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Research Article

Permanence and Periodic Solution of Predator-Prey System with Holling Type Functional Response and Impulses

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We considered a nonautonomous two dimensional predator-prey system with impulsive effect. Conditions for the permanence of the system and for the existence of a unique stable periodic solution are obtained.

1. Introduction

In biomathematics, many mathematical models have been established to describe the relationships between species and the outer environment, and the connections between different species. Among the relationships between the species living in the same outer environment, the predator-prey theory plays an important and fundamental role. The dynamic relationship between predators and their prey has long been one of the dominant theses in both ecology and mathematical ecology. Many excellent works have been done for the Lotka-Volterra type predator-prey system, for example, see [1–12]. In many situations, especially when predators have to search for food, a suitable general predator-prey theory is based on the so called ratio-dependent theory. Accordingly, researchers have proposed many ratio-dependent response functions. In [13], Holling suggested that there are three functional responses of the predator which usually are called Holling type I, Holling type II, and Holling type III. The type III response is typical of predators showing learning behavior, and

$$\varphi(x) = \frac{\mu x^2}{1+\rho x^2} \tag{1.1}$$

is usually a suitable response function. In [14], Wang and Li investigated the global existence of positive periodic solutions and permanent property of the ratio-dependent predator-prey system with Holling type III functional response and delay, which takes the form

$$\begin{aligned} x'(t) &= x(t)\left[a(t) - b(t)\int_{-\infty}^{t} k(t-s)x(s)ds\right] - \frac{c(t)x^2(t)y(t)}{x^2 + Ay^2(t)}, \\ y'(t) &= y(t)\left(-d(t) + \frac{e(t)x^2(t-\tau)}{x^2(t-\tau) + Ay^2(t-\tau_2)}\right), \end{aligned} \tag{1.2}$$

where the functional response $\varphi(u) = cu^2/(1+Au^2)$, $u = x/y$; $a(t)$, $b(t)$, $c(t)$, $e(t)$, and $d(t)$ are all positive periodic continuous functions, and $A > 0$, $\tau \geq 0$ are real constants. They found that the criteria for the permanence is exactly the same as that for the existence of the positive periodic solutions of (1.2). Other results about ratio-dependent functional response can be found in [11, 15–18].

On the other hand, biological species may undergo discrete changes of relatively short duration at a fixed time. Moreover, continuous changes in environment parameters such as temperature or rainfall can also create discontinuous outbreaks in pest population. Systems with short-term perturbations are often naturally described by impulsive differential equations.

The theory of impulsive differential equations is now being recognized to be not only richer than the corresponding theory of differential equations without impulses, but also representing a more natural framework for mathematical modeling of many real world phenomena [19–24]. Thus the wide applications naturally motivate a deeper theoretical study of the subject.

Recently, some impulsive equations have been introduced in population dynamics in relation to population ecology, chemotherapeutic treatment of disease, impulsive birth, see [25–29] and the references therein.

In this paper, we consider the nonautonomous ratio-dependent predator-prey system with Holling type III functional response and impulsive effect

$$\begin{aligned} &\left.\begin{aligned} x'(t) &= x(t)\left(a(t) - \sum_{j=1}^{m} b_j(t)x^{\lambda_j}(t)\right) - \frac{b_0(t)x^2(t)y(t)}{x^2 + Ay^2(t)}, \\ y'(t) &= y(t)\left(-d(t) + \frac{e(t)x^2(t)}{x^2(t) + Ay^2(t)}\right), \end{aligned}\right\} \quad t \neq t_k,\ t \in J, \\ &\left.\begin{aligned} x(t_k^+) &= (1+g_k)x(t_k), \\ y(t_k^+) &= y(t_k) + p, \end{aligned}\right\} \quad k \in Z^+, \end{aligned} \tag{1.3}$$

where $x(t)$ and $y(t)$ represent the densities of the prey population and predator population at time t, respectively; λ_j $(1 \leq j \leq m)$, p, A are positive constants; $a(t)$, $b_i(t)$ $(0 \leq i \leq m)$, $d(t)$, $e(t)$ are continuous ω-periodic functions and $\omega > 0$; $a(t)$ stands for prey intrinsic growth rate, $d(t)$ stands for the death rate of the predator, $b_0(t)$ and $e(t)$ stand for the conversion rates, $\sqrt{A}$ stands for half capturing saturation, the function

$x(t)[a(t) - \sum_{j=1}^{m} b_j(t)x^{\lambda_j}(t)]$ represents the specific growth rate of the prey in the absence of predator, and $x^2(t)/[x^2(t) + Ay^2(t)]$ denotes the ratio-dependent response function, which reflects the capture of the predator. $x(t_k^+)$ and $y(t_k^+)$ represent the right limit of $x(t_k)$, $y(t_k)$, respectively. $g_k > -1$ for $k \in Z^+ = \{1,2,\dots,m\}$. $J \subset R$ is an interval and $t_k < t_{k+1}$, $t_k \in J$ for $k \in Z^+$.

