Hindawi Publishing Corporation
Discrete Dynamics in Nature and Society
Volume 2007, Article ID 57491, 25 pages
doi:10.1155/2007/57491

Research Article

Stability of a Second Order of Accuracy Difference Scheme for Hyperbolic Equation in a Hilbert Space

Allaberen Ashyralyev and Mehmet Emir Koksal

Received 7 June 2007; Accepted 16 September 2007

The initial-value problem for hyperbolic equation $d^2u(t)/dt^2 + A(t)u(t) = f(t)$ $(0 \leq t \leq T)$, $u(0) = \varphi, u'(0) = \psi$ in a Hilbert space H with the self-adjoint positive definite operators $A(t)$ is considered. The second order of accuracy difference scheme for the approximately solving this initial-value problem is presented. The stability estimates for the solution of this difference scheme are established.

1. Introduction

It is known (see, e.g., [1, 2]) that various mixed problems for the hyperbolic equations can be reduced to the initial-value problem

$$\begin{gathered} \frac{d^2u(t)}{dt^2} + A(t)u(t) = f(t) \quad (0 \leq t \leq T), \\ u(0) = \varphi, \qquad u'(0) = \psi \end{gathered} \tag{1.1}$$

for differential equation in a Hilbert space H. Here, $A(t)$ are the self-adjoint positive definite operators in H with a t-independent domain $D = D(A(t))$.

A function $u(t)$ is called a solution of the problem (1.1) if the following conditions are satisfied:

(i) $u(t)$ is twice continuously differentiable on the segment $[0,T]$; the derivatives as the endpoints of the segment are understood as the appropriate unilateral derivatives;

(ii) the element $u(t)$ belongs to D for all $t \in [0,T]$, and the function $Au(t)$ is continuous on the segment $[0,T]$;

(iii) $u(t)$ satisfies the equation and the initial conditions (1.1).

A large cycle of works on difference schemes for hyperbolic partial differential equations (see, e.g., [3–6] and the references given therein), in which stability was established under the assumption that the magnitudes of the grid steps τ and h with respect to the time and space variables is connected. In abstract terms this means, in particular, that the condition $\tau \|A_{\tau,h}\| \to 0$ when $\tau \to 0$ is satisfied.

Of great interest is the study of absolute stable difference schemes of a high order of accuracy for hyperbolic partial differential equations, in which stability was established without any assumptions in respect of the grid steps τ and h. The stability inequalities for solutions of the first order of accuracy difference scheme

$$\begin{gathered}
\tau^{-2}(u_{k+1} - 2u_k + u_{k-1}) + A_k u_{k+1} = f_k, \\
A_k = A(t_k), \quad f_k = f(t_k), \quad t_k = k\tau,\ 1 \le k \le N-1,\ N\tau = T, \\
\tau^{-1}(u_1 - u_0) + iA_1^{1/2} u_1 = iA_0^{1/2} u_0 + \psi, \quad u_0 = \varphi
\end{gathered} \tag{1.2}$$

for approximately solving problem (1.1) were established without any assumtions for the first time in the paper [7].

The study of the high order of accuracy of absolute stable difference schemes for approximately solving problem (1.1) in the case of $A(t) = A$ has been studied in the papers [3, 8–11]. The second order of accuracy difference schemes

$$\begin{gathered}
\tau^{-2}(u_{k+1} - 2u_k + u_{k-1}) + Au_k + \frac{\tau^2}{4} A^2 u_{k+1} = f_k, \\
f_k = f(t_k), \quad t_k = k\tau,\ 1 \le k \le N-1, N\tau = T, \\
\tau^{-1}(u_1 - u_0) + iA^{1/2}\left(I + \frac{i\tau}{2}A^{1/2}\right) u_1 = z_1, \\
z_1 = (I + i\tau A^{1/2})\psi + \frac{\tau}{2} f_0 + (iA^{1/2} - \tau A) u_0, \quad f_0 = f(0), u_0 = \varphi, \\
\tau^{-2}(u_{k+1} - 2u_k + u_{k-1}) + \frac{1}{4} A(u_{k+1} + 2u_k + u_{k-1}) = f_k, \\
f_k = f(t_k), \quad t_k = k\tau,\ 1 \le k \le N-1, N\tau = T, \\
\tau^{-1}(u_1 - u_0) + \frac{i}{2} A^{1/2}(u_1 + u_0) = z_1, \\
z_1 = \left(I + \frac{i\tau}{2}A^{1/2}\right)\psi + \frac{\tau}{2} f_0 + \left(iA^{1/2} - \frac{\tau A}{2}\right) u_0, \quad f_0 = f(0), u_0 = \varphi
\end{gathered} \tag{1.3}$$

was presented in the paper [8]. The stability estimates for the solution of these difference schemes; and its first and second order difference derivatives were established. Unfortunately, these difference schemes are generated by the $A^{1/2}$. In paper [9], the first order of accuracy difference scheme

$$\begin{gathered}\tau^{-2}(u_{k+1}-2u_k+u_{k-1})+Au_{k+1}=f_k,\\ f_k=f(t_k),\quad t_k=k\tau,\ 1\le k\le N-1, N\tau=T,\\ \tau^{-1}(u_1-u_0)=\psi,\quad u_0=\varphi,\end{gathered}\tag{1.4}$$

and second order of accuracy difference scheme

$$\begin{gathered}\tau^{-2}(u_{k+1}-2u_k+u_{k-1})+Au_k+\frac{\tau^2}{4}A^2u_{k+1}=f_k,\\ f_k=f(t_k),\quad t_k=k\tau,\ 1\le k\le N-1, N\tau=T,\\ (I+\tau^2A)\tau^{-1}(u_1-u_0)=\frac{\tau}{2}(f_0-Au_0)+\psi,\quad f_0=f(0), u_0=\varphi,\end{gathered}\tag{1.5}$$

$$\begin{gathered}\tau^{-2}(u_{k+1}-2u_k+u_{k-1})+\frac{1}{2}Au_k+\frac{1}{4}A(u_{k+1}+u_{k-1})=f_k,\\ f_k=f(t_k),\quad t_k=k\tau,\ 1\le k\le N-1, N\tau=T,\\ (I+\tau^2A)\tau^{-1}(u_1-u_0)=\frac{\tau}{2}(f_0-Au_0)+\psi,\quad f_0=f(0), u_0=\varphi\end{gathered}\tag{1.6}$$

for approximately solving this initial-value problem were presented. These difference schemes were generated by the integer power of A. The stability estimates for the solution of these difference schemes were established.

In papers [10, 11], the high order of accuracy two-step difference schemes generated by an exact difference scheme or by the Taylor's decomposition on three points for the numerical solutions of this problem was presented. The stability estimates for the solutions of these difference schemes were established. In applications, the stability estimates for the solutions of the high order of accuracy difference schemes of the mixed type boundary value problems for hyperbolic equations were obtained.

We are interested in studying the high order of accuracy two-step difference schemes for the approximate solutions of the problem (1.1) in a Hilbert space H with self-adjoint positive definite operators $A(t)$. In paper [12], second order of accuracy difference scheme

$$\begin{aligned}&\tau^{-2}(u_{k+1}-2u_k+u_{k-1})+A_{k+1/2}4^{-1}(u_{k+1}+u_k)+A_{k+1/2}^{1/2}A_{k-1/2}^{1/2}4^{-1}(u_k+u_{k-1})\\ &\qquad+\tau^{-1}(A_{k-1/2}^{1/2}-A_{k+1/2}^{1/2})A_{k-1/2}^{-1/2}\tau^{-1}(u_k-u_{k-1})\\ &\qquad+2^{-1}\tau^{-1}(A_{k+1}^{1/2}-A_{k}^{1/2})A_{k+1/2}^{-1/2}\tau^{-1}(u_{k+1}-u_k)\\ &\qquad+A_{k+1/2}^{1/2}A_{k-1/2}^{-1/2}2^{-1}\tau^{-1}(A_{k}^{1/2}-A_{k-1}^{1/2})A_{k-1/2}^{-1/2}\tau^{-1}(u_k-u_{k-1})\\ &\quad=2^{-1}(f_{k-1/2}+f_{k+1/2})+2^{-1}(A_{k+1/2}^{1/2}-A_{k-1/2}^{1/2})A_{k-1/2}^{-1/2}f_{k-1/2},\quad 1\le k\le N-1,\ u_0=u(0)\\ &\tau^{-1}(u_1-u_0)+\frac{\tau}{2}A_{1/2}2^{-1}(u_1+u_0)+\frac{\tau}{2}(A_{1/2}^{1/2})'A_{1/2}^{-1/2}\tau^{-1}(u_1-u_0)=\frac{\tau}{2}f_{1/2}+A_{1/2}^{1/2}A_{1/2}^{-1/2}u_0'\end{aligned}\tag{1.7}$$

generated by Crank-Nicholson difference scheme was presented. The following theorems under the same smoothness assumption on $A(t)A^{-1}(0)$ (see, e.g., [12]) on the stability estimates for the solution of this difference scheme and its first and second order difference derivatives were established.

THEOREM 1.1. *Let $u(0) \in D(A^{1/2}(0))$. Then for the solution of the difference scheme (1.7), the stability estimate*

$$\left\|\left\{\frac{u_k - u_{k-1}}{\tau}\right\}_1^{N-1}\right\|_{C_\tau} + \|u^\tau\|_{C_\tau} \leq M\left[\|A^{1/2}(0)u_0\|_H + \|u_0'\|_H + \sum_{s=0}^{N-1}\|f_{s+1/2}\|_H\tau\right] \tag{1.8}$$

holds, where M does not depend on $u_0, u_0', f_{s+1/2}$ $(0 \leq s \leq N-1)$ and τ.

THEOREM 1.2. *Let $u(0) \in D(A(0))$, $u'(0) \in D(A^{1/2}(0))$. Then for the solution of the difference scheme (1.7), the stability estimat*

$$\begin{aligned}
&\left\|\left\{A^{1/2}(0)\frac{u_k - u_{k-1}}{\tau}\right\}_1^{N-1}\right\|_{C_\tau} \\
&\quad + \|A_{k+1/2}4^{-1}(u_{k+1}+u_k) + A_{k+1/2}^{1/2}A_{k-1/2}^{1/2}4^{-1}(u_k+u_{k-1}) \\
&\qquad + \tau^{-1}(A_{k-1/2}^{1/2} - A_{k+1/2}^{1/2})A_{k-1/2}^{-1/2}\tau^{-1}(u_k - u_{k-1}) \\
&\qquad + 2^{-1}\tau^{-1}(A_{k+1}^{1/2} - A_k^{1/2})A_{k+1/2}^{-1/2}\tau^{-1}(u_{k+1} - u_k) \\
&\qquad + A_{k+1/2}^{1/2}A_{k-1/2}^{-1/2}2^{-1}\tau^{-1}(A_k^{1/2} - A_{k-1}^{1/2})A_{k-1/2}^{-1/2}\tau^{-1}(u_k - u_{k-1})\|_H \\
&\quad + \|\{\tau^{-2}(u_{k+1} - 2u_k + u_{k-1})\}_1^{N-1}\|_{C_\tau} \\
&\leq M\left[\|A(0)u_0\|_H + \|A^{1/2}(0)u_0'\|_H + \max_{0\leq s\leq k}\|f_{s+1/2}\|_H + \sum_{s=0}^{n-2}\|f_{s+1/2} - f_{s-1/2}\|_H\right]
\end{aligned} \tag{1.9}$$

holds, where M does not depend on $u_0, u_0', f_{s+1/2}$ $(0 \leq s \leq N-1)$ and τ.

Note that the difference scheme (1.7) for approximately solving problem (1.1) in the case of $A(t) = A$ is (1.6). So, these stability estimates are generalization of the results of paper [9] in the general case of $A(t)$.

In the present paper, the difference scheme (1.5) for approximately solving problem (1.1) in the general case of $A(t)$ is presented. Unfortunately, the stability estimates for $\|\{(u_k - u_{k-1})/\tau\}_1^{N-1}\|_{C_\tau}$ and $\|\{u_k\}_1^{N-1}\|_{C_\tau}$ cannot be obtained for the solution of this difference scheme under the same conditions of Theorem 1.1. Nevertheless, the stability estimates for $\|\{A_{k-1/2}^{1/2}(u_k - u_{k-1}/\tau)\}_1^{N-1}\|_{C_\tau}$, $\|\{A_k u_k\}_1^{N-1}\|_{C_\tau}$ and $\|\{\tau^{-2}(u_{k+1} - 2u_k + u_{k-1})\}_1^{N-1}\|_{C_\tau}$ are obtained for the solution of this difference scheme under the same conditions.

