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Research Article

On Global Exponential Stability of Discrete-Time Hopfield Neural Networks with Variable Delays

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Global exponential stability of a class of discrete-time Hopfield neural networks with variable delays is considered. By making use of a difference inequality, a new global exponential stability result is provided. The result only requires the delay to be bounded. For this reason, the result is milder than those presented in the earlier references. Furthermore, two examples are given to show the efficiency of our result.

1. Introduction

In the past decade, the stability problem of continuous-time Hopfield neural networks (HNNs) with time delays has received considerable attention. The existence of delay in a practical neural network may induce instability, oscillation, and poor performance. So far, various stability criteria for continuous-time HNNs with delays have been presented in the literature, see, for example, and references therein [1–11]. Compared with continuous-time HNNs, less attention has been paid to discrete-time HNNs with time delays. However, when implementing the continuous-time neural networks for computer simulation or computational purposes, it is necessary to formulate a discrete-time version which is an analogue of the continuous-time neural networks. On the other hand, the discrete-time analogue may not preserve the convergence dynamics of their continuous-time counterparts since some of the numerical schemes can lead to the existence of spurious equilibria and spurious asymptotic behavior which are not present in the continuous-time counterparts [12], for details the readers can refer to [13, 14]. For this reason, it is desirable to investigate the stability of discrete-time HNNs. To the best of our knowledge, the stability results for discrete-time HNNs with delays are very few [12, 15, 16]. In this paper, by using a discrete-time version of Halanay inequality [18], a new global exponential stability result for discrete-time HNNs with variable delays is considered. The result is less conservative than those given in the previous literature since both the delay functions and the activation functions imposed on, in this paper, are milder than those earlier presented. Two numerical simulations are given to show the validity of our result.

2. Stability analysis

The dynamic behavior of discrete-time Hopfield neural networks with variable delays can be described as follows [12, 15]:

$$x_i(n+1) = c_i x_i(n) + \sum_{j=1}^{m} a_{ij} f_j(x_j(n)) + \sum_{j=1}^{m} b_{ij} f_j(x_j(n-\kappa(n))) + J_i, \tag{2.1}$$

for $i \in \{1,2,\dots,m\}$, $n \in \{0,1,2,\dots\}$, where m corresponds to the number of units in a neural networks; $x(n) = [x_1(n),\dots,x_m(n)]^T \in \mathbb{R}^m$ corresponds to the state vector; $f(x(n)) = [f_1(x_1(n)),\dots,f_m(x_m(n))]^T \in \mathbb{R}^m$ denotes the activation function of the neurons; $f(x(n-\kappa(n))) = [f_1(x_1(n-\kappa(n))),\dots,f_m(x_m(n-\kappa(n)))]^T \in \mathbb{R}^m$; and $C = \operatorname{diag}(c_i)$ ($c_i \in (0,1)$) represents the rate with which the ith unit will reset its potential to the resting state in isolation when disconnected from the network and external inputs. $A = \{a_{ij}\}$ is referred to as the feedback matrix, $B = \{b_{ij}\}$ represents the delayed feedback matrix, while $J = [J_1,\dots,J_m]^T$ is an external bias vector and $\kappa(n)$ is the transmission delay along the axon of the jth unit and satisfies $0 \le \kappa(n) \le \kappa$ (κ is a constant).

The initial conditions associated with system (2.1) are of the form:

$$x_i(l) = \varphi_i(l), \quad i \in \{1,2,\dots,m\}, \tag{2.2}$$

where l is an integer with $l \in [-\kappa, 0]$.

Definition 2.1. A vector $x^* = (x_1^*, x_2^*,\dots,x_m^*)^T$ is said to be an equilibrium point of system (2.1), if the following equation holds:

$$x_i^* = c_i x_i^* + \sum_{j=1}^{m} a_{ij} f_j(x_j^*) + \sum_{j=1}^{m} b_{ij} f_j(x_j^*) + J_i. \tag{2.3}$$

In this paper, we assume that the activation function f_i, $i = 1,2,\dots,m$, is bounded and satisfies the following assumption.

