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Research Article

On the Recursive Sequence $x_{n+1} = A + x_n^p/x_{n-1}^r$

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The paper considers the boundedness character of positive solutions of the difference equation $x_{n+1} = A + x_n^p/x_{n-1}^r$, $n \in \mathbb{N}_0$, where A, p, and r are positive real numbers. It is shown that (a) If $p^2 \geq 4r > 4$, or $p \geq 1+r$, $r \leq 1$, then this equation has positive unbounded solutions; (b) if $p^2 < 4r$, or $2\sqrt{r} \leq p < 1+r$, $r \in (0,1)$, then all positive solutions of the equation are bounded. Also, an analogous result is proved regarding positive solutions of the max type difference equation $x_{n+1} = \max\{A, x_n^p/x_{n-1}^r\}$, where A, p, $q \in (0,\infty)$.

1. Introduction

Recently, there has been an intense interest in studying nonlinear and rational difference equations (cf., [1–15] and the references therein).

In our opinion, it is of paramount importance to investigate not only rational difference equations, but also those equations which contain powers of arbitrary positive degrees.

Here, we investigate such an equation, namely, we study positive solutions of the following difference equation:

$$x_{n+1} = A + \frac{x_n^p}{x_{n-1}^r}, \quad n \in \mathbb{N}_0, \tag{1.1}$$

where A, p, and r are positive numbers.

Our aim here is to give a complete picture regarding the boundedness character of the positive solutions of (1.1). Our results extend those ones in paper [13], in which the case $p = r$ was considered. Beside some modifications of the main ideas from [13], we also use some methods which do not appear in [13] since we have some cases for which this is not possible, in particular, when $2\sqrt{r} \le p < 1+r$ and $r \in (0,1)$.

For some other closely related equations, see, for example, [1–13] and the references therein.

In the last section we study the boundedness character of the following difference equation:

$$x_{n+1} = \max\left\{A, \frac{x_n^p}{x_{n-1}^r}\right\}, \quad n \in \mathbb{N}_0, \tag{1.2}$$

where $A, p, q \in (0,\infty)$, which can be considered as a natural counterpart of (1.1) for the case of max type difference equations.

2. Boundedness character of (1.1)

In this section, we investigate the boundedness character of the positive solutions of (1.1).

2.1. Case $p^2 \ge 4r > 4$. Here, we investigate the positive solutions of (1.1) for the case $p^2 \ge 4r > 4$.

Theorem 2.1. *Assume that $p^2 \ge 4r > 4$. Then (1.1) has positive unbounded solutions.*

Proof. First, note that for every solution of (1.1), the following inequality holds:

$$x_{n+1} > \frac{x_n^p}{x_{n-1}^r}, \quad n \in \mathbb{N}_0. \tag{2.1}$$

Let $y_n = \ln x_n$. Taking the logarithm of (2.1), it follows that

$$y_{n+1} - p y_n + r y_{n-1} > 0. \tag{2.2}$$

Notice that the roots of the polynomial

$$P(\lambda) = \lambda^2 - p\lambda + r \tag{2.3}$$

are

$$\lambda_{1,2} = \frac{p \pm \sqrt{p^2 - 4r}}{2}. \tag{2.4}$$

Since $p^2 \ge 4r > 4$, we have that $\lambda_1 > 1$. On the other hand, we have that

$$\lambda_2 = \frac{2r}{p + \sqrt{p^2 - 4r}} > 0. \tag{2.5}$$

Hence, if $p^2 \ge 4r > 4$, both roots of $P(\lambda)$ are positive.