2. The construction of one difference scheme of a second order of accuracy

By papers [13, 14], we have the equivalent initial-value problem for a system of the first order linear differential equations

$$\begin{aligned}&\frac{du(t)}{dt}=iA^{1/2}(t)v(t),\quad 0<t<T,\quad u(0)=u_0,\quad u'(0)=u_0',\\&\frac{dv(t)}{dt}=iA^{1/2}(t)u(t)-A^{-1/2}(t)\left[A^{1/2}(t)\right]'v(t)-iA^{-1/2}(t)f(t).\end{aligned}\tag{2.1}$$

For construction of a two-step difference scheme, we consider the uniform grid space

$$[0,T]_\tau=\{t_k=k\tau,\ 0\le k\le N,\ N\tau=T\}.\tag{2.2}$$

Using the central difference formula for the derivative and (2.1), we can write

$$\begin{gathered}\tau^{-1}\left(u(t_k)-u(t_{k-1})\right)=iA_k^{1/2}v(t_{k-1/2})+o(\tau^2),\quad 1\le k\le N,\\ \tau^{-1}\left(v(t_k)-v(t_{k-1})\right)=iA_k^{1/2}u(t_{k-1/2})-A_k^{-1/2}\left[A_k^{1/2}\right]'v(t_{k-1/2})\\ -iA_k^{-1/2}f_k+o(\tau^2),\quad 1\le k\le N,\qquad v_0=-iA_0^{-1/2}u_0',\end{gathered}\tag{2.3}$$

where

$$\begin{gathered}A_k^{1/2}=A^{1/2}(t_{k-1/2}),\qquad \left[A_k^{1/2}\right]'=(A')^{1/2}(t_{k-1/2}),\qquad f_k=f(t_{k-1/2}),\\ t_{k-1/2}=\left(t_k-\frac{\tau}{2}\right),\qquad A_0=A(0).\end{gathered}\tag{2.4}$$

Using the Taylor expansion, we can write

$$\begin{aligned}\tau^{-1}\left(u(t_k)-u(t_{k-1})\right)&=u'(t_k)-\frac{\tau}{2}u''(t_k)+o(\tau^2),\quad 1\le k\le N,\\ w(t_{k-1/2})&=\frac{1}{2}\left(w(t_k)+w(t_{k-1})\right)+o(\tau^2),\\ w(t_{k-1/2})&=\left(w(t_k)-\frac{\tau}{2}w'(t_k)\right)+o(\tau^2).\end{aligned}\tag{2.5}$$

Applying (2.5), and the formulas

$$u'(t_k)=iA^{1/2}(t_k)v(t_k),\qquad u''(t_k)=f(t_k)-A(t_k)u(t_k),\tag{2.6}$$

we get

$$
\begin{aligned}
&\tau^{-1}(u(t_k)-u(t_{k-1}))\\
&\quad=\frac{\tau}{2}A_k u(t_k)+i\Big(A_k^{1/2}+\frac{\tau}{2}(A_k^{1/2})'\Big)v(t_k)-\frac{\tau}{2}f_k+o(\tau^2),\quad 1\le k\le N,\\
&\tau^{-1}(v(t_k)-v(t_{k-1}))\\
&\quad=iA_k^{1/2}u(t_k)+\frac{\tau}{2}A_k v(t_k)-2^{-1}A_k^{-1/2}(A_k^{1/2})'(v(t_k)+v(t_{k-1}))\\
&\qquad-iA_k^{-1/2}f_k+o(\tau^2),\quad 1\le k\le N,\qquad v_0=-iA_0^{-1/2}u_0',\\
&v(t_k)=-iA_{k+1/2}^{-1/2}\Big(\tau^{-1}(u(t_k)-u(t_{k-1}))-\frac{\tau}{2}A_k u(t_k)+\frac{\tau}{2}f_k\Big)+o(\tau^2),\quad 1\le k\le N.
\end{aligned}
\tag{2.7}
$$

Neglecting the small terms $o(\tau^2)$, we obtain the following difference scheme:

$$
\begin{aligned}
&\tau^{-1}(u_k-u_{k-1})=\frac{\tau}{2}A_k u_k+i\Big(A_k^{1/2}+\frac{\tau}{2}(A_k^{1/2})'\Big)v_k-\frac{\tau}{2}f_k,\quad u_0=u(0),\ 1\le k\le N\\
&\tau^{-1}(v_k-v_{k-1})=iA_k^{1/2}u_k+\frac{\tau}{2}A_k v_k-2^{-1}A_k^{-1/2}(A_k^{1/2})'(v_k+v_{k-1})-iA_k^{-1/2}f_k,\quad 1\le k\le N,\\
&\qquad v_k=-iA_{k+1/2}^{-1/2}\Big(\tau^{-1}(u_k-u_{k-1})-\frac{\tau}{2}A_k u_k+\frac{\tau}{2}f_k\Big),\quad 1\le k\le N,\\
&\qquad v_0=-iA_0^{-1/2}u_0'.
\end{aligned}
\tag{2.8}
$$

for the approximate solution of the initial-value problem (1.1).

Using (2.8) and eliminating v_k, collecting u_k on the left side and t_k on the right side of the equation, and rearranging the terms in (2.8), we obtain two-step difference scheme of a second order of accuracy

$$
\begin{aligned}
&\tau^{-2}(u_{k+1}-2u_k+u_{k-1})\\
&\quad=\Big[A_{k+1}^{1/2}+\frac{\tau}{2}(A_{k+1}^{1/2})'\Big]\Big\{-A_{k+1}^{1/2}u_{k+1}+\Big[\frac{\tau}{2}A_{k+1}-\frac{1}{2}A_{k+1}^{-1/2}(A_{k+1}^{1/2})'\Big]\\
&\qquad\qquad\times A_{k+3/2}^{-1/2}\Big[\tau^{-1}(u_{k+1}-u_k)-\frac{\tau}{2}A_{k+1}u_{k+1}+\frac{\tau}{2}f_{k+1}\Big]+A_{k+1}^{-1/2}f_{k+1}\Big\}\\
&\quad-\Big\{\Big[A_{k+1}^{1/2}+\frac{\tau}{2}(A_{k+1}^{1/2})'\Big]A_{k+1}^{-1/2}(A_{k+1}^{1/2})'A_{k+1/2}^{-1/2}\\
&\qquad-\tau^{-1}\Big[(A_{k+1}^{1/2}-A_k^{1/2})+\frac{\tau}{2}((A_{k+1}^{1/2})'-(A_k^{1/2})')\Big]\Big\}A_{k+1/2}^{-1/2}\\
&\quad\times\Big[\tau^{-1}(u_k-u_{k-1})-\frac{\tau}{2}A_k u_k+\frac{\tau}{2}f_k\Big]+2^{-1}(A_{k+1}u_{k+1}-A_k u_k)\\
&\quad-2^{-1}(f_{k+1}-f_k),\quad 1\le k\le N-1,\qquad u_0=u(0),\\
&\tau^{-1}(u_1-u_0)+\frac{\tau}{2}A_{1/2}2^{-1}(u_1+u_0)+\frac{\tau}{2}(A_{1/2}^{1/2})'A_{1/2}^{-1/2}\tau^{-1}(u_1-u_0)=\frac{\tau}{2}f_1+A_{1/2}^{1/2}A_{1/2}^{-1/2}u_0'
\end{aligned}
\tag{2.9}
$$

for the approximate solution of the initial-value problem (1.1). Note that the difference scheme (2.9) for approximately solving problem (1.1) in the case of $A(t) = A$ is (1.5).

Let us establish the formula for the solution of this difference scheme (2.9).

Making the transformation $\eta_k = u_k + v_k$ and $\mu_k = u_k - v_k$ in (2.8), we obtain the following system of the difference equations:

$$\begin{gathered}
\tau^{-1}(\eta_k - \eta_{k-1}) = \left(iA_k^{1/2} + \frac{\tau}{2}A_k\right)\eta_k + \varphi_k^+, \quad 2 \le k \le N, \\
\eta_1 = K(B^+u_0 + C^+u_0' + D^+f_1), \\
\tau^{-1}(\mu_k - \mu_{k-1}) = \left(-iA_k^{1/2} + \frac{\tau}{2}A_k\right)\mu_k + \varphi_k^-, \quad 2 \le k \le N, \\
\mu_1 = K(B^-u_0 + C^-u_0' + D^-f_1), \\
\varphi_k^{\pm} = i\frac{\tau}{2}(A_k^{1/2})'v_k - \frac{\tau}{2}f_k \mp A_k^{-1/2}(A_k^{1/2})'2^{-1}(v_k + v_{k-1}) \mp iA_k^{-1/2}f_k, \\
v_k = -iA_{k+1/2}^{-1/2}\left(\tau^{-1}(u_k - u_{k-1}) - \frac{\tau}{2}A_k u_k + \frac{\tau}{2}f_k\right), \quad 2 \le k \le N,
\end{gathered} \tag{2.10}$$

where

$$\begin{aligned}
K &= \left[1 + \frac{\tau^4}{4}A_1^2 + \frac{\tau}{2}A_1^{-1/2}(A_1^{1/2})' + \frac{\tau^3}{2}A_1^{1/2}(A_1^{1/2})'\right]^{-1}, \\
B^{\pm} &= 1 - \frac{\tau^2}{2}A_1 + \frac{\tau}{2}A_1^{-1/2}(A_1^{1/2})' \pm i\tau A_1^{1/2}, \\
C^{\pm} &= \tau A_1^{1/2}A_0^{-1/2} - \frac{\tau^3}{4}(A_1^{1/2})'A_1^{-1/2}(A_1^{1/2})'A_0^{-1/2} \mp iA_0^{-1/2} \\
&\qquad \pm i\frac{\tau}{2}A_1^{-1/2}(A_1^{1/2})'A_0^{-1/2} \pm i\frac{\tau^2}{2}A_1A_0^{-1/2} \mp i\frac{\tau^3}{4}A_1^{1/2}(A_1^{1/2})'A_0^{-1/2}, \\
D^{\pm} &= \frac{\tau^4}{4}A_1 - \frac{\tau^3}{4}A_1^{-1/2}(A_1^{1/2})' + \frac{3}{2}\tau^2 + \frac{\tau^3}{2}(A_1^{1/2})'A_1^{-1/2} \mp i\tau A_1^{-1/2}.
\end{aligned} \tag{2.11}$$

From this, it follows the system of recursion formulas

$$\begin{aligned}
\eta_k &= \left(I - \frac{\tau^2}{2}A_k - i\tau A_k^{1/2}\right)^{-1}\eta_{k-1} + \left(I - \frac{\tau^2}{2}A_k - i\tau A_k^{1/2}\right)^{-1}\varphi_k^+, \quad 2 \le k \le N, \\
\mu_k &= \left(I - \frac{\tau^2}{2}A_k + i\tau A_k^{1/2}\right)^{-1}\mu_{k-1} + \left(I - \frac{\tau^2}{2}A_k + i\tau A_k^{1/2}\right)^{-1}\varphi_k^-, \quad 2 \le k \le N.
\end{aligned} \tag{2.12}$$

Hence,

$$\eta_k = P_k^-(k)\eta_1 + \sum_{m=2}^{k} R_m^-(k)\varphi_m^+, \qquad \mu_k = P_k^+(k)\mu_1 + \sum_{m=2}^{k} R_m^+(k)\varphi_m^-. \tag{2.13}$$

Here,

$$P_k^{\pm}(k) = X_k^{\pm}X_{k-1}^{\pm}\cdots X_2^{\pm}, \qquad R_m^{\pm}(k) = X_k^{\pm}X_{k-1}^{\pm}\cdots X_m^{\pm}, \tag{2.14}$$

where

$$X_k^{\pm} = \left(I - \frac{\tau^2}{2}A_k \pm i\tau A_k^{1/2}\right)^{-1}. \tag{2.15}$$

