshoe imbedded in this attractor under the map H^4. Let ϱ be a subset of R^3, ϱ' denote the image of ϱ under H^4 in the following discussion.

By many attempts, we take a cuboid c in the state space with its eight vertices in term of (x_2, x_3, x_1) to be

$$\begin{aligned} C_1 &= (0.82, 1.13, 0.65), & C_2 &= (1.63, 1.13, 0.65), \\ C_3 &= (1.63, 0.94, 1.03), & C_4 &= (0.82, 0.94, 1.03), \\ C_5 &= (0.82, 1.01, 0.59), & C_6 &= (1.63, 1.01, 0.59), \\ C_7 &= (1.63, 0.82, 0.97), & C_8 &= (0.82, 0.82, 0.97), \end{aligned} \tag{3.2}$$

as illustrated in Figure 3.1, where its top surface c_t is rectangle $|C_1C_2C_3C_4|$, its bottom surface c_b is rectangle $|C_5C_6C_7C_8|$, and its side surface c_s consists of the other four rectangles.

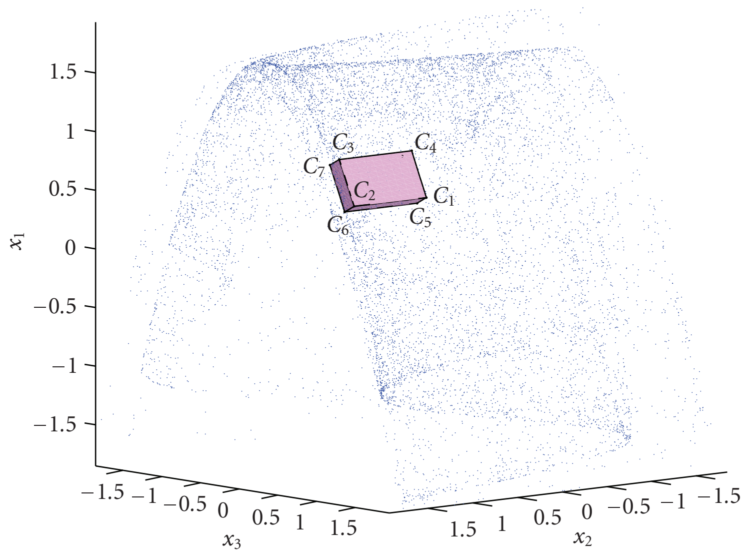


Figure 3.1. The attractor of the hyperchaotic generalized Hénon map.

