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Research Article

On Global Periodicity of a Class of Difference Equations

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We show that the difference equation $x_n = f_3(x_{n-1})f_2(x_{n-2})f_1(x_{n-3})$, $n \in \mathbb{N}_0$, where $f_i \in C[(0,\infty),(0,\infty)], i \in \{1,2,3\}$, is periodic with period 4 if and only if $f_i(x) = c_i/x$ for some positive constants c_i, $i \in \{1,2,3\}$ or if $f_i(x) = c_i/x$ when $i = 2$ and $f_i(x) = c_i x$ if $i \in \{1,3\}$, with $c_1c_2c_3 = 1$. Also, we prove that the difference equation $x_n = f_4(x_{n-1})f_3(x_{n-2})f_2(x_{n-3})f_1(x_{n-4})$, $n \in \mathbb{N}_0$, where $f_i \in C[(0,\infty),(0,\infty)]$, $i \in \{1,2,3,4\}$, is periodic with period 5 if and only if $f_i(x) = c_i/x$, for some positive constants c_i, $i \in \{1,2,3,4\}$.

1. Introduction

The study of the periodic character of solutions of rational and nonlinear difference equations has recently attracted attention; see, for example, [1–12] and the references therein. For some classical results see [13–17].

Definition 1.1. Let f be a real valued function defined on a subset of $\mathbb{R}^n$. Say that the difference equation

$$x_n = f(x_{n-1},\dots,x_{n-k}), \quad n \in \mathbb{N}_0, \tag{1.1}$$

where $k \in \mathbb{N}$, is *periodic* if every solution of (1.1) is periodic.

It is easy to see that every solution of the difference equation

$$x_n = \frac{C}{x_{n-1}x_{n-2}\cdots x_{n-k}}, \quad n \in \mathbb{N}_0, \tag{1.2}$$

is periodic with period $(k+1)$.

We also know that every solution of the difference equation

$$x_n = \frac{x_{n-1} \cdots x_{n-(2k+1)}}{x_{n-2}x_{n-4} \cdots x_{n-2k}}, \quad n \in \mathbb{N}_0, \tag{1.3}$$

is periodic with period $2k+2$. Indeed, from (1.3) it follows that

$$x_n x_{n-2} \cdots x_{n-2k} = 1, \quad n \in \mathbb{N}_0. \tag{1.4}$$

Using the changes $y_n = x_{2n}$ and $z_n = x_{2n+1}$, the last equation is reduced to (1.2), from which the statement follows.

In [18] we studied the global periodicity of (1.1) with $k = 2$. Among other results it was proved that if f separates the variables, that is, if

$$f(x,y) = f_2(x)f_1(y), \tag{1.5}$$

then every solution of (1.1) is periodic with period 3 if and only if $f(x,y) = c/xy$ where c is a positive constant.

Motivated by the method used in paper [18], in this paper we investigate the global periodicity of the positive solutions of the difference equation

$$x_n = f_k(x_{n-1}) \cdots f_1(x_{n-k}), \quad n \in \mathbb{N}_0, \tag{1.6}$$

where $k \in \{3,4\}$, $f_i \in C[(0,\infty),(0,\infty)]$, $i = 1,\ldots,k$.

We prove the following result.

Theorem 1.2. *Consider (1.6). Then the following statements hold true.*

(a) *Assume that $k = 3$. Then, every positive solution of (1.6) is periodic with period 4 if and only if $f_i(x) = c_i/x$, for some positive constants c_i, $i \in \{1,2,3\}$, or if $f_i(x) = c_i/x$, when $i = 2$ and $f_i(x) = c_i x$ if $i \in \{1,3\}$, with $c_1c_2c_3 = 1$.*

(b) *Assume that $k = 4$. Then, every positive solution of (1.6) is periodic with period 5 if and only if $f_i(x) = c_i/x$, for some positive constants c_i, $i \in \{1,2,3,4\}$.*

2. Auxiliary results

Before we give a proof of Theorem 1.2, we will prove some auxiliary results which are incorporated in the following lemmas. We say that for a mapping $f : X \to X$, $(f^{[p]})_{p\in\mathbb{N}\cup\{0\}}$ denotes the sequence of iterates of f, that is, $f^{[0]} = I$, the identity function on X, $f^{[1]} = f$ and generally $f^{[p+1]} = f \circ f^{[p]}$ for any $p \in \mathbb{N}$.