Then, using the formula $u_k = (1/2)(\eta_k + \mu_k)$, we obtain

$$\begin{aligned} u_k = 2^{-1}\Big\{&[P_k^{+}(k)KB^{-} + P_k^{-}(k)KB^{+}]u_0 + [P_k^{+}(k)KC^{-} + P_k^{-}(k)KC^{+}]u_0' \\ &+ [P_k^{+}(k)KD^{-} + P_k^{-}(k)KD^{+}]f_1 + \sum_{m=2}^{k}[R_m^{+}(k)\varphi_m^{-} + R_m^{-}(k)\varphi_m^{+}]\Big\}. \end{aligned} \tag{2.16}$$

Furthermore, by making the transformation $k-m=s$, we obtain

$$\begin{aligned} u_k = 2^{-1}\Big\{&[P_k^{+}(k)KB^{-} + P_k^{-}(k)KB^{+}]u_0 + [P_k^{+}(k)KC^{-} + P_k^{-}(k)KC^{+}]u_0' \\ &+ [P_k^{+}(k)KD^{-} + P_k^{-}(k)KD^{+}]f_1 + \sum_{s=0}^{k-2}[E_s^{+}(k)\varphi_{k-s}^{-} + E_s^{-}(k)\varphi_{k-s}^{+}]\Big\}, \end{aligned} \tag{2.17}$$

where

$$\begin{aligned} \varphi_{k-s}^{\pm} &= i\left[\pm\frac{\tau}{2}A_{k-s}^{-1/2}\left(A_{k-s}^{1/2}\right)' - i\frac{\tau^2}{2}\left(A_{k-s}^{1/2}\right)'\right]A_{k-s+1/2}^{-1/2} \\ &\quad\times\left[\tau^{-1}(u_{k-s} - u_{k-s-1}) - \frac{\tau}{2}A_{k-s}u_{k-s} + \frac{\tau}{2}f_{k-s}\right] \\ &\quad\pm i\frac{\tau}{2}A_{k-s}^{-1/2}\left(A_{k-s}^{1/2}\right)'A_{k-s-1/2}^{-1/2}\left[\tau^{-1}(u_{k-s-1} - u_{k-s-2}) - \frac{\tau}{2}A_{k-s-1}u_{k-s-1} + \frac{\tau}{2}f_{k-s-1}\right] \\ &\quad+\left(-\frac{\tau^2}{2} \mp i\tau A_{k-s}^{-1/2}\right)f_{k-s}, \\ E_s^{\pm}(k) &= X_k^{\pm}X_{k-1}^{\pm}\cdots X_{k-s}^{\pm}, \qquad E_0^{\pm}(k) = X_k^{\pm}. \end{aligned} \tag{2.18}$$

Finally, from the last formula, it follows that

$$\begin{aligned} &A_{k-1/2}^{1/2}\tau^{-1}(u_k - u_{k-1}) \\ &\quad= A_{k-1/2}^{1/2}(2\tau)^{-1}\Big\{[[P_k^{+}(k) - P_{k-1}^{+}(k-1)]KB^{-} + [P_k^{-}(k) - P_{k-1}^{-}(k-1)]KB^{+}]u_0 \\ &\qquad\qquad + [[P_k^{+}(k) - P_{k-1}^{+}(k-1)]KC^{-} + [P_k^{-}(k) - P_{k-1}^{-}(k-1)]KC^{+}]u_0' \\ &\qquad\qquad + [[P_k^{+}(k) - P_{k-1}^{+}(k-1)]KD^{-} + [P_k^{-}(k) - P_{k-1}^{-}(k-1)]KD^{+}]f_1 \\ &\qquad\qquad + [E_0^{+}(k)\varphi_k^{-} + E_0^{-}(k)\varphi_k^{+}] + \sum_{s=1}^{k-2}[E_s^{+}(k) - E_{s-1}^{+}(k-1)]\varphi_{k-s}^{-} \\ &\qquad\qquad + \sum_{s=1}^{k-2}[E_s^{-}(k) - E_{s-1}^{-}(k-1)]\varphi_{k-s}^{+}\Big\}. \end{aligned} \tag{2.19}$$

In the following section, these formulas will be used to establish the stability inequality of the difference scheme (2.9).

3. Stability of difference scheme (2.9)

First of all, let us give some subsidiary conditions for operators $A(t)$ that will be needed below. Let $A(t)$ be self-adjoint positive definite operators in H with a t-independent domain $D = D(A(t)) : A(t) \geq \delta I > 0$. Then, the following estimates hold:

$$\left\| \tau^{\alpha} A_k^{\alpha/2} \left(I + \frac{\tau^4}{4} A_k^2 \right)^{-1} \right\| \leq 1, \quad \alpha = 0,1,2, \tag{3.1}$$

$$\left\| \tau^{\alpha} A_k^{\alpha/2} \left(I + \frac{\tau^4}{4} A_k^2 \right)^{-1} \right\| \leq (4 - \sqrt{2})\alpha + 4(\sqrt{2} - 3), \quad \alpha = 3,4, \tag{3.2}$$

$$\left\| \tau^{\alpha} A_k^{\alpha/2} \left(I - \frac{\tau^2}{2} A_k \pm i\tau A_k^{1/2} \right)^{-1} \right\| \leq \frac{\alpha^2 - \alpha}{2} + 1, \quad \alpha = 0,1,2. \tag{3.3}$$

Let the operator function $A^{\rho}(t)A^{-\rho}(z)$, $\rho \in [0,2]$ satisfies the conditions

$$\|[A^{\rho}(t) - A^{\rho}(s)]A^{-\rho}(z)\| \leq M_{\rho}\|t - s\|, \tag{3.4}$$

$$\|[(A^{\rho}(t))' - (A^{\rho}(s))']A^{-\rho}(z)\| \leq M_{\rho}\|t - s\|, \tag{3.5}$$

where M_{ρ} is a positive constant independent of t, s, z for $t,s,z \in [0,T]$. From this, it follows that the operator function $A^{\rho}(t)A^{-\rho}(z)$ has a finite variation on $[0,T]$, that is, there exists a number P_{ρ} such that

$$\sum_{k=1}^{N} \|(A^{\rho}(s_k) - A^{\rho}(s_{k-1}))A^{-\rho}(z)\| \leq P_{\rho} \tag{3.6}$$

for any $0 = s_0 < s_1 < \cdots < s_N = T$. Here, P_{ρ} is a positive constant independent of s_0, $s_1,\ldots,s_N$, and z.

Furthermore, let the operator functions $(A^{\rho}(t))'A^{-\rho}(z)$ and $A^{\rho}(p)(A^{r}(t))'A^{-\rho-r}(z)$ satisfy the conditions

$$\|(A^{\rho}(t))'A^{-\rho}(z)\| \leq M_3, \tag{3.7}$$

$$\|A^{\rho}(p)(A^{r}(t))'A^{-\rho-r}(z)\| \leq M_4, \tag{3.8}$$

where M_3 and M_4 are positive constants independent of t, z for $t,z \in [0,T]$ and t, z, p for $t,z,p \in [0,T]$, respectively.

Finally, let $P_k^{\pm}(k) = X_k^{\pm}X_{k-1}^{\pm}\cdots X_2^{\pm}$ and $E_s^{\pm}(k) = X_k^{\pm}X_{k-1}^{\pm}\cdots X_{k-s}^{\pm}$ such that $X_k^{\pm} = (I-(\tau^2/2)A_k \pm i\tau A_k^{1/2})^{-1}$. We have

$$||A_k P_k^{\pm}(k)A_1^{-1}|| \le e^{M_1\sum_{i=1}^{k}\|(A_i^1-A_{i-1}^1)A_0^{-1}\|}, \tag{3.9}$$

$$\left|\left|A_k E_s^{\pm}(k)A_{1,k-s}^{\alpha/2-1}\tau^{\alpha}\right|\right| \le \left(\frac{\alpha^2-\alpha}{2}+1\right)e^{M_1\sum_{i=1}^{k}\|(A_i^1-A_{i-1}^1)A_0^{-1}\|}, \quad \alpha=0,1,2, \tag{3.10}$$

$$||A_{k-1/2}^{1/2}(2\tau)^{-1}[P_k^{\pm}(k)-P_{k-1}^{\pm}(k-1)]A_1^{-\rho}|| \le \frac{3M_{1/2}}{4}e^{M_\rho\sum_{i=1}^{k}\|(A_i^\rho-A_{i-1}^\rho)A_0^{-\rho}\|}, \tag{3.11}$$

$$\begin{aligned}&\left|\left|A_{k-s-1/2}^{1/2}(2\tau)^{-1}\left[E_s^{\pm}(k)-E_{s-1}^{\pm}(k-1)\right]\left(A_{k-s}^{\rho}\right)^{\alpha/2-1}\tau^{\alpha}\right|\right|\\ &\qquad\le \frac{3M_{1/2}}{4}e^{M_\rho\sum_{i=1}^{k}\|(A_i^\rho-A_{i-1}^\rho)A_0^{-\rho}\|}, \quad \alpha=0,1,2.\end{aligned} \tag{3.12}$$

Theorem 3.1. *Let $u(0)\in D(A^{1/2}(0))$ and $f_1\in D((A_1^{1/2})')$. Then, for the solution of the difference scheme (2.9), the stability estimate*

$$\begin{aligned}&\left|\left|\left\{A_{k-1/2}^{1/2}\frac{u_k-u_{k-1}}{\tau}\right\}_1^N\right|\right|_{C_\tau}+||\{A_ku_k\}_1^N||_{C_\tau}\\ &\quad\le M\left[||A(0)u_0||_H+||A^{1/2}(0)u_0'||_H+\max_{1\le s\le N}||f_k||+||\tau^2(A_1^{1/2})'f_1||_H+\sum_{s=1}^{N}||f_s-f_{s-1}||_H\right]\end{aligned} \tag{3.13}$$

holds, where M does not depend on u_0, u_0', f_s $(1\le s\le N)$, and τ.

Proof. Firstly, the estimate $\|\{A_{k-1/2}^{1/2}(u_k-u_{k-1}/\tau)\}_1^N\|_{C_\tau}$ will be obtained. Applying formula (2.19), we can write

$$A_{k-1/2}^{1/2}\tau^{-1}(u_k-u_{k-1}) = J_{1k}+J_{2k}+J_{3k}+J_{4k}+J_{5k}, \tag{3.14}$$

where

$$\begin{aligned}J_{1k} &= A_{k-1/2}^{1/2}(2\tau)^{-1}\{[P_k^+(k)-P_{k-1}^+(k-1)]KB^-+[P_k^-(k)-P_{k-1}^-(k-1)]KB^+\}u_0,\\ J_{2k} &= A_{k-1/2}^{1/2}(2\tau)^{-1}\{[P_k^+(k)-P_{k-1}^+(k-1)]KC^-+[P_k^-(k)-P_{k-1}^-(k-1)]KC^+\}u_0',\\ J_{3k} &= A_{k-1/2}^{1/2}(2\tau)^{-1}\{[P_k^+(k)-P_{k-1}^+(k-1)]KD^-+[P_k^-(k)-P_{k-1}^-(k-1)]KD^+\}f_1,\\ J_{4k} &= A_{k-1/2}^{1/2}(2\tau)^{-1}[E_0^+(k)\varphi_k^-+E_0^-(k)\varphi_k^+],\\ J_{5k} &= A_{k-1/2}^{1/2}(2\tau)^{-1}\left\{\sum_{s=1}^{k-2}[E_s^+(k)-E_{s-1}^+(k-1)]\varphi_{k-s}^-+\sum_{s=1}^{k-2}[E_s^-(k)-E_{s-1}^-(k-1)]\varphi_{k-s}^+\right\}.\end{aligned} \tag{3.15}$$