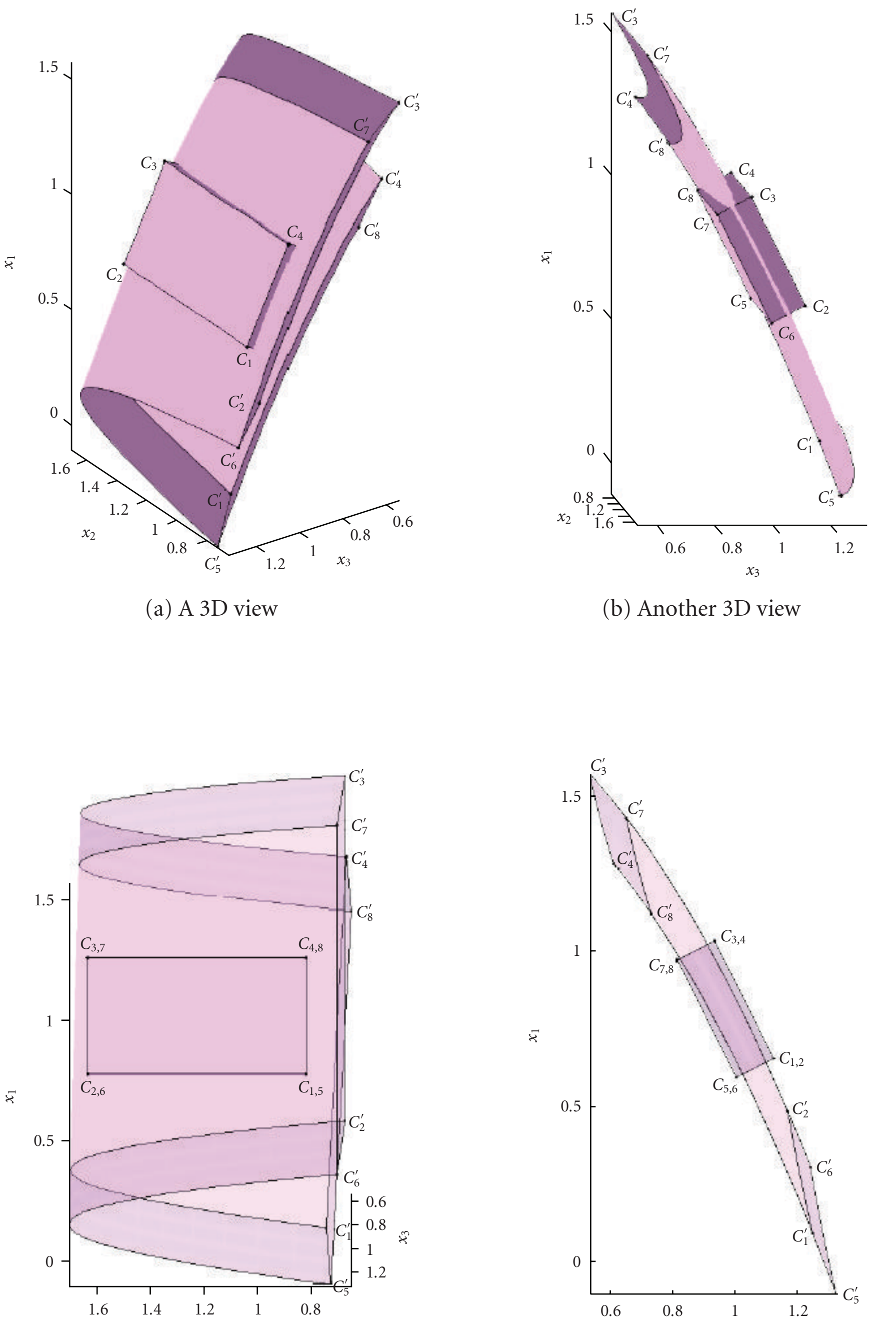
Each point of c can be estimating the accuracy by utilizing the technique of interval arithmetic [8] with INTLAB (an interval arithmetic package for MATLAB [9]). By sampling thousands of equally distributed points from each surface of c, the image of c is computed with the maximal error of all points less than 10^{-10}, as shown in Figure 3.2 from four different angles of view. In Figures 3.2(c) and 3.2(d), all surfaces are half transparent. So Figure 3.2(c) indicates that c_s' is mapped outside of c_s. With additional consideration of Figure 3.2(d), c_t' and c_b' transversely intersect c twice between c_t and c_b. If there is a plane paralleling $|C_3C_4C_8C_7|$ and intersecting with c and c', it is easy to see that the intersection must look like a Smale horseshoe. Thereby, it is not hard to have the following statement.

Theorem 3.1. *For the map H corresponding to the cuboid c, there exists a closed invariant set $c_I \subset c$ for which $H^4 \mid c_I$ is semiconjugate to 2-shift dynamics, then* $\operatorname{ent}(f) \geq (1/4)\log 2$.

Proof. In view of Theorem 2.2, we only need to show that for two disjoint compact subsets of c, such as a and b, there exists an H^4-connected family.