Assumption 2.2.

$$|f_i(\xi_1) - f_i(\xi_2)| \le L_i |\xi_1 - \xi_2|, \quad \forall \xi_1, \xi_2 \in \mathbb{R}. \tag{2.4}$$

This type of activation functions is clearly more general than both the usual sigmoid activation functions and the piecewise linear function (PWL): $f_i(x) = (1/2)(|x+1| - |x-1|)$ which is used in [10].

Assumption 2.3.

$$0 \le (f_i(\xi_1) - f_i(\xi_2))(\xi_1 - \xi_2) \le L_i (\xi_1 - \xi_2)^2, \quad \forall \xi_1, \xi_2 \in \mathbb{R}. \tag{2.5}$$

Remark 2.4. In [16], under the assumptions that the bounded activation function f_i, $i = 1,2,\dots,m$, satisfies Assumption 2.3, and $a_{ij} = 0$ in (2.1), the authors studied the global exponential stability for (2.1). It is obvious that Assumption 2.3 is more restrictive than Assumption 2.2. Hence, due to this fact, the result in this paper improves that of [16].

Remark 2.5. Due to the boundedness of the activation functions under Assumption 2.2, by using the well-known Brouwer fixed point theorem and the same way as in [17], we can easily know that there must exist an equilibrium point of system (2.1).

Let $x^* = (x_1^*, x_2^*, \dots, x_n^*)^T$ be an equilibrium point of (2.1), one can derive from (2.1) that the transformation $y_i(n) = x_i(n) - x_i^*$ transforms system (2.1) into the following system:

$$y_i(n+1) = c_i y_i(n) + \sum_{j=1}^{m} a_{ij} g_j\big(y_j(n)\big) + \sum_{j=1}^{m} b_{ij} g_j\big(y_j(n-\kappa(n))\big) \tag{2.6}$$

or equivalently,

$$y(n+1) = Cy(n) + Ag\big(y(n)\big) + Bg\big(y(n-\kappa(n))\big), \tag{2.7}$$

where $y(n) = (y_1(n), y_2(n), \dots, y_m(n))^T$, $C = \operatorname{diag}(c_1, c_2, \dots, c_m)$, $A = (a_{ij})_{m\times m}$, $B = (b_{ij})_{m\times m}$, $g(y(n)) = (g_1(y_1(n)), g_2(y_2(n)), \dots, g_m(y_m(n)))^T$, and $g_j(y_j(n)) = f_j(y_j(n) + x_j^*) - f_j(x_j^*)$.

Since each function $f_j(\cdot)$ satisfies Assumption 2.2, hence, each $g_j(\cdot)$ satisfies

$$\big|g_j(\xi_j)\big| \le L_j \big|\xi_j\big|. \tag{2.8}$$

To prove the stability of x^* of (2.1), it is sufficient to prove the stability of the trivial solution of (2.6).

In the following, denote $\|u\|$ as a vector norm of the vector $u \in \mathbb{R}^m$ and $\|A\|_p$ is the p-norm of A induced by the vector norm.

Definition 2.6. The trivial solution of (2.6) is said to be globally exponentially stable if for any solution $x_i(n, \varphi_i)$ with the initial condition $x_i(l) = \varphi_i(l)$ for $l \in [-\kappa, 0]$ from $\mathbb{R}^m$ there exist constants $\varepsilon \in (0,1)$ and $M \ge 1$ such that

$$\|y(n)\| \le M\|\phi\|\varepsilon^n, \quad \forall n \ge 0, \tag{2.9}$$

where $\|\phi\| = \max_{l\in[-\kappa,0]}\{\|\phi(l)\|\}$, $\phi(l) = (\varphi_1(l), \varphi_2(l), \dots, \varphi_m(l))^T$.