Now, let us estimate the terms $\|J_{mk}\|_H$, $m=\overline{1,5}$, separately. Let $m=1$. Then applying estimates (3.1), (3.2), (3.4), (3.8), and (3.11), we get

$$\begin{aligned}
\|J_{1k}\|_H &\le 2\|A_{k-1/2}^{1/2}(2\tau)^{-1}[P_k^+(k)-P_{k-1}^+(k-1)]KB^-u_0\|_H\\
&\le 2\|A_{k-1/2}^{1/2}(2\tau)^{-1}[P_k^+(k)-P_{k-1}^+(k-1)]A_1^{-1}\|\|A_1kb^-A_1^{-1}\|\|A_1A_0^{-1}\|\|A_0u_0\|_H\\
&\le 2\|A_{k-1/2}^{1/2}(2\tau)^{-1}[P_k^+(k)-P_{k-1}^+(k-1)]A_1^{-1}\|\\
&\quad\times\left\|A_1\left[1+\frac{\tau^4}{4}A_1^2+\frac{\tau}{2}A_1^{-1/2}(A_1^{1/2})'+\frac{\tau^3}{2}A_1^{1/2}(A_1^{1/2})'\right]^{-1}\right.\\
&\qquad\left.\times\left[1-\frac{\tau^2}{2}A_1+\frac{\tau}{2}A_1^{-1/2}(A_1^{1/2})'-i\tau A_1^{1/2}\right]\right\|\|A_1A_0^{-1}\|\|A_0u_0\|_H\\
&\le\frac{3}{4}(M_{1/2}+1)M_1\left[\frac{1}{1-\tau M_4}\left(1+\frac{5\tau}{4}M_4\right)+\frac{3}{2}\right]e^{M_1P_1}\|A_0u_0\|_H=c_1\|A_0u_0\|_H.
\end{aligned}\tag{3.16}$$

Let $m=2$. Then applying estimates (3.1), (3.2), (3.4), (3.7), (3.8), and (3.11), we get

$$\begin{aligned}
\|J_{2k}\|_H &\le 2\|A_{k-1/2}^{1/2}(2\tau)^{-1}[P_k^+(k)-P_{k-1}^+(k-1)]KC^-u_0'\|_H\\
&\le 2\|A_{k-1/2}^{1/2}(2\tau)^{-1}[P_k^+(k)-P_{k-1}^+(k-1)]A_1^{-1}\|\|A_1KC^-A_0^{-1/2}\|\|A_0^{1/2}u_0'\|_H\\
&\le 2\|A_{k-1/2}^{1/2}(2\tau)^{-1}[P_k^+(k)-P_{k-1}^+(k-1)]A_1^{-1}\|\\
&\quad\times\left\|A_1\left[1+\frac{\tau^4}{4}A_1^2+\frac{\tau}{2}A_1^{-1/2}(A_1^{1/2})'+\frac{\tau^3}{2}A_1^{1/2}(A_1^{1/2})'\right]^{-1}\right.\\
&\qquad\times\left[\tau A_1^{1/2}A_0^{-1/2}-\frac{\tau^3}{4}(A_1^{1/2})'A_1^{-1/2}(A_1^{1/2})'A_0^{-1/2}-iA_0^{-1/2}+i\frac{\tau}{2}A_1^{-1/2}(A_1^{1/2})'A_0^{-1/2}\right.\\
&\qquad\left.\left.+i\frac{\tau^2}{2}A_1A_0^{-1/2}-i\frac{\tau^3}{4}A_1^{1/2}(A_1^{1/2})'A_0^{-1/2}\right]A_0^{-1/2}\right\|\|A_0^{1/2}u_0'\|_H\\
&\le\frac{3}{2}M_{1/2}\left[\left(M_1+\frac{M_3}{2}\right)+\frac{\tau}{4}\frac{M_3^2}{\sqrt{\delta}}+\frac{1}{1-\tau M_4}\right.\\
&\qquad\left.\times\left(\frac{3M_1}{2}+\frac{\tau}{4}M_4(1+3M_1)+\frac{3\tau^2}{8}M_4\left(\frac{M_3^2}{\sqrt{\delta}}+M_4\right)\right)\right]e^{M_1P_1}\|A_0^{1/2}u_0\|_H\\
&=C_2\|A_0^{1/2}u_0\|_H.
\end{aligned}\tag{3.17}$$

Let $m=3$. Then applying estimates (3.1), (3.2), (3.4), (3.7), (3.8), and (3.11), we get

$$\begin{aligned}
\|J_{3k}\|_H &\le 2\|A_{k-1/2}^{1/2}(2\tau)^{-1}[P_k^+(k)-P_{k-1}^+(k-1)]KD^-f_1\|_H\\
&\le 2\|A_{k-1/2}^{1/2}(2\tau)^{-1}[P_k^+(k)-P_{k-1}^+(k-1)]A_1^{-1}\|\\
&\quad\times\left\|A_1\left[1+\frac{\tau^4}{4}A_1^2+\frac{\tau}{2}A_1^{-1/2}(A_1^{1/2})'+\frac{\tau^3}{2}A_1^{1/2}(A_1^{1/2})'\right]^{-1}\right.\\
&\qquad\left.\times\left[\frac{\tau^4}{4}A_1-\frac{\tau^3}{4}A_1^{-1/2}(A_1^{1/2})'+\frac{3\tau^2}{2}+\frac{\tau^3}{2}(A_1^{1/2})'A_1^{-1/2}-i\tau A_1^{-1/2}\right]\right\|\|f_1\|_H
\end{aligned}$$

$$\begin{aligned}&\leq \frac{3}{2}(M_{1/2}+1)+\left[\frac{5}{2}+\frac{\tau}{2}(M_4+M_3)+\frac{5\tau}{4}\frac{M_4}{1-\tau M_4}+\frac{3\tau^2}{4}\frac{M_4(M_4+M_3)}{1-\tau M_4}\right]e^{M_1P_1}\\&\quad\times(\|f_1\|_H+\|\tau^2(A_1^{1/2})'f_1\|_H)=C_3[\|f_1\|_H+\|\tau^2(A_1^{1/2})'f_1\|_H].\end{aligned}\tag{3.18}$$

Let $m=4$. We have that

$$\begin{aligned}J_{4k}&=A_{k-1/2}^{1/2}(2\tau)^{-1}[E_0^+(k)\varphi_k^-+E_0^-(k)\varphi_k^+]\\&=\left[i\frac{1}{4}A_{k-1/2}^{1/2}(X_k^--X_k^+)A_k^{-1/2}(A_k^{1/2})'+\frac{\tau}{4}A_{k-1/2}^{1/2}(X_k^-+X_k^+)(A_k^{1/2})'\right]\\&\quad\times A_{k+1/2}^{-1/2}\frac{u_k-u_{k-1}}{\tau}+i\frac{1}{4}A_{k-1/2}^{1/2}(X_k^--X_k^+)A_k^{-1/2}(A_k^{1/2})'A_{k-1/2}^{-1/2}\frac{u_{k-1}-u_{k-2}}{\tau}\\&\quad+\left[i\frac{\tau}{8}A_{k-1/2}^{1/2}(X_k^+-X_k^-)A_k^{-1/2}+\frac{\tau^2}{8}A_{k-1/2}^{1/2}(X_k^++X_k^-)\right](A_k^{1/2})'A_{k+1/2}^{-1/2}A_ku_k\\&\quad+i\frac{\tau}{8}A_{k-1/2}^{1/2}(X_k^+-X_k^-)A_k^{-1/2}(A_k^{1/2})'A_{k-1/2}^{-1/2}A_{k-1}u_{k-1}\\&\quad+\frac{\tau}{8}A_{k-1/2}^{1/2}[X_k^+(-iA_k^{-1/2}+\tau)+X_k^-(iA_k^{-1/2}+\tau)](A_k^{1/2})'A_{k+1/2}^{-1/2}f_k\\&\quad+A_{k-1/2}^{1/2}(2\tau)^{-1}\left[X_k^+\left(-\frac{\tau^2}{2}+i\tau A_k^{-1/2}\right)+X_k^-\left(-\frac{\tau^2}{2}-i\tau A_k^{-1/2}\right)\right]f_k\\&\quad+A_{k-1/2}^{1/2}i\frac{\tau}{8}(X_k^--X_k^+)A_k^{-1/2}(A_k^{1/2})'A_{k-1/2}^{-1/2}f_{k-1}.\end{aligned}\tag{3.19}$$

Then applying estimates (3.3), (3.4), (3.7), and (3.8), we get

$$\begin{aligned}\|J_{4k}\|_H&\leq\frac{\tau}{2}\|A_{k-1/2}^{1/2}A_k^{-1/2}\|(\|X_k^-\|+1)\|X_k^+\|\|A_k^{1/2}(a_k^{1/2})'A_{k+1/2}^{-1}\|\\&\quad\times\|A_{k+1/2}^{1/2}A_{k-1/2}^{-1/2}\|\left\|A_{k-1/2}^{1/2}\frac{u_k-u_{k-1}}{\tau}\right\|_H+\frac{\tau}{2}\|A_{k-1/2}^{1/2}A_k^{-1/2}\|\|X_k^-\|\|X_k^+\|\\&\quad\times\|A_k^{1/2}(A_k^{1/2})'A_{k-1/2}^{-1}\|\|A_{k-1/2}^{1/2}A_{k-3/2}^{-1/2}\|\left\|A_{k-3/2}^{1/2}\frac{u_k-u_{k-1}}{\tau}\right\|_H\\&\quad+\frac{\tau}{4}(\|A_{k-1/2}^{1/2}A_k^{-1/2}\|\|X_k^+\|+\|A_k^{1/2}\tau X_k^+\|)\|(A_k^{1/2})'A_{k+1/2}^{-1/2}\|\|A_ku_k\|_H\\&\quad+\frac{\tau}{4}\|A_{k-1/2}^{1/2}A_k^{-1/2}\|\|X_k^+\|\|(A_k^{1/2})'A_{k+1/2}^{-1/2}\|\|A_{k-1}u_{k-1}\|_H\\&\quad+\frac{\tau^2}{2}\|A_{k-1/2}^{1/2}A_k^{-1/2}\|(\|X_k^+\|+2\|A_k^{1/2}\tau X_k^+\|)\|(A_k^{1/2})'A_{k+1/2}^{-1/2}\|\|f_k\|_H\\&\quad+\tau\|A_{k-1/2}^{1/2}A_k^{-1/2}\|(2\|X_k^+\|+\|A_k^{1/2}\tau X_k^+\|)\|f_k\|_H\\&\quad+\frac{\tau^2}{2}\|A_{k-1/2}^{1/2}A_k^{-1/2}\|\|X_k^+\|\|(A_k^{1/2})'A_{k+1/2}^{-1/2}\|\|f_{k-1}\|_H\\&\leq\tau C_4\left[\left\|A_{k-1/2}^{1/2}\frac{u_k-u_{k-1}}{\tau}\right\|_H+\left\|A_{k-3/2}^{1/2}\frac{u_{k-1}-u_{k-2}}{\tau}\right\|_H\right.\\&\qquad\left.+\|A_ku_k\|_H+\|A_{k-1}u_{k-1}\|_H+\|f_k\|_H+\|f_{k-1}\|_H+\frac{3}{2}M_{1/2}\|f_k\|_H\right],\end{aligned}\tag{3.20}$$

where

$$C_4 = \max\left\{ (M_{1/2}+1)^2 M_4, \frac{1}{2}(M_{1/2}+1)M_3 \right\}. \tag{3.21}$$