By many attempts, we take two cuboid in c. The first one is a with its eight vertices in term of (x_2, x_3, x_1) to be

$$\begin{aligned} A_1 &= (0.84, 1.13, 0.65), & A_2 &= (1.205, 1.13, 0.65), \\ A_3 &= (1.205, 0.94, 1.03), & A_4 &= (0.84, 0.94, 1.03), \\ A_5 &= (0.84, 1.01, 0.59), & A_6 &= (1.205, 1.01, 0.59), \\ A_7 &= (1.205, 0.82, 0.97), & A_8 &= (0.84, 0.82, 0.97), \end{aligned} \tag{3.3}$$



(a) A 3D view

(b) Another 3D view

(c) The top view

(d) The side view

Figure 3.2. Cuboid c and its image under H^4.

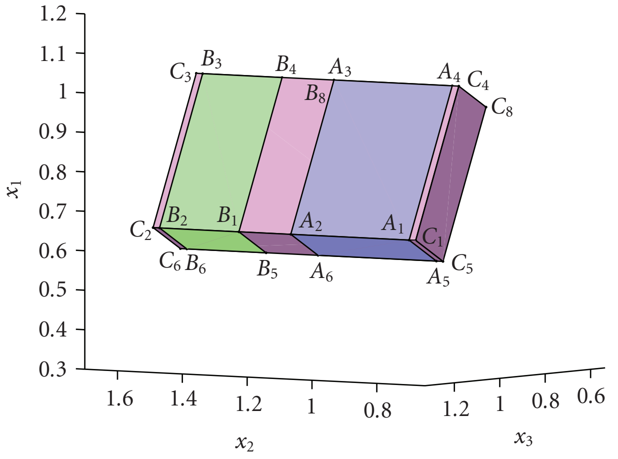


Figure 3.3. The location of cuboid a and cuboid b.

and the second one is b with its eight vertices in term of (x_2, x_3, x_1) to be

$$\begin{aligned} B_1 &= (1.365, 1.13, 0.65), & B_2 &= (1.61, 1.13, 0.65), \\ B_3 &= (1.61, 0.94, 1.03), & B_4 &= (1.365, 0.94, 1.03), \\ B_5 &= (1.365, 1.01, 0.59), & B_6 &= (1.61, 1.01, 0.59), \\ B_7 &= (1.61, 0.82, 0.97), & B_8 &= (1.365, 0.82, 0.97), \end{aligned} \tag{3.4}$$

as shown in Figure 3.3. It is obvious that $H^4 \mid a$ and $H^4 \mid b$ are both continuous. a_t, a_b, and a_s denote the top rectangle $|A_1A_2A_3A_4|$, the bottom rectangle $|A_5A_6A_7A_8|$, and the other four rectangles of a, respectively. For cuboid b, it has the same situation with a.

After a long time of computation by means of interval analysis, the images of a and b are shown in Figures 3.4 and 3.5.

For cuboid a, it is easy to see from Figure 3.4 that H^4 sends a to its image a' as follows: a_t and a_b are almost contracted together while both of them expanded in two directions, and transversely intersect cuboid a between a_t and a_b and cuboid b between b_t and b_b; a_s is mapped outside of a_s and b_s. In this case, we say that the image $a'^4(a)$ lies wholly across the cuboids a and b with respect to a_s and b_s.

For cuboid b, it has the same situation with a, as shown in Figure 3.5, the image $b'^4(b)$ lies wholly across the cuboids a and b with respect a_s and b_s.

It is easy to see from the wholecrossness of $H^4(a)$ and $H^4(b)$ with respect to the sides of a and b that there exists an H^4-connected family with respect to a and b. In view of Theorem 2.2, this means that H^4 is semiconjugate to 2-shift map. □

This theorem is similar with the Smale's conclusion which we have mentioned in Section 1, and indicates that the map shown in Figure 3.2 is a topological horseshoe. Since