Lemma 2.7 [19]. *Let $r > 0$ be a natural number, and let $\{z(n)\}_{n\ge -r}$ be a sequence of real numbers satisfying the inequality:*

$$\Delta z(n) \le -pz(n) + q\max\{z(n), z(n-1), \dots, z(n-r)\}, \tag{2.10}$$

where $\Delta z(n) = z(n+1) - z(n)$ and $n \ge 0$. If $0 < q < p \le 1$, then there exists a constant $\lambda_0 \in (0,1)$ such that

$$z(n) \le \max\{0, z(0), z(-1), \dots, z(-r)\}\lambda_0^n, \quad n \ge 0. \tag{2.11}$$

Moreover, λ_0 *can be chosen as the smallest root in the interval* $(0, 1)$ *of equation*

$$\lambda^{r+1} + (p-1)\lambda^r - q = 0. \tag{2.12}$$

Now we will present a sufficient condition for ensuring global exponential stability of (2.1).

Remark 2.8. Lemma 2.7 can be regarded as discrete Halanay-type inequalities corresponding to the original [18].

Theorem 2.9. *Under Assumption 2.2, the equilibrium point of (2.1) is globally exponentially stable if*

$$c_{\max} + L_{\max}(\|A\| + \|B\|) < 1, \tag{2.13}$$

where $c_{\max} = \max_j(c_j)$ *and* $L_{\max} = \max_j(L_j)$.

Proof. Consider the following function:

$$z(n) = ||y(n)||, \tag{2.14}$$

then, we get

$$\begin{aligned} z(n+1) &= ||y(n+1)|| \\ &= ||Cy(n) + Ag(y(n)) + Bg(y(n-\kappa(n)))|| \\ &\le ||Cy(n)|| + ||Ag(y(n))|| + ||Bg(y(n-\kappa(n)))|| \\ &\le c_{\max}||y(n)|| + \|A\|L_{\max}||y(n)|| + \|B\|L_{\max}||y(n-\kappa(n))|| \\ &\le (c_{\max} + L_{\max}\|A\|)||y(n)|| + L_{\max}\|B\|||y(n-\kappa(n))||. \end{aligned} \tag{2.15}$$

Calculating $\Delta z(n)$ along (2.7), we obtain

$$\begin{aligned} \Delta z(n) &= z(n+1) - z(n) \\ &\le (c_{\max} + L_{\max}\|A\|)||y(n)|| + L_{\max}\|B\|||y(n-\kappa(n))|| - ||y(n)|| \\ &= (c_{\max} - 1 + L_{\max}\|A\|)||y(n)|| + L_{\max}\|B\|||y(n-\kappa(n))|| \\ &= -(1 - c_{\max} - L_{\max}\|A\|)z(n) + L_{\max}\|B\|z(n-\kappa(n)) \\ &\le -pz(n) + q\max\{z(n), z(n-1), \dots, z(n-\kappa)\}, \end{aligned} \tag{2.16}$$

where

$$\begin{aligned} p &= 1 - c_{\max} - L_{\max}\|A\|, \\ q &= L_{\max}\|B\|. \end{aligned} \tag{2.17}$$

By Lemma 2.7, we know that if

$$0 < q < p \le 1 \tag{2.18}$$

then, there exists a constant $\lambda_0 \in (0,1)$ such that

$$\begin{aligned} z(n) &= ||y(n)|| \\ &\leq \max\{0, z(0), z(-1), \dots, z(-\kappa)\}\lambda_0^n \\ &= \max\left\{0, ||y(0)||, ||y(-1)||, \dots, ||y(-\kappa)||\right\}\lambda_0^n \\ &\leq \|\phi\|\lambda_0^n. \end{aligned} \tag{2.19}$$

It directly follows from Definition 2.6 that the unique equilibrium point of (2.1) is global exponential stability under Assumption 2.2.