Let $m = 5$. It is clear that

$$J_{5k} = S_{1k} + S_{2k} + S_{3k} + S_{4k} + S_{5k} + S_{6k} + S_{7k}, \tag{3.22}$$

where

$$\begin{aligned}
S_{1k} &= A_{k-1/2}^{1/2}(2\tau)^{-1}\sum_{s=1}^{k-2}\left\{ [E_s^+(k) - E_{s-1}^+(k-1)]\left(-i\frac{\tau}{2}A_{k-s}^{-1/2} + \frac{\tau^2}{2}\right) + [E_s^-(k) - E_{s-1}^-(k-1)] \right.\\
&\qquad\qquad \left. \times\left(i\frac{\tau}{2}A_{k-s}^{-1/2} + \frac{\tau^2}{2}\right)\right\}(A_{k-s}^{1/2})' A_{k-s+1/2}^{-1/2}\tau^{-1}(u_{k-s} - u_{k-s-1}),\\
S_{2k} &= A_{k-1/2}^{1/2}(2\tau)^{-1} i \sum_{s=1}^{k-2}\{ -[E_s^+(k) - E_{s-1}^+(k-1)] + [E_s^-(k) - E_{s-1}^-(k-1)]\}\\
&\quad \times \frac{\tau}{2}A_{k-s}^{-1/2}(A_{k-s}^{1/2})' A_{k-s-1/2}^{-1/2}\tau^{-1}(u_{k-s-1} - u_{k-s-2}),\\
S_{3k} &= A_{k-1/2}^{1/2}(2\tau)^{-1}\sum_{s=1}^{k-2}\left\{ [E_s^+(k) - E_{s-1}^+(k-1)]\left(-i\frac{\tau}{2}A_{k-s}^{-1/2} + \frac{\tau^2}{2}\right)[E_s^-(k) - E_{s-1}^-(k-1)] \right.\\
&\qquad\qquad \left. \times\left(i\frac{\tau}{2}A_{k-s}^{-1/2} + \frac{\tau^2}{2}\right)\right\}(A_{k-s}^{1/2})' A_{k-s+1/2}^{-1/2}\frac{\tau}{2}A_{k-s}u_{k-s},\\
S_{4k} &= A_{k-1/2}^{1/2}(2\tau)^{-1} i \sum_{s=1}^{k-2}\{ -[E_s^+(k) - E_{s-1}^+(k-1)] + [E_s^-(k) - E_{s-1}^-(k-1)]\}\\
&\quad \times \frac{\tau}{2}A_{k-s}^{-1/2}(A_{k-s}^{1/2})' A_{k-s-1/2}^{-1/2}\frac{\tau}{2}A_{k-s-1}u_{k-s-1},\\
S_{5k} &= A_{k-1/2}^{1/2}(2\tau)^{-1}\sum_{s=1}^{k-2}\left\{ [E_s^+(k) - E_{s-1}^+(k-1)]\left(-i\frac{\tau}{2}A_{k-s}^{-1/2} + \frac{\tau^2}{2}\right)\right.\\
&\qquad\qquad \left. + [E_s^-(k) - E_{s-1}^-(k-1)]\left(i\frac{\tau}{2}A_{k-s}^{-1/2} + \frac{\tau^2}{2}\right)\right\}(A_{k-s}^{1/2})' A_{k-s+1/2}^{-1/2}\frac{\tau}{2}f_{k-s},\\
S_{6k} &= A_{k-1/2}^{1/2}(2\tau)^{-1}\sum_{s=1}^{k-2}\left\{ [E_s^+(k) - E_{s-1}^+(k-1)]\left(-\frac{\tau^2}{2} - i\tau A_{k-s}^{-1/2}\right)\right.\\
&\qquad\qquad \left. + [E_s^-(k) - E_{s-1}^-(k-1)]\left(-\frac{\tau^2}{2} + i\tau A_{k-s}^{-1/2}\right)\right\}f_{k-s},\\
S_{7k} &= A_{k-1/2}^{1/2}(2\tau)^{-1} i \sum_{s=1}^{k-2}\{ -[E_s^+(k) - E_{s-1}^+(k-1)] + [E_s^-(k) - E_{s-1}^-(k-1)]\}\\
&\quad \times\left(i\frac{\tau}{2}A_{k-s}^{-1/2}(A_{k-s}^{1/2})'\right)A_{k-s-1/2}^{-1/2}\frac{\tau}{2}f_{k-s-1}.
\end{aligned} \tag{3.23}$$

Now, let us estimate the terms $\|S_{mk}\|_H$, $m=\overline{1,7}$, separately. Let $m=1$. Then applying estimates (3.4), (3.8), and (3.12), we get

$$\begin{aligned}||S_{1k}||_H \le{}& 2\left\|\sum_{s=1}^{k-2} A_{k-1/2}^{1/2}(2\tau)^{-1}\left[E_s^{+}(k)-E_{s-1}^{+}(k-1)\right]A_{k-s}^{-1/2}\right.\\ &\left.\times\left(-i\frac{\tau}{2}+\frac{\tau^2}{2}A_{k-s}^{1/2}\right)\left(A_{k-s}^{1/2}\right)' A_{k-s+1/2}^{-1/2}\tau^{-1}\left(u_{k-s}-u_{k-s-1}\right)\right\|_H\\ \le{}& 2\sum_{s=1}^{k-2}\left[||A_{k-1/2}^{1/2}(2\tau)^{-1}\left[E_s^{+}(k)-E_{s-1}^{+}(k-1)\right]A_{k-s}^{-1}||\right.\\ &\left.+||A_{k-1/2}^{1/2}(2\tau)^{-1}\left[E_s^{+}(k)-E_{s-1}^{+}(k-1)\right]A_{k-s}^{-1/2}\tau||\right]\\ &\times||A_{k-s}^{1/2}\left(A_{k-s}^{1/2}\right)' A_{k-s+1/2}^{-1}||\,||A_{k-s+1/2}^{1/2}A_{k-s-1/2}^{-1/2}||\,||A_{k-s-1/2}^{1/2}\tau^{-1}\left(u_{k-s}-u_{k-s-1}\right)||_H\\ \le{}& \tau C_5\sum_{s=1}^{k-2}||A_{k-s-1/2}^{1/2}\tau^{-1}\left(u_{k-s}-u_{k-s-1}\right)||_H,\end{aligned} \tag{3.24}$$

where

$$C_5=\frac{3}{2}\left(M_{1/2}+1\right)^2 M_4 e^{M_1 P_1}. \tag{3.25}$$

Let $m=2$. Then applying estimates (3.4), (3.8), and (3.12), we get

$$\begin{aligned}||S_{2k}||_H \le{}& 2\left\|\sum_{s=1}^{k-2} A_{k-1/2}^{1/2}(2\tau)^{-1}\left[E_s^{+}(k)-E_{s-1}^{+}(k-1)\right]\tau A_{k-s}^{-1/2}\left(A_{k-s}^{1/2}\right)'\right.\\ &\left.\times A_{k-s-1/2}^{-1/2}\tau^{-1}\left(u_{k-s-1}-u_{k-s-2}\right)\right\|_H\\ \le{}& 2\tau\sum_{s=1}^{k-2}||A_{k-1/2}^{1/2}(2\tau)^{-1}\left[E_s^{+}(k)-E_{s-1}^{+}(k-1)\right]A_{k-s}^{-1}||\\ &\times||A_{k-s}^{1/2}\left(A_{k-s}^{1/2}\right)' A_{k-s-1/2}^{-1}||\,||A_{k-s-1/2}^{1/2}A_{k-s-3/2}^{-1/2}||\,||A_{k-s-3/2}^{1/2}\tau^{-1}\left(u_{k-s-1}-u_{k-s-2}\right)||_H,\\ ||S_{2k}||_H \le{}& \frac{\tau}{2}C_5\sum_{s=1}^{k-2}||A_{k-s-3/2}^{1/2}\tau^{-1}\left(u_{k-s-1}-u_{k-s-2}\right)||_H.\end{aligned} \tag{3.26}$$

Let $m=3$. Then applying estimates (3.7) and (3.12), we get

$$\begin{aligned}||S_{3k}||_H \le{}& 2\left\|\sum_{s=1}^{k-2} A_{k-1/2}^{1/2}(2\tau)^{-1}\left[E_s^{+}(k)-E_{s-1}^{+}(k-1)\right]A_{k-s}^{-1/2}\right.\\ &\left.\times\left[i\frac{\tau}{2}A_{k-s}^{-1/2}\left(A_{k-s}^{1/2}\right)'-\frac{\tau^2}{2}\left(A_{k-s}^{1/2}\right)'\right]A_{k-s+1/2}^{-1/2}\frac{\tau}{2}A_{k-s}u_{k-s}\right\|_H\end{aligned}$$

$$\begin{aligned}
&\leq 2\sum_{s=1}^{k-2}||A_{k-1/2}^{1/2}(2\tau)^{-1}[E_s^+(k)-E_{s-1}^+(k-1)]A_{k-s}^{-1/2}\tau|| \\
&\quad\times||(A_{k-s}^{1/2})'A_{k-s+1/2}^{-1}||||A_{k-s}u_{k-s}||_H \leq \tau C_6\sum_{s=1}^{k-2}||A_{k-s}u_{k-s}||_H,
\end{aligned} \tag{3.27}$$

where

$$C_6=\frac{3}{4}(M_{1/2}+1)M_4e^{M_1P_1}. \tag{3.28}$$

Let $m=4$. Then applying estimates (3.7) and (3.12), we get

$$\begin{aligned}
||S_{4k}||_H &\leq 2\left\|\sum_{s=1}^{k-2}A_{k-1/2}^{1/2}(2\tau)^{-1}[E_s^+(k)-E_{s-1}^+(k-1)]A_{k-s}^{-1/2}\right. \\
&\qquad\left.\times\left[i\frac{\tau}{2}A_{k-s}^{-1/2}(A_{k-s}^{1/2})'-\frac{\tau^2}{2}(A_{k-s}^{1/2})'\right]A_{k-s+1/2}^{-1/2}\frac{\tau}{2}A_{k-s}u_{k-s}\right\|_H \\
&\leq 2\tau\sum_{s=1}^{k-2}||A_{k-1/2}^{1/2}(2\tau)^{-1}[E_s^+(k)-E_{s-1}^+(k-1)]A_{k-s}^{-1/2}\tau|| \\
&\quad\times||(A_{k-s}^{1/2})'A_{k-s-1/2}^{-1}||||A_{k-s-1}u_{k-s-1}||_H\leq\frac{\tau}{2}C_6\sum_{s=1}^{k-2}||A_{k-s-1}u_{k-s-1}||_H.
\end{aligned} \tag{3.29}$$

Let $m=5$. Then applying estimates (3.7) and (3.12), we get

$$\begin{aligned}
||S_{5k}||_H &\leq \tau\sum_{s=1}^{k-2}[||A_{k-1/2}^{1/2}(2\tau)^{-1}[E_s^+(k)-E_{s-1}^+(k-1)]A_{k-s}^{-1/2}\tau|| \\
&\qquad+||A_{k-1/2}^{1/2}(2\tau)^{-1}[E_s^+(k)-E_{s-1}^+(k-1)]\tau^2||]||(A_{k-s}^{1/2})'A_{k-s-1/2}^{-1}||||f_{k-s}||_H \\
&\leq \tau C_6\sum_{s=1}^{k-2}||f_{k-s}||_H.
\end{aligned} \tag{3.30}$$

Let $m=6$. It is easy to show that $-(\tau^2/2)A_{k-s}+i\tau A_{k-s}^{1/2}=-I+(X_{k-s}^+)^{-1}$. Making $s-1=m$ for the first term in the parenthesis in S_6, we get

$$\begin{aligned}
&A_{k-1/2}^{1/2}(2\tau)^{-1}\sum_{s=1}^{k-2}\{-[E_s^+(k)-E_{s-1}^+(k-1)]A_{k-s}^{-1/2}+[E_s^+(k)-E_{s-1}^+(k-1)](X_{k-s}^+)^{-1}\}f_{k-s} \\
&\quad=A_{k-1/2}^{1/2}(2\tau)^{-1}[E_0^+(k)-E_{-1}^+(k-1)]f_{k-1} \\
&\qquad+A_{k-1/2}^{1/2}(2\tau)^{-1}[E_{k-2}^+(k)-E_{k-3}^+(k-1)]f_1 \\
&\qquad+A_{k-1/2}^{1/2}(2\tau)^{-1}\sum_{s=1}^{k-2}[E_s^+(k)-E_{s-1}^+(k-1)](f_{k-s-1}-f_{k-s}).
\end{aligned} \tag{3.31}$$

Then applying estimates (3.3) and (3.4), we get

$$\begin{aligned}||S_{6k}||_H &\leq \frac{1}{2}||A_{k-1/2}^{1/2}A_k^{-1/2}||(||\tau A_k^{1/2}X_k^+||+2||X_k^+||)\\ &\quad\times(||f_{k-1}||_H+||X_{k-1}^+||\cdots||X_2^+||||f_1||_H)\\ &\quad+\sum_{s=1}^{k-2}||A_{k-1/2}^{1/2}A_k^{-1/2}||(2^{-1}||\tau A_k^{1/2}X_k^+||+||X_k^+||)\\ &\quad\times||X_{k-1}^+||\cdots||X_{k-s}^+||||f_{k-s}-f_{k-s-1}||_H\\ &\leq\frac{3}{4}(M_{1/2}+1)\left[||f_{k-1}||_H+||f_1||_H+\sum_{s=1}^{k-2}||f_{k-s}-f_{k-s-1}||_H\right].\end{aligned}\tag{3.32}$$