The following lemma is folklore and can be found, for example, in [19] (see also [20]). We give a proof of the lemma for the benefit of the reader.

Lemma 2.1. *Assume that $f : I \to I$ is a continuous function on the open (or closed) interval $I \subset \mathbb{R}$ satisfying the equation*

$$f^{[p]}(x) = x, \quad x \in I, \tag{2.1}$$

for some $p \in \mathbb{N}$. Then $f(x) \equiv x$, $x \in I$ or $f^{[2]}(x) = x$.

Proof. Assume that $f \in C[I,I]$ is such that $f^{[p]}(x) = x$ for every $x \in I$. Then, if $f(x) = f(y)$, it follows that

$$x = f^{[p]}(x) = f^{[p]}(y) = y \tag{2.2}$$

which implies that the function f must be $1-1$. Since f is a continuous function, we have that f must be strictly monotone.

First assume that f is strictly increasing. If there is a point $x_0 \in I$ such that $x_0 < f(x_0)$, then by the monotonicity of f we have

$$x_0 < f(x_0) < f^{[2]}(x_0) < \cdots < f^{[p]}(x_0) = x_0 \tag{2.3}$$

which is a contradiction. If $x_0 > f(x_0)$, then we have

$$x_0 > f(x_0) > f^{[2]}(x_0) > \cdots > f^{[p]}(x_0) = x_0 \tag{2.4}$$

arriving again at a contradiction.

From this it follows that $f(x) = x$ for every $x \in I$.

Assume now that f is strictly decreasing. Then the function $g(x) = f^{[2]}(x)$ is strictly increasing and according to the first case we have that

$$g^{[p]}(x) = \left(f^{[2]}\right)^{[p]}(x) = \left(f^{[p]}\right)^{[2]}(x) = x \circ x = x, \tag{2.5}$$

that is, $f^{[2]}(x) \equiv x$, finishing the proof of the lemma. □

Lemma 2.2. *Assume that f is a decreasing continuous function which maps the interval $(0,\infty)$ into itself, and satisfies the following conditions*

$$\begin{gathered}\lim_{z \to +0} f(z) = \infty, \qquad \lim_{z \to \infty} f(z) = 0, \\ f(z) f\left(\frac{1}{z}\right) = 1, \quad z \in (0,\infty),\end{gathered} \tag{2.6}$$

$$f(z) = f^{-1}(z) \quad z \in (0,\infty). \tag{2.7}$$

Then $f(z) = 1/z$.

Proof. Assume that $f(z) \not\equiv 1/z$, $z \in (0,\infty)$, then there is a $z_0 \in (0,\infty)$ such that $f(z_0) > 1/z_0$ or $f(z_0) < 1/z_0$. From (2.6) and positivity of the function f it follows that $f(1) = 1$. Hence $z_0 \neq 1$.

First, assume that $f(z_0) > 1/z_0$ and $z_0 < 1$. From this and (2.6) it follows that

$$f\left(\frac{1}{z_0}\right) = \frac{1}{f(z_0)} < z_0 < 1 < \frac{1}{z_0} < f(z_0). \tag{2.8}$$

On the other hand, the point $(f(1/z_0), 1/z_0)$ belongs to the graph of the curve $y = f(z)$, since f is self-invertible. Hence the points $(f(1/z_0), 1/z_0)$, $(1,1)$, and $(z_0, f(z_0))$ belong to the graph of the curve $y = f(z)$. We know that f is decreasing and from (2.8) we have $f(1/z_0) < z_0 < 1$, thus we obtain $f(f(1/z_0)) > f(z_0) > f(1)$, that is, $1/z_0 > f(z_0) > 1$. The last statement contradicts (2.8).

Now, assume that $f(z_0) > 1/z_0$, $1 < z_0$, and $f(z_0) < 1$. Note that the points $(1/z_0, f(1/z_0))$, $(1,1)$, and $(f(z_0), z_0)$ are on the graph of f. Since $1/z_0 < f(z_0) < 1$ and f is decreasing it follows that $f(1/z_0) > f(f(z_0)) > f(1)$, that is, $1/f(z_0) > z_0 > 1$, which is a contradiction.