Throughout this paper, we always assume system (1.3) satisfies the following conditions.

(H) $b_i(t) \geq 0$ for $0 \leq i \leq m$, $e(t) \geq 0$ and there is at least $i_0 \in \{1,\dots,m\}$ such that $b_{i_0}(t) > 0$. There exists an integer $q > 0$ such that $g_{k+q} = g_k$, $t_{k+q} = t_k + \omega$.

Denote, by $PC(J,R)$, the set of function $\rho : J \to R$ which are continuous for $t \in J$, $t \neq t_k$ are continuous from the left for $t \in J$, and have discontinuities of the first kind at the point t_k. Denote, by $PC^1(J,R)$, the set of function $\rho : J \to R$ with a derivative $(d\rho/dt) \in PC(J,R)$. The solution of the system (1.3) is a piecewise continuous function $u = \mathrm{col}(x(t), y(t)) : J \to R^2$, $x(t) \in PC^1(J,R)$, $y(t) \in PC^1(J,R)$ satisfying (1.2).

With the model (1.3), we can take into account the possible exterior effects under which the population densities change very rapidly. For instance, impulsive effect of the pest population density is possible after its partial destruction by catching, poisoning with chemicals used in agriculture ($-1 < g_k < 0$), or after its increase because of migration of the outside pest population ($g_k > 0$). An impulsive increase of the predator population density is possible by artificially breeding the species or releasing some species ($p > 0$).

We will also investigate the asymptotic behavior of nonnegative solution for system (1.3). Note that according to biological interpretation of the solutions $x(t)$ and $y(t)$, they must be nonnegative. Our results extend the ideas in [30]. The organization of this paper is as follows. In the next section, we present necessary preliminaries and consider the dynamics of a single species model. We obtain the sufficient and necessary condition for permanence in Section 3. In Section 4, we discuss the existence and attractivity of the periodic solution of system (1.3).

2. Preliminary lemmas

In this section, we first consider the nonlinear single species model

$$\begin{aligned} x'(t) &= x(t)\left(\alpha(t) - \sum_{j=1}^{l} \beta_j(t)x^{\gamma_j}(t)\right), \quad t \neq t_k, \\ x(t_k^+) &= (1 + h_k)x(t_k), \end{aligned} \tag{2.1}$$

where $\alpha(t)$, $\beta_i(t)$ $(1 \leq i \leq l)$ are continuous ω-periodic functions; $\beta_i(t) \geq 0$ $(1 \leq i \leq l)$; γ_i is a positive constant; there exists an integer $q > 0$ such that $h_{k+q} = h_k$, $t_{k+q} = t_k + \omega$, and $h_k > -1$ for all $k \in Z^+$.

In order to explore the existence of periodic solutions of (2.1), for the reader's, we first summarize below a few concepts and results from [31] that will be used in this section.

Let X, Z be a normed space, $L : \mathrm{Dom}\, L \subset X \to Z$ be a linear mapping, and $N : X \to Z$ be a continuous mapping. The mapping L will be called a Fredholm mapping of index if $\dim\mathrm{Ker}\, L = \mathrm{codim}\mathrm{Im}\, L < +\infty$ and $\mathrm{Im}\, L$ is closed in Z. If L is a Fredholm mapping of index zero, there exist continuous projectors $P : X \to X$ and $Q : Z \to Z$ such that $\mathrm{Im}\, P = \mathrm{Ker}\, L$,

$\operatorname{Ker} Q = \operatorname{Im} P$. It follows that $L|_{\operatorname{Dom} L \cap \operatorname{Ker} P} \to \operatorname{Im} L$ is invertible. We denote the inverse of that map by K_P. If Ω is an open bounded subset of X, the mapping N will be called L-compact on $\overline{\Omega}$ if $QN(\overline{\Omega})$ is bounded and $K_P(I-Q)N : \overline{\Omega} \to X$ is compact. Since $\operatorname{Im} Q$ is isomorphic to $\operatorname{Ker} L$, there exists an isomorphism $J : \operatorname{Im} Q \to \operatorname{Ker} L$.