Let $m=7$. Then applying estimates (3.7) and (3.12), we get

$$\begin{aligned}||S_{7k}||_H &\leq \tau\sum_{s=1}^{k-2}||A_{k-1/2}^{1/2}(2\tau)^{-1}[E_s^+(k)-E_{s-1}^+(k-1)]A_{k-s}^{-1/2}\tau||\\ &\quad\times||(A_{k-s}^{1/2})'A_{k-s-1/2}^{-1}||||f_{k-s-1}||_H\leq\frac{\tau}{2}C_6\sum_{s=1}^{k-2}||f_{k-s}||_H.\end{aligned}\tag{3.33}$$

Using formula (3.22), the triangle inequality, and the last seven estimates, we obtain

$$\begin{aligned}||J_{5k}||_H &\leq \tau C_7\sum_{s=2}^{k-1}[||A_{s-1/2}^{1/2}\tau^{-1}(u_s-u_{s-1})||_H+||A_{s-3/2}^{1/2}\tau^{-1}(u_{s-1}-u_{s-2})||_H\\ &\quad+||A_su_s||_H+||A_{s-1}u_{s-1}||_H+||f_s||_H+||f_{s-1}||_H]\\ &\quad+3(M_{1/2}+1)\left[||f_{k-1}||_H+||f_1||_H+\sum_{s=2}^{k-1}||f_s-f_{s-1}||_H\right],\end{aligned}\tag{3.34}$$

where

$$C_7=\max\left\{C_5,\frac{3}{2}C_6\right\}.\tag{3.35}$$

Using formula (3.14), the triangle inequality, and the estimates for $\|J_{mk}\|_H$, $m = \overline{1,5}$, we obtain

$$
\begin{aligned}
&\left\| A_{k-1/2}^{1/2} \frac{u_k - u_{k-1}}{\tau} \right\|_H \\
&\quad \le C_1 \|A_0 u_0\|_H + C_2 \|A_0^{1/2} u_0'\|_H + C_3 \left[\|f_1\|_H + \|\tau^2 (A_1^{1/2})' f_1\|_H \right] \\
&\qquad + \tau C_4 \left[\|A_{k-1/2}^{1/2} \tau^{-1} (u_k - u_{k-1})\|_H + \|A_k u_k\|_H + \|A_{k-3/2}^{1/2} \tau^{-1} (u_{k-1} - u_{k-2})\|_H \right. \\
&\qquad\qquad \left. + \|A_{k-1} u_{k-1}\|_H + \|f_k\|_H + \|f_{k-1}\|_H \right] + \frac{3}{2} M_{1/2} \|f_k\|_H \\
&\qquad + \tau C_7 \sum_{s=2}^{k-1} \left[\|A_{s-1/2}^{1/2} \tau^{-1} (u_s - u_{s-1})\|_H + \|A_s u_s\|_H + \|A_{s-3/2}^{1/2} \tau^{-1} (u_{s-1} - u_{s-2})\|_H \right. \\
&\qquad\qquad \left. + \|A_{s-1} u_{s-1}\|_H + \|f_s\|_H + \|f_{s-1}\|_H \right] \\
&\qquad + 3 M_{1/2} \left[\|f_{k-1}\|_H + \|f_1\|_H + \sum_{s=2}^{k-1} \|f_s - f_{s-1}\|_H \right].
\end{aligned} \tag{3.36}
$$

From the above result, it follows that

$$
\begin{aligned}
&\left\| A_{k-1/2}^{1/2} \frac{u_k - u_{k-1}}{\tau} \right\|_H \\
&\quad \le C_8 \left[\|A_0 u_0\|_H + \|A_0^{1/2} u_0'\|_H + \|\tau^2 (A_1^{1/2})' f_1\|_H + \max_{1 \le s \le k} \|f_s\|_H \right. \\
&\qquad \left. + \tau \sum_{s=1}^{k} \left(\|A_{s-1/2}^{1/2} \tau^{-1} (u_s - u_{s-1})\|_H + \|A_s u_s\|_H \right) + \sum_{s=2}^{k} \|f_s - f_{s-1}\|_H \right],
\end{aligned} \tag{3.37}
$$

where

$$
C_8 = \max\{C_1, C_2, C_3 + 3(M_{1/2} + 1), C_4 + C_7\}. \tag{3.38}
$$

Second, let us estimate $\|\{A_k u_k\}_1^N\|_H$. Applying formula (2.17), we can write

$$
A_k u_k = Y_{1k} + Y_{2k} + Y_{3k} + Y_{4k}, \tag{3.39}
$$

where

$$
\begin{aligned}
J_{1k} &= A_k 2^{-1} \left[P_k^+(k) K B^- + P_k^-(k) K B^+ \right] u_0, \\
J_{2k} &= A_k 2^{-1} \left[P_k^+(k) K C^- + P_k^-(k) K C^+ \right] u_0', \\
J_{3k} &= A_k 2^{-1} \left[P_k^+(k) K D^- + P_k^-(k) K D^+ \right] f_1, \\
J_{4k} &= A_k 2^{-1} \sum_{s=0}^{k-2} \left[E_s^+(k) \varphi_{k-s}^- + E_s^-(k) \varphi_{k-s}^+ \right].
\end{aligned} \tag{3.40}
$$

Now, let us estimate the terms $\|Y_{mk}\|_H$, $m=\overline{1,4}$, separately. Let $m=1$. Then applying estimates (3.1), (3.2), (3.4), (3.8), and (3.9), we get

$$\begin{aligned}||Y_{1k}||_H &\le ||A_{k-1/2}^{1/2}P_k^+(k)KB^-u_0||_H\\&\le ||A_kP_k^+(k)A_1^{-1}||||A_1KB^-A_1^{-1}||||A_1A_0^{-1}||||A_0u_0||_H\\&\le M_1\left[\frac{1}{1-\tau M_4}\left(1+\frac{5\tau}{4}M_4\right)+\frac{3}{2}\right]e^{M_1P_1}||A_0u_0||_H=C_9||A_0u_0||_H.\end{aligned}\tag{3.41}$$

Let $m=2$. Then applying estimates (3.1), (3.2), (3.3), (3.4), (3.7), (3.8), and (3.9), we get

$$\begin{aligned}||Y_{2k}||_H &\le ||A_{k-1/2}^{1/2}P_k^+(k)KC^-u_0'||_H\le ||A_kP_k^+(k)A_1^{-1}||||A_1KC^-u_0'||_H\\&\le\left[\frac{5}{4}\left(M_1+\frac{1}{2}M_3\right)+\frac{\tau}{4}\left(\frac{15}{8}M_4+\frac{1}{2}M_3\right)+\frac{3}{4}\tau^2\left(M_4^2+M_4M_3\right)\right]\\&\quad\times e^{M_1P_1}||A_0^{1/2}u_0'||_H=C_{10}||A_0^{1/2}u_0'||_H.\end{aligned}\tag{3.42}$$

Let $m=3$. Then applying estimates (3.1), (3.2), (3.4), (3.7), (3.8), and (3.9), we get

$$\begin{aligned}||Y_{3k}||_H &\le ||A_kP_k^+(k)KD^-f_1||_H\le ||A_kP_k^+(k)A_1^{-1}||||A_1KD^-||||f_1||_H\\&\le\left[\frac{5}{4}+\tau\left(\frac{15}{8}M_4+\frac{1}{2}M_3\right)+\frac{3}{4}\tau^2\left(M_4^2+M_4M_3\right)\right]e^{M_1P_1}\\&\quad\times\left(||f_1||_H+||\tau^2(A_1^{1/2})'f_1||_H\right)=C_{11}\left[||f_1||_H+||\tau^2(A_1^{1/2})'f_1||_H\right].\end{aligned}\tag{3.43}$$

Let $m=4$. We have that

$$Y_4=Q_{1k}+Q_{2k}+Q_{3k}+Q_{4k}+Q_{5k}+Q_{6k}+Q_{7k},\tag{3.44}$$

where

$$\begin{aligned}Q_{1k}&=A_k2^{-1}\sum_{s=1}^{k-2}\left[E_s^+(k)\left(-i\frac{\tau}{2}A_{k-s}^{-1/2}+\frac{\tau^2}{2}\right)\right.\\&\qquad\left.+E_s^-(k)\left(i\frac{\tau}{2}A_{k-s}^{-1/2}+\frac{\tau^2}{2}\right)\right](A_{k-s}^{1/2})'A_{k-s+1/2}^{-1/2}\tau^{-1}(u_{k-s}-u_{k-s-1}),\\Q_{2k}&=A_k2^{-1}\sum_{s=1}^{k-2}i[-E_s^+(k)+E_s^-(k)]\frac{\tau}{2}A_{k-s}^{-1/2}(A_{k-s}^{1/2})'A_{k-s-1/2}^{-1/2}\tau^{-1}(u_{k-s-1}-u_{k-s-2}),\\Q_{3k}&=-A_k2^{-1}\sum_{s=1}^{k-2}\left[E_s^+(k)\left(-i\frac{\tau}{2}A_{k-s}^{-1/2}+\frac{\tau^2}{2}\right)\right.\\&\qquad\left.+E_s^-(k)\left(i\frac{\tau}{2}A_{k-s}^{-1/2}+\frac{\tau^2}{2}\right)\right](A_{k-s}^{1/2})'A_{k-s+1/2}^{-1/2}\frac{\tau}{2}A_{k-s}u_{k-s},\end{aligned}$$

$$
\begin{aligned}
Q_{4k} &= -A_k 2^{-1}\sum_{s=1}^{k-2}\Big[E_s^{+}(k)\Big(-i\frac{\tau}{2}A_{k-s}^{-1/2}+\frac{\tau^2}{2}\Big)\\
&\qquad\qquad + E_s^{-}(k)\Big(i\frac{\tau}{2}A_{k-s}^{-1/2}+\frac{\tau^2}{2}\Big)\Big](A_{k-s}^{1/2})' A_{k-s-1/2}^{-1/2}\frac{\tau}{2}A_{k-s-1}u_{k-s-1},\\
Q_{5k} &= A_k 2^{-1}\sum_{s=1}^{k-2}\Big[E_s^{+}(k)\Big(-i\frac{\tau}{2}A_{k-s}^{-1/2}+\frac{\tau^2}{2}\Big)+E_s^{-}(k)\Big(i\frac{\tau}{2}A_{k-s}^{-1/2}+\frac{\tau^2}{2}\Big)\Big](A_{k-s}^{1/2})'\frac{\tau}{2}f_{k-s},\\
Q_{6k} &= A_k 2^{-1}\sum_{s=1}^{k-2}\Big[E_s^{+}(k)\Big(-i\frac{\tau}{2}A_{k-s}^{-1/2}+\frac{\tau^2}{2}\Big)\\
&\qquad\qquad + E_s^{-}(k)\Big(i\frac{\tau}{2}A_{k-s}^{-1/2}+\frac{\tau^2}{2}\Big)\Big](A_{k-s}^{1/2})' A_{k-s-1/2}^{-1/2}\frac{\tau}{2}f_{k-s-1},\\
Q_{7k} &= A_k 2^{-1}\sum_{s=1}^{k-2}\Big[E_s^{+}(k)\Big(i\tau A_{k-s}^{-1/2}-\frac{\tau^2}{2}\Big)+E_s^{-}(k)\Big(-i\tau A_{k-s}^{-1/2}-\frac{\tau^2}{2}\Big)\Big]f_{k-s}.
\end{aligned}
\tag{3.45}
$$

