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Research Article

Dynamics of a Class of Higher Order Difference Equations

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We prove that all positive solutions of the autonomous difference equation $x_n = \alpha x_{n-k}/(1+x_{n-k}+f(x_{n-1},\dots,x_{n-m}))$, $n\in\mathbb{N}_0$, where $k,m\in\mathbb{N}$, and f is a continuous function satisfying the condition $\beta\min\{u_1,\dots,u_m\}\le f(u_1,\dots,u_m)\le\beta\max\{u_1,\dots,u_m\}$ for some $\beta\in(0,1)$, converge to the positive equilibrium $\overline{x}=(\alpha-1)/(\beta+1)$ if $\alpha>1$.

1. Introduction

In this paper, we investigate the global stability of positive solutions of the following autonomous difference equation:

$$x_n = \frac{\alpha x_{n-k}}{1+x_{n-k}+f(x_{n-1},\dots,x_{n-m})},\quad n\in\mathbb{N}_0 := \{0,1,2,\dots\},\tag{1.1}$$

where $k,m\in\mathbb{N}$, and f is a continuous function satisfying the condition

$$\beta\min\{u_1,\dots,u_m\}\le f(u_1,\dots,u_m)\le\beta\max\{u_1,\dots,u_m\}\tag{1.2}$$

for some $\beta\in(0,1)$ (the case $\beta=0$ is not of some interest since in the case equation it turned as Riccati' one).

Note that in a view of relations (1.2), $\overline{x}=(\alpha-1)/(\beta+1)$ is a unique positive equilibrium of (1.1), if $\alpha>1$.

Further, note that the behaviour of positive solutions of (1.1) for the case $\alpha\in(0,1)$ is quite simple. Namely, in this case, we have $x_n\le\alpha x_{n-k}$, so that the sequences $(x_{lk+r})_{l\in\mathbb{N}}$, $r\in\{0,1,\dots,k-1\}$ converge to zero, and consequently, the sequence x_n does. The case

$\alpha = 1$ is slightly complicated. In this case, the sequences $(x_{lk+r})_{l\in\mathbb{N}}$, $r \in \{0,1,\dots,k-1\}$ are still convergent, as positive and nonincreasing. If we replace n in (1.1) by $kl, l \in \mathbb{N}$, and then let $l \to \infty$, we obtain

$$\phi_0 = \frac{\phi_0}{1+\phi_0+f(\phi_1,\dots,\phi_m)}, \tag{1.3}$$

where $\phi_i := \lim_{l\to\infty} x_{kl+i}$, $i \in \{0,1,\dots,k-1\}$. Without loss of generality, we may assume that $\phi_0 \neq 0$. From (1.3), we have that $\phi_0 + f(\phi_1,\dots,\phi_m) = 0$, which implies $\phi_0 = 0$, a contradiction. Hence, every positive solution of (1.1) converges to zero, also in this case.

Equation (1.1) for the case $\alpha \in (0,1]$ is a particular case of the difference equation

$$x_n = g(x_{n-1},\dots,x_{n-s}), \tag{1.4}$$

where the function g satisfies the condition

$$g(u_1,\dots,u_s) \le \max\{u_1,\dots,u_s\}. \tag{1.5}$$

Equation (1.4), whose function g satisfies condition (1.5) or the following condition:

$$\lim_{x\to\infty} \frac{g(x,\dots,x)}{x} = 1, \tag{1.6}$$

has been extensively studied by many authors (see, e.g., [9, 14–21, 25]).

In the proof of the result, we use the method of so-called *"frame" sequences*, that is, a discrete analog of the method of frame curves, commonly used in the theory of differential equations. This method and closely related methods have been used in the literature for many times; see, for example, [21, 1–5, 7, 10, 11, 22–24] and the related references therein. Our motivation stems from [10–12]. Recently, there has been a great interest in studying nonlinear difference equations and systems, in particular those which model some real-life situations in population biology and ecology (see, e.g., [18, 20, 21, 25, 10, 6, 8, 13] and the references cited therein).

2. The global stability of (1.1)

We prove the main result of this paper in this section. Before this, we need a lemma.