Assume that $f(z_0) > 1/z_0$, $1 < z_0$ and $f(z_0) > 1$. In the case the points $(z_0, f(z_0))$, $(1,1)$ and $(f(z_0), z_0)$ are on the graph of f. If $1 < z_0 < f(z_0)$, then we obtain that $1 > f(z_0) > z_0$, a contradiction. If $1 < f(z_0) \le z_0$, then it follows that $1 > z_0 \ge f(z_0)$, which is again a contradiction.

The case $f(z_0) < 1/z_0$ can be treated similarly so we omit the proof of this part of the lemma. □

3. Proof of the main result

In this section we give a proof of Theorem 1.2. Before this we present some formulae which are of some interest not only for these two cases in Theorem 1.2, but also for all $k \ge 3$.

Hence, assume that all positive solutions of (1.1) are periodic with period $(k+1)$. Then for every $x_1, \ldots, x_k \in (0,\infty)$ we have that the following system of functional relationships holds:

$$\begin{aligned}
u &= f_k(x_k) f_{k-1}(x_{k-1}) \cdots f_2(x_2) f_1(x_1),\\
x_1 &= f_k(u) f_{k-1}(x_k) \cdots f_2(x_3) f_1(x_2),\\
x_2 &= f_k(x_1) f_{k-1}(u) \cdots f_2(x_4) f_1(x_3),\\
&\ \ \vdots\\
x_k &= f_k(x_{k-1}) f_{k-1}(x_{k-2}) \cdots f_2(x_1) f_1(u).
\end{aligned} \tag{3.1}$$

From (3.1) it follows that

$$
\begin{aligned}
x_1 &= f_k\left(\prod_{j=1}^{k} f_j(x_j)\right) f_{k-1}(x_k)\cdots f_2(x_3)f_1(x_2),\\
x_2 &= f_k(x_1)f_{k-1}\left(\prod_{j=1}^{k} f_j(x_j)\right)\cdots f_2(x_4)f_1(x_3),\\
x_3 &= f_k(x_2)f_{k-1}(x_1)f_{k-2}\left(\prod_{j=1}^{k} f_j(x_j)\right)\cdots f_2(x_5)f_1(x_4),\\
&\ \vdots\\
x_{k-1} &= f_k(x_{k-2})f_{k-1}(x_{k-3})\cdots f_2\left(\prod_{j=1}^{k} f_j(x_j)\right)f_1(x_k),\\
x_k &= f_k(x_{k-1})f_{k-1}(x_{k-2})\cdots f_2(x_1)f_1\left(\prod_{j=1}^{k} f_j(x_j)\right).
\end{aligned}
\tag{3.2}
$$

In each of the k equations in (3.2) we choose that all variables, except the jth which is arbitrary, are equal to 1, and use the changes

$$
g_j(x) = f_j(x)\prod_{i=1,i\neq j}^{k} f_i(1), \quad j = 1,\dots,k. \tag{3.3}
$$

Then, we obtain

$$
\begin{aligned}
&g_k(g_1(z)) = z, \quad g_k(g_j(z))g_{j-1}(z) = C, \quad 2\le j\le k;\\
&g_{k-1}(g_1(z))g_k(z) = C, \quad g_{k-1}(g_2(z)) = z, \quad g_{k-1}(g_j(z))g_{j-2}(z) = C, \quad 3\le j\le k;\\
&g_{k-2}(g_j(z))g_{j+k-2}(z) = C, \quad j = 1,2,\\
&g_{k-2}(g_3(z)) = z, \quad g_{k-2}(g_j(z))g_{j-3}(z) = C, \quad 4\le j\le k;\\
&\qquad\vdots\\
&g_2(g_j(z))g_{j+2}(z) = C, \quad 1\le j\le k-2, \quad g_2(g_{k-1}(z)) = z, \quad g_2(g_k(z))g_1(z) = z,\\
&g_1(g_j(z))g_{j+1}(z) = C, \quad 1\le j\le k-1, \quad g_1(g_k(z)) = z,
\end{aligned}
\tag{3.4}
$$

where $C = \prod_{i=1}^{k} f_i(1)$.