Now, let us estimate the terms $\|Q_{mk}\|_H$, $m=\overline{1,7}$, separately. Let $m=1$. Then applying estimates (3.4), (3.8), and (3.10), we get

$$
\begin{aligned}
\|Q_{1k}\|_H &\le \sum_{s=1}^{k-2}\frac{\tau}{2}\big[\|A_kE_s^{+}(k)A_{k-s}^{-1}\|+\|A_kE_s^{+}(k)A_{k-s}^{-1/2}\tau\|\big]\\
&\quad\times\|A_{k-s}^{1/2}(A_{k-s}^{1/2})'A_{k-s+1/2}^{-1}\|\|A_{k-s+1/2}^{1/2}A_{k-s-1/2}^{-1/2}\|\|A_{k-s-1/2}^{1/2}\tau^{-1}(u_{k-s}-u_{k-s-1})\|_H,\\
\|Q_{1k}\|_H &\le \frac{2\tau}{3}C_6\sum_{s=1}^{k-2}\|A_{k-s-1/2}^{1/2}\tau^{-1}(u_{k-s}-u_{k-s-1})\|_H.
\end{aligned}
\tag{3.46}
$$

Let $m=2$. Then applying estimates (3.4), (3.8), and (3.10), we get

$$
\begin{aligned}
\|Q_{2k}\|_H &\le \sum_{s=1}^{k-2}\frac{\tau}{2}\|A_kE_s^{+}(k)A_{k-s}^{-1}\|\|A_{k-s}^{1/2}(A_{k-s}^{1/2})'A_{k-s+1/2}^{-1}\|\\
&\quad\times\|A_{k-s+1/2}^{1/2}A_{k-s-1/2}^{-1/2}\|\|A_{k-s-1/2}^{1/2}\tau^{-1}(u_{k-s}-u_{k-s-1})\|_H.\\
\|Q_{2k}\|_H &\le \frac{2\tau}{3}C_6\sum_{s=1}^{k-2}\|A_{k-s-3/2}^{1/2}\tau^{-1}(u_{k-s-1}-u_{k-s-2})\|_H.
\end{aligned}
\tag{3.47}
$$

Let $m=3$. Then applying estimates (3.7) and (3.10), we get

$$
\begin{aligned}
\|Q_{3k}\|_H &\le \sum_{s=1}^{k-2}\frac{\tau}{8}\big[\|A_kE_s^{+}(k)A_{k-s}^{-1/2}\tau\|+\|A_kE_s^{+}(k)\tau^2\|\big]\\
&\quad\times\|(A_{k-s}^{1/2})'A_{k-s+1/2}^{-1}\|\|A_{k-s}u_{k-s}\|_H \le \tau C_{12}\sum_{s=1}^{k-2}\|A_{k-s}u_{k-s}\|,
\end{aligned}
\tag{3.48}
$$

where

$$C_{12} = \frac{3}{2} M_3 e^{M_1 P_1}. \tag{3.49}$$

Let $m = 4$. Then applying estimates (3.7) and (3.10), we get

$$\begin{aligned} \|Q_{4k}\|_H &\leq \sum_{s=1}^{k-2} \frac{\tau}{4} \|A_k E_s^+(k) A_{k-s}^{-1/2} \tau\| \|(A_{k-s}^{1/2})' A_{k-s-1/2}^{-1}\| \|A_{k-s-1} u_{k-s-1}\|_H \\ &\leq \frac{\tau}{3} C_{12} \sum_{s=1}^{k-2} \|A_{k-s-1} u_{k-s-1}\|_H. \end{aligned} \tag{3.50}$$

Let $m = 5$. Then applying estimates (3.7) and (3.10), we get

$$\begin{aligned} \|Q_{5k}\|_H &\leq \sum_{s=1}^{k-2} \frac{\tau}{8} [\|A_k E_s^+(k) A_{k-s}^{-1/2} \tau\| + \|A_k E_s^+(k) \tau^2\|] \|(A_{k-s}^{1/2})' A_{k-s+1/2}^{-1}\| \|f_{k-s}\|_H \\ &\leq \tau C_{12} \sum_{s=1}^{k-2} \|f_{k-s}\|_H. \end{aligned} \tag{3.51}$$

Let $m = 6$. Then applying estimates (3.7) and (3.10), we get

$$\begin{aligned} \|Q_{6k}\|_H &\leq \sum_{s=1}^{k-2} \frac{\tau}{4} \|A_k E_s^+(k) A_{k-s}^{-1/2} \tau\| \|(A_{k-s}^{1/2})' A_{k-s-1/2}^{-1/2}\| \|f_{k-s-1}\|_H \\ &\leq \frac{\tau}{3} C_{12} \sum_{s=1}^{k-2} \|f_{k-s-1}\|_H. \end{aligned} \tag{3.52}$$

Let $m = 7$. We have

$$Q_{7k} = A_k 2^{-1} \sum_{s=1}^{k-2} \left[E_s^+(k) \left(-\frac{\tau^2}{2} + i\tau A_{k-s}^{-1/2} \right) + E_s^-(k) \left(-\frac{\tau^2}{2} - i\tau A_{k-s}^{-1/2} \right) \right] f_{k-s}. \tag{3.53}$$

Using similar manner in Q_{6k}, we get

$$\begin{aligned} \|Q_{7k}\|_H &\leq \sum_{s=1}^{k-2} \frac{\tau}{4} \|A_k E_s^+(k) A_{k-s}^{-1/2} \tau\| \|(A_{k-s}^{1/2})' A_{k-s-1/2}^{-1/2}\| \|f_{k-s-1}\|_H \\ &\leq \frac{1}{2} \left[\|f_k\|_H + e^{M_1 P_1} \|f_1\|_H + e^{M_1 P_1} \sum_{s=2}^{k} \|f_s - f_{s-1}\|_H \right]. \end{aligned} \tag{3.54}$$

Using formula (3.22), the triangle inequality, and the last seven estimates, we obtain

$$\begin{aligned}||Y_{4k}||_H \le \tau\sum_{s=2}^{k-1}\Big[&\frac{2}{3}C_6\big(||A_{s-1/2}^{1/2}\tau^{-1}(u_s-u_{s-1})||_H+||A_{s-3/2}^{1/2}\tau^{-1}(u_{s-1}-u_{s-2})||_H\big)\\&+\frac{1}{3}C_{12}\big(3||A_su_s||_H+||A_{s-1}u_{s-1}||_H+3||f_s||_H+||f_{s-1}||_H\big)\Big]\\&+\frac{1}{2}\Big[||f_k||_H+e^{M_1P_1}||f_1||_H+e^{M_1P_1}\sum_{s=2}^{k}||f_s-f_{s-1}||_H\Big].\end{aligned}\tag{3.55}$$

Using formula (3.14), the triangle inequality, and the estimates $\|Y_{mk}\|_H$, $m=\overline{1,4}$, we obtain

$$\begin{aligned}||A_ku_k||_H \le{}& C_9||A_0u_0||_H+C_{10}||A_0^{1/2}u_0'||_H+C_{11}\big[||f_1||_H+||\tau^2(A_1^{1/2})'f_1||_H\big]\\&+\tau\sum_{s=2}^{k-1}\Big[\frac{2}{3}C_6\big(||A_{s-1/2}^{1/2}\tau^{-1}(u_s-u_{s-1})||_H+||A_{s-3/2}^{1/2}\tau^{-1}(u_{s-1}-u_{s-2})||_H\big)\\&\qquad+C_{12}\Big(||A_su_s||_H+\frac{1}{3}||A_{s-1}u_{s-1}||_H+||f_s||_H+\frac{2}{3}||f_{s-1}||_H\Big)\Big]\\&+\frac{1}{2}\Big[||f_k||_H+e^{M_1P_1}||f_1||_H+e^{M_1P_1}\sum_{s=2}^{k}||f_s-f_{s-1}||_H\Big].\end{aligned}\tag{3.56}$$

From the above result, it follows that

$$\begin{aligned}||A_ku_k||_H \le C_{13}\Big[&||A_0u_0||_H+||A_0^{1/2}u_0'||_H+||\tau^2(A_1^{1/2})'f_1||_H+\max_{1\le s\le k}||f_s||_H\\&+\tau\sum_{s=1}^{k}\big(||A_{s-1/2}^{1/2}\tau^{-1}(u_s-u_{s-1})||_H+||A_su_s||_H\big)+\sum_{s=2}^{k}||f_s-f_{s-1}||_H\Big],\end{aligned}\tag{3.57}$$

where

$$C_{13}=\max\Big\{C_9,C_{10},C_{11}+e^{M_1P_1},\frac{4}{3}C_6,2C_{12}\Big\}.\tag{3.58}$$

Combining estimates (3.37) and (3.57), we get

$$\begin{aligned}&\Big|\Big|A_{k-1/2}^{1/2}\frac{u_k-u_{k-1}}{\tau}\Big|\Big|_H+||A_ku_k||_H\\&\quad\le c_{14}\Big[||A_0u_0||_H+||A_0^{1/2}u_0'||_H+||\tau^2(A_1^{1/2})'f_1||_H\\&\qquad+\max_{1\le s\le k}||f_s||_H+\sum_{s=1}^{k}\big(||A_{s-1/2}^{1/2}\tau^{-1}(u_s-u_{s-1})||_H+||A_su_s||_H\big)+\sum_{s=2}^{k}||f_s-f_{s-1}||_H\Big]\end{aligned}\tag{3.59}$$

for any $k, 1 \leq k \leq N$. Here,

$$C_{14} = \frac{1}{1 - \tau(C_{13} + C_8)}. \tag{3.60}$$

Applying difference analogy of the integral inequality, we obtain

$$\begin{aligned}
&\left\| \left\{ A_{k-1/2}^{1/2} \frac{u_k - u_{k-1}}{\tau} \right\}_{k=1}^{N} \right\|_{C_\tau} + \| \{ A_k u_k \}_{k=1}^{N} \|_{C_\tau} \\
&\qquad \leq C_{15} \left[\| A_0 u_0 \|_H + \| A_0^{1/2} u_0' \|_H + \| \tau^2 (A_1^{1/2})' f_1 \|_H + \max_{1 \leq s \leq N} \| f_s \|_H + \sum_{s=1}^{n} \| f_s - f_{s-1} \|_H \right],
\end{aligned} \tag{3.61}$$

where

$$C_{15} = C_{14} e^{k\tau C_{14}}. \tag{3.62}$$

Theorem 3.1 is proved. □

Theorem 3.2. *Let $u(0) \in D(A(0))$, $u'(0) \in D(A^{1/2}(0))$, and $f_{k+1} \in D$. Then for the solution of the difference scheme (2.9), the stability estimate*

$$\begin{aligned}
\| \{ \tau^{-2} (u_{k+1} - 2u_k + u_{k-1}) \}_1^{N-1} \|_{C_\tau} \leq M \Bigg[& \| A(0) u_0 \|_H + \| A^{1/2}(0) u_0' \|_H + \max_{1 \leq k \leq N} \| f_k \|_H \\
& + \max_{1 \leq k \leq N} \| \tau^2 A_{k+1} f_{k+1} \|_H + \sum_{s=1}^{N} \| f_s - f_{s-1} \|_H \Bigg]
\end{aligned} \tag{3.63}$$

holds, where M does not depend on u_0, u_0', f_s $(1 \leq s \leq N)$, and τ.