Lemma 2.1. *Assume that $\alpha > 1$, $\beta \in (0,1)$, $\varepsilon \in (0,(\alpha-1)(1-\beta)/(1+\beta))$, and that $(m_n)_{n\in\mathbb{N}}$ and $(M_n)_{n\in\mathbb{N}}$ are sequences defined as follows:*

$$m_n = \alpha - 1 - \beta M_{n-1} - \frac{\varepsilon}{2^{n-1}}, \qquad M_n = \alpha - 1 - \beta m_n + \frac{\varepsilon}{2^{n-1}}, \tag{2.1}$$

for $n \ge 2$, with initial values

$$m_1 = (\alpha-1) - \beta(\alpha-1+\varepsilon) - \varepsilon, \qquad M_1 = \alpha - 1 + \varepsilon. \tag{2.2}$$

Then

$$\lim_{n\to\infty} m_n = \lim_{n\to\infty} M_n = \frac{\alpha-1}{1+\beta}. \tag{2.3}$$

Proof. From (2.1) we obtain the following linear first-order difference equation:

$$M_n = \beta^2 M_{n-1} + (\alpha-1)(1-\beta) + (2\beta+1)\frac{\varepsilon}{2^{n-1}}, \quad n \geq 2, \tag{2.4}$$

whose general solution is

$$M_n = \beta^{2n-2} M_1 + (\alpha-1)(1-\beta)\frac{\beta^{2n-2}-1}{\beta^2-1} + \frac{(2\beta+1)\varepsilon}{2^{n-1}}\sum_{j=0}^{n-2}(2\beta^2)^j. \tag{2.5}$$

Letting $n \to \infty$ in (2.5), using the assumption $\beta \in (0,1)$ and Stoltz theorem, it follows that $\lim_{n\to\infty} M_n = (\alpha-1)/(1+\beta)$.

From this and (2.1), it easily follows that $\lim_{n\to\infty} m_n = (\alpha-1)/(1+\beta)$ too, as claimed. □

Now, we are able to formulate and to prove our main result.

Theorem 2.2. *Assume that $\alpha > 1$, and f is a continuous function satisfying condition (1.2) for some $\beta \in (0,1)$. Then, every positive solution of (1.1) converges to the positive equilibrium $\overline{x} = (\alpha-1)/(\beta+1)$.*

Proof. From (1.1), we have that

$$x_n = \frac{\alpha x_{n-k}}{1+x_{n-k}+f(x_{n-1},\ldots,x_{n-m})} \leq \frac{\alpha x_{n-k}}{1+x_{n-k}}, \quad n \in \mathbb{N}. \tag{2.6}$$

Assume that u_n is a solution of the following difference equation:

$$u_n = \frac{\alpha u_{n-k}}{1+u_{n-k}}, \tag{2.7}$$

with initial values $u_0 = x_0, \ldots, u_{-k} = x_{-k}$. It is clear that (2.7) can be reduced into k-independent Riccati equations of the form $z_n = \alpha z_{n-1}/(1+z_{n-1})$. It is well known that for $\alpha > 1$, there is finite limit $\lim_{n\to\infty} z_n$ (which is equal to $\alpha - 1$). From this and since in the light of the monotonicity of the function $f(x) = \alpha x/(1+x)$, we have that $x_n \leq u_n$ for $n \geq -k$. By letting $n \to \infty$, it follows that

$$S = \limsup_{n\to\infty} x_n \leq \alpha - 1 = \lim_{n\to\infty} u_n. \tag{2.8}$$

From (2.8), we have that for every $\varepsilon \in (0, (\alpha-1)(1-\beta)/(1+\beta))$,

$$x_n \leq \alpha - 1 + \varepsilon, \tag{2.9}$$

for $n \geq n_0$. From (1.1), condition (1.2), and relation (2.9), it follows that

$$\frac{\alpha x_{n-k}}{1+x_{n-k}+\beta(\alpha-1+\varepsilon)} \leq \frac{\alpha x_{n-k}}{1+x_{n-k}+f(x_{n-1},\ldots,x_{n-m})} = x_n \tag{2.10}$$

for every $n \geq n_0 + m$. Assume that (y_n) is a solution of the following difference equation:

$$y_n = \frac{\alpha y_{n-k}}{1+y_{n-k}+\beta(\alpha-1+\varepsilon)} \tag{2.11}$$

with initial values $y_{n_0} = x_{n_0}, \dots, y_{n_0+k-1} = x_{n_0+k-1}$. Then, since the function $g(x) = \alpha x/(1+\beta(\alpha-1+\varepsilon)+x)$ is increasing on the interval $(0,\infty)$, it is easy to see by the induction that $y_n \le x_n$ for $n \ge n_0$, and that

$$\lim_{n\to\infty} y_n = (\alpha-1) - \beta(\alpha-1+\varepsilon). \tag{2.12}$$

Hence, we obtain that

$$0 < (\alpha-1) - \beta(\alpha-1+\varepsilon) \le \liminf_{n\to\infty} x_n = I. \tag{2.13}$$

In this way, we formed two frame sequences (y_n) and (u_n) such that $y_n \le x_n \le u_n$ for $n \ge n_0 + m$.

Now, let $\varepsilon \in (0, (\alpha-1)(1-\beta)/(1+\beta))$ and sequences $(m_n)_{n\in\mathbb{N}}$ and $(M_n)_{n\in\mathbb{N}}$ be defined by (2.1) with (2.2).

Then we have

$$0 < m_1 \le I \le S \le M_1. \tag{2.14}$$

On the other hand, similar to (2.6)–(2.13), for each $t \in \mathbb{N}\setminus\{1\}$ fixed, we can form the sequences $(y_n^{(t)})$ and $(u_n^{(t)})$ defined by

$$u_n^{(t)} = \frac{\alpha u_{n-k}^{(t)}}{1+u_{n-k}^{(t)}+\beta m_{t-1}}, \qquad y_n^{(t)} = \frac{\alpha y_{n-k}^{(t)}}{1+y_{n-k}^{(t)}+\beta M_t}, \tag{2.15}$$

and easily show that

$$\begin{gathered} \lim_{n\to\infty} u_n^{(t)} = \alpha - 1 - \beta m_{t-1}, \qquad \lim_{n\to\infty} y_n^{(t)} = \alpha - 1 - \beta M_t, \\ \alpha - 1 - \beta M_t - \frac{\varepsilon}{2^{t-1}} < y_n^{(t)} \le x_n \le u_n^{(t)} < \alpha - 1 - \beta m_{t-1} + \frac{\varepsilon}{2^{t-1}}, \quad n \ge n_t. \end{gathered} \tag{2.16}$$

From this and Lemma 2.1, it follows that

$$m_t \le I \le S \le M_t \tag{2.17}$$

for every $t \in \mathbb{N}$. Letting $t \to \infty$ in relations (2.17), the result follows. □

By Theorem 2.2 and the change of variables $x_n = y_n/c$, we obtain the following corollary.

Corollary 2.3. *Assume that $k, m \in \mathbb{N}$, α_j, $j \in \{1,\dots,m\}$, are nonnegative numbers such that $\sum_{j=1}^m \alpha_j = 1$, $\alpha > 1$, $c > 0$, and $\beta \in (0,c)$. Then, every positive solution of the difference equation*

$$x_n = \frac{\alpha x_{n-k}}{1 + c x_{n-k} + \beta \sum_{j=1}^m \alpha_j x_{n-j}}, \quad n \in \mathbb{N}_0, \tag{2.18}$$

converges to the positive equilibrium $\overline{x} = (\alpha-1)/(\beta+c)$.

In the following example, we show that the function f need not be a linear one.

Example 2.4. Let

$$f(u_1,\ldots,u_m) = \sqrt[a]{\frac{\sum_{j=1}^{m} u_j^a}{m}}, \tag{2.19}$$

where $a > 0$; then this function satisfies conditions of Theorem 2.2. Hence, every positive solution of the difference equation

$$x_n = \frac{\alpha x_{n-k}}{1 + x_{n-k} + \beta\sqrt[a]{(x_{n-1}^a + x_{n-2}^a + \cdots + x_{n-m}^a)/m}} \tag{2.20}$$

converges to the positive equilibrium $\overline{x} = (\alpha - 1)/(\beta + 1)$.

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