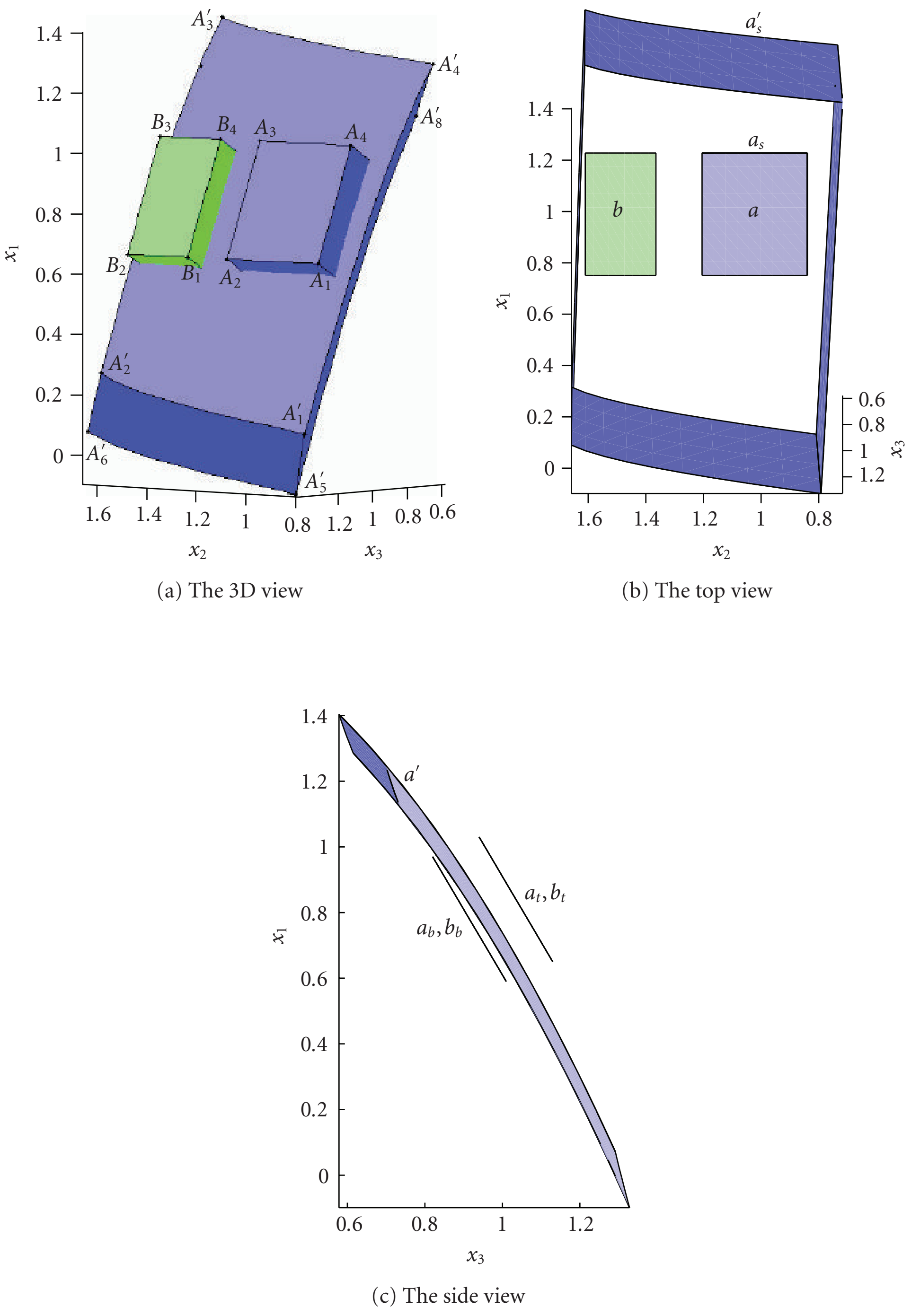
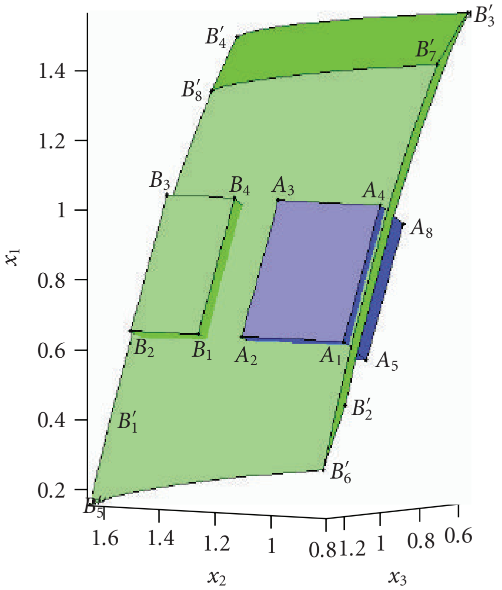
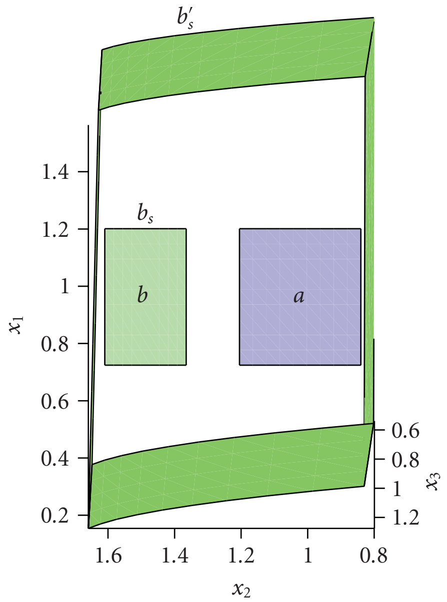