From (3.4) it follows that

$$
g_j\circ g_{k+1-j}(z) = z, \quad j = 1,\dots,k, \tag{3.5}
$$

$$
g_j\circ g_i(z) = g_i\circ g_j(z), \tag{3.6}
$$

if $i \neq j$ and $i+j \neq k+1$, and

$$\begin{gathered} g_k^2(z)g_{k-1}(z) = C, \qquad g_{k-1}^2(z)g_{k-3}(z) = C, \qquad g_{k-2}^2(z)g_{k-5}(z) = C, \\ \vdots \\ g_2^2(z)g_4(z) = C, \qquad g_1^2(z)g_2(z) = C. \end{gathered} \tag{3.7}$$

Proof of Theorem 1.2. The sufficiency part of the theorem follows from (1.3) and (1.2). Hence, we need only prove the necessity.

First, assume that $k = 3$. Then (3.5)–(3.7) are

$$\begin{aligned} &g_3(g_1(z)) = z, \qquad g_3(g_2(z))g_1(z) = C, \qquad g_3(g_3(z))g_2(z) = C, \\ &g_2(g_1(z))g_3(z) = C, \qquad g_2(g_2(z)) = z, \qquad g_2(g_3(z))g_1(z) = C, \\ &g_1(g_1(z))g_2(z) = C, \qquad g_1(g_2(z))g_3(z) = C, \qquad g_1(g_3(z)) = z. \end{aligned} \tag{3.8}$$

From (3.8) we have

$$g_1(g_3(z)) = g_3(g_1(z)) = z, \qquad g_2(g_2(z)) = z, \tag{3.9}$$

which implies that

$$g_3(z) = g_1^{-1}(z), \qquad g_2(z) = g_2^{-1}(z), \tag{3.10}$$

and that the functions g_1, g_2, and g_3 map the interval $(0,\infty)$, "$1-1$" and onto itself.

Further, from the third and seventh identity in (3.8) we have that

$$g_1(g_1(z)) = g_3(g_3(z)). \tag{3.11}$$

From (3.10) and (3.11) it follows that

$$g_1^{[4]}(z) = z. \tag{3.12}$$

Lemma 2.1 implies that

$$g_1(z) = z \quad \text{or} \quad g_1^{[2]}(z) = z. \tag{3.13}$$

If $g_1(z) = z$, then (3.10) implies $g_3(z) = z$, from this and the second identity in (3.8) we obtain that $g_2(z) = C/z$. Hence, the equation becomes

$$x_n = C\frac{x_{n-1}x_{n-3}}{x_{n-2}}. \tag{3.14}$$

By some simple calculations it is shown that C must be equal to 1 in order that all solutions of the equation are periodic with period four, from which the result follows in this case.

If $g_1^{[2]}(z) = z$, then

$$g_1(z) = g_1^{-1}(z). \tag{3.15}$$

Equations (3.10) and (3.15) imply that $g_1 = g_3$. From this and the sixth identity in (3.8) it follows that

$$g_2\big(g_1(z)\big) = \frac{C}{g_1(z)} \tag{3.16}$$

and by the change $g_1(z) \to z$, we have that

$$g_2(z) = \frac{C}{z}. \tag{3.17}$$

Substituting (3.17) into the eight identity in (3.8) we obtain

$$g_1(z)g_1\left(\frac{C}{z}\right) = C. \tag{3.18}$$

Using the change $h_1(z) = (1/\sqrt{C})g_1(\sqrt{C}z)$ we see that the function h_1 satisfies the following relationships:

$$h_1(z)h_1\left(\frac{1}{z}\right) = 1, \quad h_1(z) = h_1^{-1}(z). \tag{3.19}$$

From this we see that the function h_1 satisfies the conditions of Lemma 2.2, which implies that $h_1(z) = 1/z$. Hence $g_1(z) = C/z$ and consequently

$$g_3(z) = \frac{C}{z}, \qquad g_2(z) = \frac{C}{z}, \tag{3.20}$$

form which the result follows.