Proof. Using (2.9), we get

$$\begin{aligned}
&\left\| \frac{u_{k+1} - 2u_k + u_{k-1}}{\tau^2} \right\|_H \\
&\quad \leq \left[\frac{1}{4} + \frac{\tau}{8} \| (A_{k+1}^{1/2})' A_{k+1}^{-1/2} \| \right] \| A_{k+1}^{3/2} A_{k+3/2}^{-3/2} \| \| A_{k+3/2} A_{k+1}^{-1} \| \| \tau^2 A_{k+1}^2 u_{k+1} \|_H \\
&\qquad + \left[\frac{1}{2} + \frac{\tau}{4} \| (A_{k+1}^{1/2})' A_{k+3/2}^{-1/2} \| + \frac{\tau}{2} \| (A_{k+1}^{1/2})' A_{k+1}^{-1/2} \| \right. \\
&\qquad\qquad \left. + \frac{\tau^2}{8} \| (A_{k+1}^{1/2})' A_{k+1}^{-1/2} \| \| (A_{k+1}^{1/2})' A_{k+3/2}^{-1/2} \| \right] \| A_{k+1} u_{k+1} \|_H \\
&\qquad + \left[\frac{1}{2} + \frac{\tau}{2} \| (A_{k+1}^{1/2})' A_{k+3/2}^{-1/2} \| + \frac{\tau^2}{4} \| (A_{k+1}^{1/2})' A_{k+1}^{-1/2} \| \| (A_{k+1}^{1/2})' A_{k+1/2}^{-1/2} \| \right. \\
&\qquad\qquad \left. + \frac{1}{2} \| (A_{k+1}^{1/2} - A_k^{1/2}) A_{k+1}^{-1/2} \| + \frac{\tau}{2} \| [(A_{k+1}^{1/2})' - (A_k^{1/2})'] A_{k+1/2}^{-1/2} \| \right] \| A_k u_k \|_H
\end{aligned}$$

$$
\begin{aligned}
&+\left[\frac{1}{2}\|A_{k+1}^{3/2}A_{k+3/2}^{-3/2}\|\|A_{k+3/2}A_{k+1}^{-1}\|+\frac{\tau}{4}\|(A_{k+1}^{1/2})'A_{k+1}^{-1/2}\|\right.\\
&\qquad\left.\times\|A_{k+1}^{3/2}A_{k+3/2}^{-3/2}\|\|A_{k+3/2}A_{k+1}^{-1}\|\right]\left\|\tau A_{k+1}\frac{u_{k+1}-u_k}{\tau}\right\|_H\\
&+\left[\frac{1}{2}\|(A_{k+1}^{1/2})'A_{k+3/2}^{-1/2}\|+\frac{\tau}{4}\|(A_{k+1}^{1/2})'A_{k+1}^{-1/2}\|\|(A_{k+1}^{1/2})'A_{k+3/2}^{-1/2}\|\right]\\
&\times\|A_{k+1/2}^{-1/2}\|\left\|A_{k+1/2}^{1/2}\frac{u_{k+1}-u_k}{\tau}\right\|_H\\
&+\left[\|(A_{k+1}^{1/2})'A_{k+1/2}^{-1/2}\|+\frac{\tau}{2}\|(A_{k+1}^{1/2})'A_{k+1}^{-1/2}\|\|(A_{k+1}^{1/2})'A_{k+1/2}^{-1/2}\|\right.\\
&\qquad\left.+\frac{1}{\tau}\|(A_{k+1}^{1/2}-A_k^{1/2})A_{k+1/2}^{-1/2}\|\frac{1}{2}\|[(A_{k+1}^{1/2})'-(A_k^{1/2})']A_{k+1/2}^{-1/2}\|\right]\\
&\times\|A_{k-1/2}^{-1/2}\|\left\|A_{k-1/2}^{1/2}\frac{u_k-u_{k-1}}{\tau}\right\|_H\\
&+\left[\frac{1}{4}\|A_{k+1}^{3/2}A_{k+3/2}^{-3/2}\|\|A_{k+3/2}A_{k+1}^{-1}\|\right.\\
&\qquad\left.+\frac{\tau}{8}\|(A_{k+1}^{1/2})'A_{k+1}^{-1/2}\|\|A_{k+1}^{3/2}A_{k+3/2}^{-3/2}\|\|A_{k+3/2}A_{k+1}^{-1}\|\right]\|\tau^2A_{k+1}f_{k+1}\|_H\\
&+\left[1+\frac{\tau}{4}\|(A_{k+1}^{1/2})'A_{k+3/2}^{-1/2}\|+\frac{\tau^2}{4}\|(A_{k+1}^{1/2})'A_{k+1}^{-1/2}\|\|(A_{k+1}^{1/2})'A_{k+3/2}^{-1/2}\|\right.\\
&\qquad\left.+\frac{\tau}{2}\|(A_{k+1}^{1/2})'A_{k+1}^{-1/2}\|\right]\|f_{k+1}\|_H\\
&+\left[\frac{\tau}{2}\|(A_{k+1}^{1/2})'A_{k+1/2}^{-1/2}\|+\frac{\tau^2}{4}\|(A_{k+1}^{1/2})'A_{k+1}^{-1/2}\|\|(A_{k+1}^{1/2})'A_{k+1/2}^{-1/2}\|\right.\\
&\qquad\left.+\frac{1}{2}\|(A_{k+1}^{1/2}-A_k^{1/2})A_{k+1/2}^{-1/2}\|+\frac{\tau}{4}\|[(A_{k+1}^{1/2})'-(A_k^{1/2})']A_{k+1/2}^{-1/2}\|\right]\|f_k\|_H
\end{aligned}
\tag{3.64}
$$

for any $k, 1 \le k \le N$. In a similar manner as the proof of estimates (3.37) and (3.57), we get

$$
\begin{aligned}
&\left\|\tau A_{k+1}\frac{u_k-u_{k-1}}{\tau}\right\|_H\\
&\quad\le 2C_8\left[\|A_0u_0\|_H+\|A_0^{1/2}u_0'\|_H+\|\tau^2(A_1^{1/2})'f_1\|_H+\max_{1\le s\le k}\|f_s\|_H\right.\\
&\qquad\left.+\tau\sum_{s=1}^{k}\left(\|A_{s-1/2}^{1/2}\tau^{-1}(u_s-u_{s-1})\|_H+\|A_su_s\|_H\right)+\sum_{s=2}^{k}\|f_s-f_{s-1}\|_H\right],
\end{aligned}
$$

$$
\begin{aligned}
&\|\tau^2 A_{k+1}^2 u_k\|_H \\
&\quad \le 2C_{13}\Bigg[\|A_0u_0\|_H + \|A_0^{1/2}u_0'\|_H + \|\tau^2 (A_1^{1/2})' f_1\|_H + \max_{1\le s\le k}\|f_s\|_H \\
&\qquad + \tau\sum_{s=1}^{k}\left(\|A_{s-1/2}^{1/2}\tau^{-1}(u_s-u_{s-1})\|_H + \|A_s u_s\|_H\right) + \sum_{s=2}^{k}\|f_s - f_{s-1}\|_H\Bigg],
\end{aligned} \tag{3.65}
$$

respectively. Now, putting them in (3.64) and applying estimates (3.4), (3.5), and (3.7), we get

$$
\begin{aligned}
\left\|\frac{u_{k+1}-2u_k+u_{k-1}}{\tau^2}\right\|_H \le C_{16}\Bigg[&\|\tau^2 A_{k+1}^2 u_{k+1}\|_H + \left\|\tau A_{k+1}\frac{u_{k+1}-u_k}{\tau}\right\|_H + \|A_{k+1}u_{k+1}\|_H \\
&+ \left\|A_{k+1/2}^{1/2}\frac{u_{k+1}-u_k}{\tau}\right\|_H + \|A_k u_k\|_H + \left\|A_{k-1/2}^{1/2}\frac{u_k-u_{k-1}}{\tau}\right\|_H \\
&+ \|\tau^2 A_{k+1} f_{k+1}\|_H + \|f_{k+1}\|_H + \|f_k\|_H\Bigg],
\end{aligned} \tag{3.66}
$$

where

$$
\begin{aligned}
C_{16} = \max\Bigg\{&\frac{1}{4}+\frac{\tau}{2}M_3, \frac{1}{2}+\frac{3\tau}{4}M_3+\frac{\tau^2}{8}M_3^2, \frac{1}{2}+\frac{\tau}{2}M_3+\frac{\tau^2}{4}M_3^2+\tau M_{1/2}, \\
&\frac{1}{2}(M_{3/2}+1)M_1 + \frac{\tau}{4}M_3(M_{3/2}+1)M_1, \frac{1}{\sqrt{\delta}}\left(M_3+\frac{\tau}{2}M_3^2+\frac{3}{2}(M_{1/2}+1)\right), \\
&\frac{1}{\sqrt{\delta}}\left(\frac{1}{2}M_3+\frac{\tau}{4}M_3^2\right), 1+\frac{3\tau}{4}M_3+\frac{\tau^2}{4}M_3^2, \frac{\tau}{2}(M_3+M_{1/2})+\frac{\tau^2}{4}(M_3^2+M_{1/2})\Bigg\}.
\end{aligned} \tag{3.67}
$$

Using estimate (3.37), we get

$$
\begin{aligned}
&\max_{1\le k\le N-1}\left\|\frac{u_{k+1}-2u_k+u_{k-1}}{\tau^2}\right\|_H \\
&\quad\le C_{17}\Bigg[\|A_0u_0\|_H + \|A_0^{1/2}u_0'\|_H + \max_{1\le k\le N}\|\tau^2 A_{k+1} f_{k+1}\|_H + \max_{1\le s\le N}\|f_s\|_H + \sum_{s=1}^{N}\|f_s-f_{s-1}\|_H\Bigg],
\end{aligned} \tag{3.68}
$$

where

$$
C_{17} = 5C_{15}C_{16}. \tag{3.69}
$$

Theorem 3.2 is proved. □

Acknowledgment

The authors would like to thank Professor Dr. Pavel Sobolevskii (Jerusalem, Israel) for his helpful suggestions to the improvement of this paper.

References

[1] H. O. Fattorini, *Second Order Linear Differential Equations in Banach Spaces*, vol. 108 of *North-Holland Mathematics Studies*, North-Holland, Amsterdam, The Netherlands, 1985.

[2] S. G. Kreĭn, *Linear Differential Equations in a Banach Space*, Nauka, Moscow, Russia, 1967.

[3] A. Ashyralyev, "Difference schemes of a higher order of accuracy for second-order evolution equations," in *Functional-Differential Equations (Russian)*, pp. 145–150, Perm Polytechnic Institute, Perm, Russia, 1989.

[4] S. I. Piskarëv, "Approximation of holomorphic semigroups," *Tartu Riikliku Ülikooli Toimetised*, no. 492, pp. 3–14, 1979.

[5] S. I. Piskarëv, "Stability of difference schemes in Cauchy problems with almost periodic solutions," *Differentsial'nye Uravneniya*, vol. 20, no. 4, pp. 689–695, 1984.

[6] S. I. Piskarëv, "Principles of discretization methods III," Tech. Rep., Acoustic Institute, Academy of Science USSR, Moscow, Russia, 1986.

[7] P. E. Sobolevskiĭ and L. M. Čebotareva, "Approximate solution of the Cauchy problem for an abstract hyperbolic equation by the method of lines," *Izvestija Vysših Učebnyh Zavedeniĭ Matematika*, no. 5(180), pp. 103–116, 1977 (Russian).

[8] A. Ashyralyev and I. Muradov, "On one difference scheme of a second order of accuracy for hyperbolic equations," in *Proceeding of the IMM of AS of Turkmenistan*, pp. 58–63, Ashgabat, Turkmenistan, June 1995.

[9] A. Ashyralyev and P. E. Sobolevskii, "A note on the difference schemes for hyperbolic equations," *Abstract and Applied Analysis*, vol. 6, no. 2, pp. 63–70, 2001.

[10] A. Ashyralyev and P. E. Sobolevskii, *New Difference Schemes for Partial Differential Equations*, vol. 148 of *Operator Theory: Advances and Applications*, Birkhäuser, Basel, Switzerland, 2004.

[11] A. Ashyralyev and P. E. Sobolevskii, "Two new approaches for construction of the high order of accuracy difference schemes for hyperbolic differential equations," *Discrete Dynamics in Nature and Society*, vol. 2005, no. 2, pp. 183–213, 2005.

[12] A. Ashyralyev and M. E. Koksal, "On the second order of accuracy difference scheme for hyperbolic equations in a Hilbert space," *Numerical Functional Analysis and Optimization*, vol. 26, no. 7-8, pp. 739–772, 2005.

[13] P. E. Sobolevskiĭ and V. A. Pogorelenko, "Hyperbolic equations in a Hilbert space," *Sibirskiĭ Matematičeskiĭ Žurnal*, vol. 8, pp. 123–145, 1967.

[14] Yu. L. Daletskii, "Path integrals associated with the operator evolution equations," *Uspekhi Matematicheskikh Nauk*, vol. 17, no. 5, pp. 3–115, 1958.

Allaberen Ashyralyev: Department of Mathematics, Fatih University, 34900 Buyukcekmece, Istanbul, Turkey
Email address: aashyr@fatih.edu.tr

Mehmet Emir Koksal: Graduate Institute of Sciences and Engineering, Fatih University, 34900 Buyukcekmece, Istanbul, Turkey; Department of Mathematics, Gebze Institute of Technology, 41400 Gebze/Kocaeli, Turkey
Email address: mekoksal@fatih.edu.tr