Remark 2.10. The uniqueness of the equilibrium point of (2.1) will follow from the global exponential stability of the system, see, for example, [18]. □

COROLLARY 2.11. *Under Assumption 2.2, the unique equilibrium point of (2.1) is globally exponentially stable if*

$$c_{\max} + L_{\max}\left(\max_{1\leq j\leq m}\sum_{i=1}^{m}|a_{ij}| + \max_{1\leq j\leq m}\sum_{i=1}^{m}|b_{ij}|\right) < 1. \tag{2.20}$$

COROLLARY 2.12. *Under Assumption 2.2, the unique equilibrium point of (2.1) is globally exponentially stable if*

$$c_{\max} + L_{\max}\left(\max_{1\leq i\leq m}\sum_{j=1}^{m}|a_{ij}| + \max_{1\leq i\leq m}\sum_{j=1}^{m}|b_{ij}|\right) < 1. \tag{2.21}$$

Proof. Taking the induced norm as 1-norm $\|A\|_1$ and infinity norm $\|A\|_\infty$ in Theorem 2.9 respectively, one can easily obtain Corollaries 2.11 and 2.12. □

Remark 2.13. In [12], the authors studied the global exponential stability of discrete-time cellular neural networks with constant delays and obtained a sufficient condition. However, the results in [12] cannot be applied to the neural networks with variable delays.

Remark 2.14. In [15], under the assumption that the bounded delay satisfies $1 < \kappa(n+1) < 1 + \kappa(n)$, the authors obtained some global exponential stability criteria for system (2.1). In this paper, our Theorem 2.9 removes this restriction and only requires the delays to be bounded. Hence, our result is less restrictive than that given in [15].

3. Examples

In this section, we will give two examples to show that the condition given in this paper is less restrictive than that given in some earlier literature.

Example 3.1. Consider the following discrete-time delayed Hopfield neural networks with two neurons:

$$\begin{aligned} x_1(n+1) &= 0.2x_1(n) + 0.1f(x_1(n)) + 0.2f(x_2(n)) \\ &\quad - 0.2f(x_1(n-\kappa(n))) + 0.1f(x_2(n-\kappa(n))) - 3, \\ x_2(n+1) &= 0.1x_2(n) - 0.25f(x_1(n)) + 0.15f(x_2(n)) \\ &\quad - 0.15f(x_1(n-\kappa(n))) + 0.3f(x_2(n-\kappa(n))) - 1; \end{aligned} \tag{3.1}$$

that is,

$$C = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.1 \end{bmatrix}, \qquad A = \begin{bmatrix} 0.1 & 0.2 \\ -0.25 & 0.15 \end{bmatrix}, \qquad B = \begin{bmatrix} -0.2 & 0.1 \\ -0.15 & 0.3 \end{bmatrix}, \tag{3.2}$$

and the nonlinear input-output function is chosen as $f(x) = \tanh(x)$. It can be verified that this function satisfies Assumption 2.2 with $L_1 = L_2 = 1$. The unique equilibrium point of system (3.1) can be obtained as

$$x^* = (x_1^*, x_2^*)^T = (-3.9197, -1.0598)^T. \tag{3.3}$$

Furthermore, one can easily check that

$$\|A\|_1 + \|B\|_1 = 0.75 < 1 - c_{\max} = 0.8. \tag{3.4}$$

Therefore, from Corollary 2.11, it follows that the equilibrium point of system (3.1) is globally exponentially stable.

For the convenience of numerical simulation, let $\kappa(n) = 1$ in (3.1). It is obvious that $\kappa(n) = 1$ does not satisfy the condition $1 < \kappa(n+1) < 1 + \kappa(n)$ which is needed in [15]. Therefore, the results in [15] cannot be applied to this example.

The initial state $(\varphi_1(l), \varphi_2(l))^T = (-3, 1)^T$ for $l \in [-1, 0]$. The numerical simulation is illustrated in Figure 3.1.

Example 3.2. Consider the following discrete-time delayed Hopfield neural networks with two neurons:

$$\begin{aligned} x_1(n+1) &= 0.2x_1(n) + 0.25f(x_1(n)) + 0.1f(x_2(n)) \\ &\quad - 0.2f(x_1(n-\kappa(n))) + 0.1f(x_2(n-\kappa(n))) + 2, \\ x_2(n+1) &= 0.15x_2(n) - 0.2f(x_1(n)) + 0.15f(x_2(n)) \\ &\quad + 0.1f(x_1(n-\kappa(n))) + 0.3f(x_2(n-\kappa(n))) + 3; \end{aligned} \tag{3.5}$$