Assume now that $k = 4$. Then (3.5)–(3.7) are

$$\begin{array}{llll} g_4\big(g_1(z)\big) = z, & g_4\big(g_2(z)\big)g_1(z) = C, & g_4\big(g_3(z)\big)g_2(z) = C, & g_4\big(g_4(z)\big)g_3(z) = C, \\ g_3\big(g_1(z)\big)g_4(z) = C, & g_3\big(g_2(z)\big) = z, & g_3\big(g_3(z)\big)g_1(z) = C, & g_3\big(g_4(z)\big)g_2(z) = C, \\ g_2\big(g_1(z)\big)g_3(z) = C, & g_2\big(g_2(z)\big)g_4(z) = C, & g_2\big(g_3(z)\big) = z, & g_2\big(g_4(z)\big)g_1(z) = C, \\ g_1\big(g_1(z)\big)g_2(z) = C, & g_1\big(g_2(z)\big)g_3(z) = C, & g_1\big(g_3(z)\big)g_4(z) = C, & g_1\big(g_4(z)\big) = z. \end{array} \tag{3.21}$$

From (3.21) we have

$$g_1\big(g_4(z)\big) = g_4\big(g_1(z)\big) = z, \qquad g_2\big(g_3(z)\big) = g_3\big(g_2(z)\big) = z, \tag{3.22}$$

which implies

$$g_4(z) = g_1^{-1}(z), \qquad g_3(z) = g_2^{-1}(z), \tag{3.23}$$

and consequently that the functions g_1, g_2, g_3, and g_4 map the interval $(0,\infty)$, "$1-1$" and onto itself. Also, we have

$$g_i(g_j(z)) = g_j(g_i(z)), \quad \text{when } i+j \neq 5,\ i \neq j, \tag{3.24}$$

$$\begin{aligned} g_4(g_4(z))g_3(z) &= C, \qquad & g_3(g_3(z))g_1(z) &= C, \\ g_2(g_2(z))g_4(z) &= C, \qquad & g_1(g_1(z))g_2(z) &= C. \end{aligned} \tag{3.25}$$

From (3.24) and (3.25), it follows that

$$g_4^{[2]} \circ g_2(z) = g_3^{[2]} \circ g_4(z) = g_2^{[2]} \circ g_1(z) = g_1^{[2]} \circ g_3(z) = \frac{C}{z}. \tag{3.26}$$

From (3.24) and (3.26), it follows that

$$g_3^{[3]}(z) = g_4(z), \qquad g_4^{[3]}(z) = g_2(z), \qquad g_1^{[3]}(z) = g_3(z), \qquad g_2^{[3]}(z) = g_1(z). \tag{3.27}$$

For example, if we replace in the first equality in (3.26) z by $g_3(z)$ and use (3.22) and (3.24), we obtain

$$g_4^{[2]}(z) = g_4^{[2]} \circ g_2 \circ g_3(z) = g_3^{[2]} \circ g_4 \circ g_3(z) = g_3^{[3]} \circ g_4(z), \tag{3.28}$$

from which it follows that

$$g_3^{[3]}(z) = g_4(z). \tag{3.29}$$

Using (3.27), we obtain that

$$g_i^{[81]}(z) = g_i(z), \quad i \in \{1,2,3,4\}, \tag{3.30}$$

and consequently

$$g_i^{[80]}(z) = z, \quad i \in \{1,2,3,4\}. \tag{3.31}$$

By Lemma 2.1, we have that

$$g_i(z) = z \quad \text{or} \quad g_i^{[2]}(z) = z. \tag{3.32}$$

If $g_1(z) = z$, then $g_4(z) = z$. From this and the fourth equality in (3.21), it follows that

$$g_3(z) = \frac{C}{z}. \tag{3.33}$$

On the other hand, from (3.33) and the seventh equality in (3.21), it follows that

$$g_3\left(\frac{C}{z}\right) = \frac{C}{z}, \tag{3.34}$$

which implies that $g_3(z) = z$, a contradiction. Similar, if $g_i(z) = z$ for some $i \in \{2,3,4\}$, we obtain a contradiction.

Hence, $g_i^{[2]}(z) = z$, for every $i \in \{1,2,3,4\}$. From this and (3.25) it follows that

$$g_1(z) = g_2(z) = g_3(z) = g_4(z) = \frac{C}{z}, \tag{3.35}$$

finishing the proof of the theorem. □

Remark 3.1. We believe that Theorem 1.2 can be extended in a natural way for every $k \geq 2$, and that the proof of the corresponding result can be obtained by some modifications of the proof of Theorem 1.2. We leave the solution of the problem to the reader.

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