Now, note that (2.2) can be written in the following form:

$$y_{n+1} - \lambda_1 y_n - \lambda_2 (y_n - \lambda_1 y_{n-1}) > 0. \tag{2.6}$$

If we turn back to the variable x_n, we obtain

$$\frac{x_{n+1}}{x_n^{\lambda_1}} > \left(\frac{x_n}{x_{n-1}^{\lambda_1}}\right)^{\lambda_2}. \tag{2.7}$$

From (2.7), it follows that

$$\frac{x_n}{x_{n-1}^{\lambda_1}} > \left(\frac{x_0}{x_{-1}^{\lambda_1}}\right)^{\lambda_2^n}. \tag{2.8}$$

Choose x_{-1} and x_0 so that

$$x_0 > 1, \qquad x_0 = x_{-1}^{\lambda_1}. \tag{2.9}$$

From this and (2.8), it follows that

$$x_n > \left(\frac{x_0}{x_{-1}^{\lambda_1}}\right)^{\lambda_2^n} x_{n-1}^{\lambda_1} = x_{n-1}^{\lambda_1} > \cdots > x_0^{\lambda_1^n}, \tag{2.10}$$

and consequently,

$$x_n > x_0^{\lambda_1^n}, \quad n \in \mathbb{N}. \tag{2.11}$$

Letting $n \to \infty$ in (2.11), it follows that

$$x_n \longrightarrow +\infty, \quad \text{as } n \longrightarrow \infty, \tag{2.12}$$

from which the result follows. □

2.2. Case $p^2 < 4r$. Here, we investigate the behavior of the positive solutions of (1.1) for the case $p^2 < 4r$.

THEOREM 2.2. *Assume that $p^2 < 4r$. Then all positive solutions of (1.1) are bounded.*

Proof. From (1.1), we have that for every $k \in \mathbb{N}$, the following holds:

$$x_{n+1} = A + \frac{x_n^p}{x_{n-1}^r} = A + \left(\frac{A}{x_{n-1}^{r/p}} + \frac{x_{n-1}^{p-r/p}}{x_{n-2}^r}\right)^p \tag{2.13}$$

$$= A + \left(\frac{A}{x_{n-1}^{r/p}} + \left(\frac{x_{n-1}}{x_{n-2}^{r/(p-r/p)}}\right)^{p-r/p}\right)^p$$

$$= A + \left(\frac{A}{x_{n-1}^{r/p}} + \left(\frac{A}{x_{n-2}^{r/(p-r/p)}} + \frac{x_{n-2}^{p-r/(p-r/p)}}{x_{n-3}^r}\right)^{p-r/p}\right)^p$$

$$= A + \left(\frac{A}{x_{n-1}^{r/p}} + \left(\frac{A}{x_{n-2}^{r/(p-r/p)}} + \left(\frac{x_{n-2}}{x_{n-3}^{r/(p-r/(p-r/p))}}\right)^{p-r/(p-r/p)}\right)^{p-r/p}\right)^p$$

$$= \cdots$$

$$= A + \left(\frac{A}{x_{n-1}^{r/p}} + \left(\frac{A}{x_{n-2}^{r/(p-r/p)}} + \left(\cdots + \left(\frac{A}{x_{n-k}^{p_k}} + \frac{x_{n-k}^{p-p_k}}{x_{n-k-1}^r}\right)^{p-p_{k-1}} \cdots\right)^{p-r/(p-r/p)}\right)^{p-r/p}\right)^p, \tag{2.14}$$

where the sequence p_k is defined by

$$p_{k+1} = \frac{r}{p - p_k}, \qquad p_0 = 0. \tag{2.15}$$

First, assume that $p^2 \le r$, then from (2.13), it follows that for $n \ge 3$,

$$x_{n+1} = A + \left(\frac{A}{x_{n-1}^{r/p}} + \frac{1}{x_{n-1}^{r/p-p} x_{n-2}^r}\right)^p \le A + \left(\frac{1}{A^{r/p-1}} + \frac{1}{A^{r/p-p+r}}\right)^p, \tag{2.16}$$

from which the boundedness follows in this case.