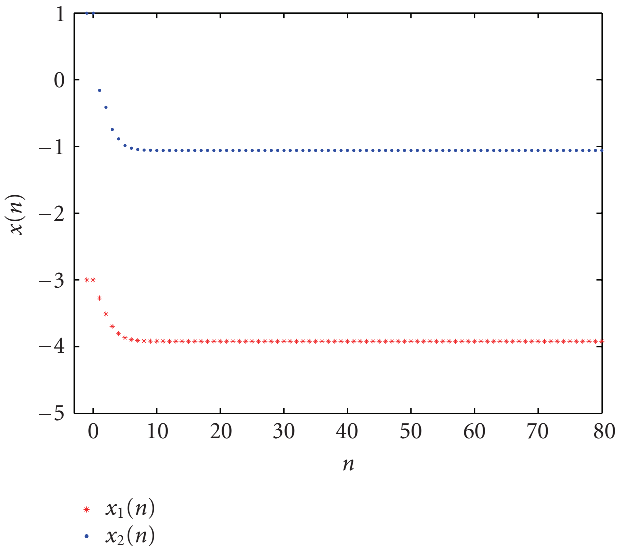

Figure 3.1. The solution of (3.1).

that is,

$$C = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.15 \end{bmatrix}, \qquad A = \begin{bmatrix} 0.25 & 0.1 \\ -0.2 & 0.15 \end{bmatrix}, \qquad B = \begin{bmatrix} -0.2 & 0.1 \\ 0.1 & 0.3 \end{bmatrix}, \tag{3.6}$$

and the activation function is $f(x) = (1/2)(|x-1| - |x+1|)$. Since this type of function is not nondecreasing, it does not satisfy Assumption 2.3 which is used in [16]. Therefore, the results in [16] cannot be applied to this example. On the other hand, it can be checked that this function satisfies Assumption 2.2 with $L_1 = L_2 = 1$. The unique equilibrium point of system (3.1) can be obtained as

$$x^* = (x_1^*, x_2^*)^T = (2.1875, 3.1176)^T. \tag{3.7}$$

Furthermore, one can easily check that

$$\|A\|_\infty + \|B\|_\infty = 0.75 < 1 - c_{\max} = 0.8. \tag{3.8}$$

Therefore, from Corollary 2.12, it follows that the equilibrium point of system (3.5) is globally exponentially stable.

In the following numerical simulation, let $\kappa(n) = [0.5 + 2n/(n+7)]$ in (3.5), where $[r]$ denotes the integer part of the real number r. It is obvious that $0 \le \kappa(n) \le 2$ is bounded but does not satisfy the condition $1 < \kappa(n+1) < 1 + \kappa(n)$ which is needed in [15]. Therefore, the results in [15] cannot be applied to this example.

The initial state $(\varphi_1(l), \varphi_2(l))^T = (1,2)^T$ for $l \in [-1,0]$. The numerical simulation is illustrated in Figure 3.2.

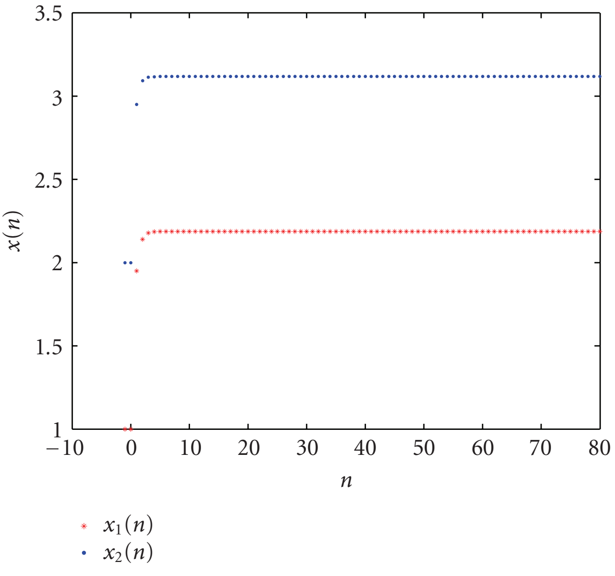

Figure 3.2. The solution of (3.5).

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