Lemma 2.1 [31]. *Let $\Omega \subset X$ be an open bounded set and L be a Fredholm mapping of index zero. Assume that $N : X \to Z$ is a continuous operator and L-compact on $\overline{\Omega}$. Suppose*

(a) *$Lx \neq \lambda Nx$ for all $\lambda \in (0,1)$ and $x \in \partial\Omega \cap \operatorname{Dom} L$;*
(b) *$QNx \neq 0$ for $x \in \partial\Omega \cap \operatorname{Ker} L$;*
(c) *$\deg\{JQN, \Omega \cap \operatorname{Ker} L, 0\} \neq 0$.*

Then $Lx = Nx$ has at least one solution in $\overline{\Omega} \cap \operatorname{Dom} L$.

Lemma 2.2. *(1) All solutions $x(t)$ of (2.1) with positive initial value satisfy $\lim_{t\to\infty} x(t) = 0$, provided*

$$\int_0^{\omega} \alpha(s)ds + \ln \prod_{i=1}^{q} (1+h_i) < 0. \tag{2.2}$$

(2) There is at least $l_0 \in \{1,\dots,l\}$ such that $\beta_{l_0}(t) > 0$, then system (2.1) admits a unique positive ω-periodic solution if and only if

$$\int_0^{\omega} \alpha(s)ds + \ln \prod_{i=1}^{q} (1+h_i) > 0, \tag{2.3}$$

which, moreover, is globally asymptotically stable.

Proof. (1) Without loss of generality, we consider (2.1) on $[0,\infty)$ and let $t_1 > 0$. Now assume that $x(0) > 0$. It is easy to show that $x(t) > 0$ for $t \geq 0$. Then from (2.1), we have that

$$x'(t) \leq \alpha(t)x(t). \tag{2.4}$$

And so

$$0 < x(t) \leq x(0) \prod_{0<t_k<t} (1+h_k) \exp\left(\int_0^t \alpha(s)ds\right). \tag{2.5}$$

The condition that $\int_0^{\omega} \alpha(s)ds + \ln \prod_{i=1}^{q}(1+h_i) < 0$ implies that

$$\lim_{t\to\infty} \prod_{0<t_k<t} (1+h_k) \exp\left(\int_0^t \alpha(s)ds\right) = 0. \tag{2.6}$$

Hence $\lim_{t\to\infty} x(t) = 0$.

(2) Making the change of variable $x(t) = \exp u(t)$, (2.1) is transformed into

$$\begin{gathered} u'(t) = \alpha(t) - \sum_{i=1}^{l} \beta_j(t) \exp(\gamma_j u(t)), \quad t \neq t_k, \\ u(t_k^+) = u(t_k) + \ln(1+h_k). \end{gathered} \tag{2.7}$$

Let $X = \{u \in PC(R,R), u(t+\omega) = u(t)\}$ with the norm $\|u\| = \sup_{t\in[0,\omega]} |u(t)|$. Let $Y = PC(R,R) \times R^q$ with the norm $\|(u,a_1,\dots,a_q)\| = (\|u\|^2 + a_1^2 + \cdots + a_q^2)^{1/2}$. Then X and Y are Banach spaces. □

Define mappings L and N by

$$L : \mathrm{Dom}\, L \subset X \longrightarrow Y, \qquad u \longrightarrow (u', \Delta u(t_1), \Delta u(t_2), \dots, \Delta u(t_q)),$$
$$N : X \longrightarrow Y,\ u \longrightarrow \left(\alpha(t) - \sum_{j=1}^{l} \beta_j(t)\exp(\gamma_j u(t)), \ln(1+h_1), \dots, \ln(1+h_k)\right). \tag{2.8}$$

It is easy to check that

$$\mathrm{Ker}\, L = \{u \in X : u = c \in R\},$$
$$\mathrm{Im}\, L = \left\{(u,a_1,\dots,a_q) : \int_0^\omega u(s)ds + \sum_{i=1}^{q} a_i = 0\right\}. \tag{2.9}$$