Now, assume that $p^2 > r$. We show that there is a $k_0 \in \mathbb{N}$ such that $p_{k_0-1} < p$ and $p_{k_0} \ge p$. Assume, to the contrary, that $p_k < p$ for every $k \in \mathbb{N}_0$. Since $0 = p_0 < p_1 = r/p$, and the function $f(x) = r/(p-x)$ is strictly increasing for $x < p$, we have that the sequence p_k is strictly increasing. Since the sequence p_k is bounded above by p, it converges, say, to p^*, and it is a solution of the equation

$$x^2 - px + r = 0. \tag{2.17}$$

However, in view of the assumption $p^2 < 4r$, the equation does not have real solutions. Hence, there is the least $k_0 \in \mathbb{N}$ such that $p_{k_0-1} < p$ and $p_{k_0} \ge p$. From this, (2.14) with

$k = k_0$, and by using the fact $x_n > A$ for $n \geq 1$, it follows that

$$
\begin{aligned}
x_{n+1} &= A + \left(\frac{A}{x_{n-1}^{r/p}} + \left(\frac{A}{x_{n-2}^{r/(p-r/p)}} + \left(\cdots + \left(\frac{A}{x_{n-k_0}^{p_{k_0}}} + \frac{1}{x_{n-k_0}^{p_{k_0}-p} x_{n-k_0-1}^{r}} \right)^{p-p_{k_0-1}} \cdots \right)^{p-r/(p-r/p)} \right)^{p-r/p} \right)^{p} \\
&\leq A + \left(\frac{1}{A^{r/p-1}} + \left(\frac{1}{A^{r/(p-r/p)-1}} + \left(\cdots + \left(\frac{1}{A^{p_{k_0}-1}} + \frac{1}{A^{p_{k_0}-p+r}} \right)^{p-p_{k_0-1}} \cdots \right)^{p-r/(p-r/p)} \right)^{p-r/p} \right)^{p},
\end{aligned}
\tag{2.18}
$$

for $n \geq k_0 + 2$, finishing the proof of the theorem. □

2.3. Case $p^2 \geq 4r$, $r \leq 1$. Here, we study the case $p^2 \geq 4r$, $r \leq 1$.

Assume first that $p > 1 + r$. Then

$$\lambda_1 = \frac{p + \sqrt{p^2 - 4r}}{2} > \frac{1 + r + |1 - r|}{2} = 1. \tag{2.19}$$

Hence, similar to the proof of Theorem 2.1, we can obtain the following result.

Theorem 2.3. *Assume that $p > 1 + r$, $r \leq 1$. Then (1.1) has positive unbounded solutions.*

The next theorem concerns the case $p = r + 1$, $r \in (0, 1]$.

Theorem 2.4. *Assume that $p = r + 1$ and $r \in (0, 1]$. Then (1.1) has positive unbounded solutions.*

Proof. Let $x_0 > x_{-1}$. Note that (1.1) in this case is

$$x_{n+1} = A + \frac{x_n^{r+1}}{x_{n-1}^{r}}. \tag{2.20}$$

Equation (2.20) can be written in the form

$$\frac{x_{n+1}}{x_n} = \frac{A}{x_n} + \left(\frac{x_n}{x_{n-1}} \right)^{r}, \tag{2.21}$$

from which it follows that

$$\frac{x_{n+1}}{x_n} > \left(\frac{x_n}{x_{n-1}} \right)^{r} > \cdots > \left(\frac{x_0}{x_{-1}} \right)^{r^{n+1}} > 1, \quad n \in \mathbb{N}_0. \tag{2.22}$$

Hence, $x_{n+1} > x_n$ for $n \in \mathbb{N}_0$. Assume that x_n is bounded. Then, there is a finite positive $\lim_{n\to\infty} x_n = c$. Letting $n \to \infty$ in (2.20), we obtain $c = A + c$, which is a contradiction. Hence, all the solutions of (2.20) with $x_0 > x_{-1}$ are unbounded. □

Now, we assume that $2\sqrt{r} \leq p < 1 + r$ and $r \in (0, 1)$. The following theorem holds true.

Theorem 2.5. *Assume that $2\sqrt{r} \leq p < 1 + r$ and $r \in (0, 1)$. Then, every positive solution of (1.1) is bounded.*

Proof. First, note that in this case, both roots of the polynomial $P(\lambda)$ are real, and moreover,

$$0<\lambda_2<\lambda_1<1. \tag{2.23}$$

Further, (1.1) can be written in the following form:

$$x_{n+1}=A+\frac{x_n^{\lambda_1+\lambda_2}}{x_{n-1}^{\lambda_1\lambda_2}}. \tag{2.24}$$

From (2.24), it follows that

$$\frac{x_{n+1}}{x_n^{\lambda_1}}=\frac{A}{x_n^{\lambda_1}}+\left(\frac{x_n}{x_{n-1}^{\lambda_1}}\right)^{\lambda_2}\le A^{1-\lambda_1}+\left(\frac{x_n}{x_{n-1}^{\lambda_1}}\right)^{\lambda_2}, \tag{2.25}$$

for $n\in\mathbb{N}$.