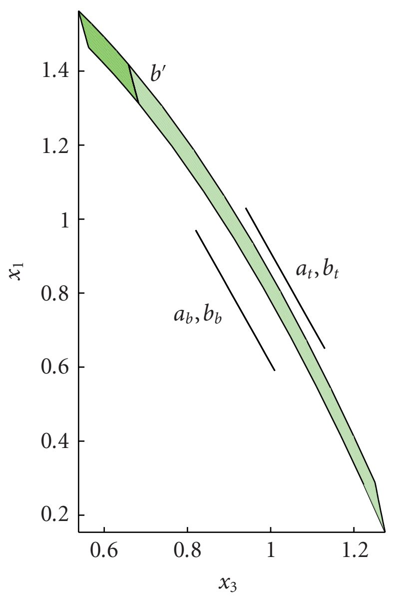
Figure 3.4. $a'^4(a)$ wholly across a and b.



(a) The 3D view



(b) The top view



(c) The side view

Figure 3.5. $b'^4(b)$ wholly across a and b

the whole map looks like a planar Smale horseshoe with an additional vertical expansion, we call it a 3D Smale horseshoe.

The global picture of the images $H^4(a)$ and $H^4(b)$ suggests that $H^4 \mid a$ and $H^4 \mid b$ both expand in two directions. The local expansions of H^4 on a and b can be partially confirmed by numerically studying the Jacobian matrix of $H^4 =$

$$\begin{bmatrix} 4(1.76 - x_1^2 - 0.1x_2)x_1 & 0.352 - 0.2x_1^2 + 0.18x_2 & 0.01 \\ -0.1 & 4(1.76 - x_2^2 - 0.1x_2)x_2 & 0.352 - 0.2x_2^2 - 0.02x_2 \\ -2x_1 & -0.1 & 0 \end{bmatrix}. \tag{3.5}$$

At randomly chosen points in the intersection set of a and b and their images, we numerically find that the matrix has one eigenvalue lying in the interior of the unit circle and two eigenvalues that are located outside of the unit circle. This implies that $H^4 \mid c_I$ has two-directional expansions. Thereby it justifiably indicates a very immediate evidence that the attractor illustrated in Figure 3.1 is hyperchaotic.

4. Conclusions

We have presented a 3D Smale horseshoe in the hyperchaotic generalized Hénon map. Numerical studies suggest that there exist two-directional expansions in this horseshoe map. In this way, a computer-assisted verification of hyperchaos has been provided by virtue of topological horseshoe theory.

Acknowledgments

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