Let

$$y_n=\frac{x_n}{x_{n-1}^{\lambda_1}}. \tag{2.26}$$

Let $(z_n)_{n\in\mathbb{N}}$ be the solution of the difference equation

$$z_n=A^{1-\lambda_1}+z_{n-1}^{\lambda_2}, \tag{2.27}$$

with $z_0=y_0$. By (2.27) and induction, we see that $y_n\le z_n$, $n\in\mathbb{N}$. Hence, it is enough to prove that the sequence $(z_n)_{n\in\mathbb{N}}$ is bounded. Since the function

$$f(x)=A^{1-\lambda_1}+x^{\lambda_2},\quad x\in(0,\infty), \tag{2.28}$$

is increasing and concave (we use here the condition $\lambda_2\in(0,1)$), it follows that there is a unique fixed point x^* of the equation $f(x)=x$ and that the function f satisfies the condition

$$(f(x)-x)(x-x^*)<0,\quad x\in(0,\infty)\setminus\{x^*\}. \tag{2.29}$$

Using this fact, it is easy to see that if $z_0\in(0,x^*]$, the sequence $(z_n)_{n\in\mathbb{N}}$ is nondecreasing and bounded above by x^*, and if $z_0\ge x^*$, it is nonincreasing and bounded below by x^*. Thus, for every $z_0\in(0,\infty)$, the sequence $(z_n)_{n\in\mathbb{N}}$ is bounded. Hence, there is a positive constant M such that

$$x_n\le Mx_{n-1}^{\lambda_1},\quad n\in\mathbb{N}. \tag{2.30}$$

From (2.30), it follows that

$$x_n\le M^{(1-\lambda_1^n)/(1-\lambda_1)}x_0^{\lambda_1^n}\le\max\{1,M\}^{1/(1-\lambda_1)}\max\{1,x_0\}, \tag{2.31}$$

from which the result follows. □

In the next theorem, we summarize Theorems 2.1–2.5 into a result regarding the boundedness character of positive solutions of (1.1).

Theorem 2.6. *Consider (1.1) where A, p, and r are positive real numbers. The following statements are true.*

(a) *If $p^2 \geq 4r > 4$, or $p \geq 1+r$, $r \in (0,1]$, then (1.1) has positive unbounded solutions.*

(b) *If $p^2 < 4r$, or $2\sqrt{r} \leq p < 1+r$, $r \in (0,1)$, then all positive solutions of (1.1) are bounded.*

The following result gives some more information of some unbounded solutions of (1.1) for the case $p = r+1$, $r \in (0,1)$.

Theorem 2.7. *Assume that $p = r+1$ and $r \in (0,1)$. Then every positive solution of (2.20) diverging to $+\infty$ satisfies the following condition*

$$\lim_{n\to\infty} \frac{x_n}{x_{n-1}} = 1. \tag{2.32}$$

Proof. First, note that we can assume that $A = 1$ in (2.20) since, by using the change $x_n = Au_n$, the equation is reduced to this case.

Hence, from (2.20), we have that

$$\frac{x_{n+1}}{x_n} \leq 1 + \left(\frac{x_n}{x_{n-1}}\right)^r, \quad n \in \mathbb{N}. \tag{2.33}$$

In view of the assumption $r \in (0,1)$, similar to the proof of Theorem 2.5, the boundedness of the sequence x_n/x_{n-1} can be proved.

Since x_n tend to $+\infty$, it follows that

$$\lim_{n\to\infty} \frac{1}{x_n} = 0. \tag{2.34}$$

Hence, the sequence $\varepsilon_n = 1/x_n$ is a zero sequence. Now, consider the difference equations

$$y_n = y_{n-1}^r, \qquad z_n = \varepsilon + z_{n-1}^r, \tag{2.35}$$

with $x_0/x_{-1} = y_0 = z_0$. Without loss of generality, we may assume that $\varepsilon_n < \varepsilon$ for every $n \in \mathbb{N}_0$. It is easy to see by induction that

$$y_n \leq \frac{x_n}{x_{n-1}} \leq z_n, \quad n \in \mathbb{N}. \tag{2.36}$$

Since $y_n = y_0^{r^n}$, we have that $\lim_{n\to\infty} y_n = 1$. On the other hand, as in Theorem 2.5, we have that z_n converges to the positive solution $x^*(\varepsilon)$ of the equation $x - \varepsilon - x^r = 0$. Because the function $x - \varepsilon - x^r$ is continuous on $\mathbb{R}^2$, it follows that $x^*(\varepsilon) \to x^*(0) = 1$, as $\varepsilon \to 0$. This finishes the proof. □

3. On the equation $x_{n+1} = \max\{A, x_n^p/x_{n-1}^r\}$

In this section, we study the boundedness character of positive solutions of (1.2), where $A, p, r \in (0,\infty)$, which is closely related to (1.1).

The following result is main in this section. Since the proof of the result is similar to the proofs of Theorems 2.1–2.5, we will only point out important differences.

THEOREM 3.1. *Consider (1.2), where A, p, and r are positive real numbers. The following statements are true.*

(a) *If $p^2 \geq 4r > 4$, or $p \geq 1+r$, $r \in (0,1]$, then (1.2) has positive unbounded solutions.*

(b) *If $p^2 < 4r$, or $2\sqrt{r} \leq p < 1+r$, $r \in (0,1)$, then all positive solutions of (1.2) are bounded.*

Proof. (a) The proof of this statement is a direct consequence of the proofs of Theorems 2.1, 2.3, and 2.4. It should be only noticed that for a solution (x_n) of (1.2), inequalities (2.1) and (2.22) hold for the corresponding values of parameters p and r in Theorems 2.1, 2.3, and 2.4.

(b) Assume first that $p^2 < 4r$. Similar to (2.14), it can be proved that

$$x_{n+1} = \max\left\{A, \left\{\frac{A}{x_{n-1}^{r/p}}, \left\{\frac{A}{x_{n-2}^{r/(p-r/p)}}, \left\{\cdots\left\{\frac{A}{x_{n-k}^{p_k}}, \frac{x_{n-k}^{p-p_k}}{x_{n-k-1}^{r}}\right\}^{p-p_{k-1}}\cdots\right\}^{p-r/(p-r/p)}\right\}^{p-r/p}\right\}^{p}\right\}, \tag{3.1}$$

where $(p_k)_{k\in\mathbb{N}_0}$ is defined in (2.15), from which the boundedness of (x_n) follows by making use of arguments similar to those ones in the proof of Theorem 2.2.

Assume that $2\sqrt{r} \leq p < 1+r$ and $r \in (0,1]$. Let λ_1 and λ_2 be as in the proof of Theorem 2.5. From (1.2), it follows that

$$\frac{x_{n+1}}{x_n^{\lambda_1}} = \max\left\{\frac{A}{x_n^{\lambda_1}}, \left(\frac{x_n}{x_{n-1}^{\lambda_1}}\right)^{\lambda_2}\right\} \leq \max\left\{A^{1-\lambda_1}, \left(\frac{x_n}{x_{n-1}^{\lambda_1}}\right)^{\lambda_2}\right\}, \tag{3.2}$$

for $n \in \mathbb{N}$.

Let $u_n = x_n/x_{n-1}^{\lambda_1}$ and $(v_n)_{n\in\mathbb{N}_0}$ be the solution of the difference equation

$$v_n = \max\left\{A^{1-\lambda_1}, v_{n-1}^{\lambda_2}\right\}, \quad n \in \mathbb{N}, \tag{3.3}$$

with $v_0 = u_0$. Similar to the proof of Theorem 2.5, it follows that the sequence (v_n) is bounded, and as a consequence that solutions of (1.2) are bounded in the case. □

Remark 3.2. Theorem 3.1 extends Theorems 1 and 2 in [14].

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