Since $\mathrm{Im}\, L$ is closed in Z and $\dim\mathrm{Ker}\, L = \mathrm{codim}\mathrm{Im}\, L = 1$, L is a Fredholm mapping of index zero. There exist continuous projectors $P : X \to X$ and $Q : Y \to Y$ by

$$Pu = \frac{1}{\omega}\int_0^\omega u(s)ds,$$
$$Q(u,a_1,\dots,a_q) = \left(\frac{1}{\omega}\left[\int_0^\omega u(s)ds + \sum_{i=0}^{q} a_i\right], 0, \dots, 0\right). \tag{2.10}$$

The generalized inverse $K_P : \mathrm{Im}\, L \to \mathrm{Dom}\, L \cap \mathrm{Ker}\, P$ is given,

$$K_P(u,a_1,\dots,a_q) = \int_0^\omega u(s)ds + \sum_{t>t_i} a_i - \frac{1}{\omega}\int_0^\omega\int_0^t u(s)ds\,dt - \sum_{i=1}^{q} a_i + \frac{1}{\omega}\sum_{i=1}^{q} a_i t_i. \tag{2.11}$$

Clearly, QN and $K_P(I-Q)N$ are continuous. It is not difficult to show that $K_P(I-Q)N(\overline{\Omega})$ is compact for any open bounded set $\Omega \subset X$. So N is L-compact.

We consider operator equation $Lu = \lambda Nu$, $\lambda \in (0,1)$, that is,

$$u'(t) = \lambda\left[\alpha(t) - \sum_{j=1}^{l} \beta_j(t)\exp(\gamma_i u(t))\right], \quad t \neq t_k,$$
$$u(t_k^+) = u(t_k) + \lambda \ln(1+h_k). \tag{2.12}$$

Let $u(t)$ be a ω-periodic solution of (2.7); integrating (2.12), we obtain

$$\int_0^\omega \alpha(s)ds + \ln\prod_{i=1}^{q}(1+h_k) = \int_0^\omega \sum_{j=1}^{l} \beta_j(t)\exp(\gamma_i u(t))dt. \tag{2.13}$$

From (2.12) and (2.13), we have that

$$\begin{aligned}\int_0^\omega |u'(t)|\,dt &\leq \lambda \int_0^\omega \left| \alpha(t)dt - \sum_{j=1}^{l} \beta_j(t) \exp\left(\gamma_i u(t)\right) \right| dt \\ &\leq \lambda \int_0^\omega |\alpha(t)|\,dt + \int_0^\omega \sum_{j=1}^{l} \beta_j(t) \exp\left(\gamma_i u(t)\right) dt \\ &\leq \lambda \int_0^\omega |\alpha(t)|\,dt + \int_0^\omega \alpha(t)dt + \ln \prod_{i=1}^{q} (1+h_k) =: M_1.\end{aligned} \tag{2.14}$$

Since $u \in X$, there exist $\xi, \eta \in [0,\omega]$ such that

$$u(\xi) = \min_{t\in[0,\omega]} x(t), \qquad u(\eta) = \max_{t\in[0,\omega]} x(t). \tag{2.15}$$

From (2.13), we have

$$\omega \beta_{l_0}^L \exp\left(\gamma^L u(\xi)\right) \leq \int_0^\omega \alpha(s)ds + \ln \prod_{i=1}^{q} (1+h_k) \leq \omega \sum_{j=1}^{l} \beta_j^M \exp\left(\gamma^M u(\eta)\right), \tag{2.16}$$

where

$$\begin{aligned}\gamma^L &= \min\{\gamma_1, \ldots, \gamma_l\}, \qquad \gamma^M = \max\{\gamma_1, \ldots, \gamma_l\}, \\ \beta_j^M &= \max_{t\in[0,\omega]} \beta_j(t), \qquad \beta_{l_0}^L = \min_{t\in[0,\omega]} \beta_{l_0}(t).\end{aligned} \tag{2.17}$$

Thus

$$\begin{aligned}u(\eta) &\geq \frac{1}{\gamma^M} \ln \frac{1}{\omega \sum_{j=1}^{l} \beta_j^M} \left(\int_0^\omega \alpha(s)ds + \ln \prod_{i=1}^{q} (1+h_k) \right), \\ u(\xi) &\leq \frac{1}{\gamma^L} \ln \frac{1}{\omega \beta_{l_0}^L} \left(\int_0^\omega \alpha(s)ds + \ln \prod_{i=1}^{q} (1+h_k) \right).\end{aligned} \tag{2.18}$$

Hence we have

$$\begin{aligned}u(t) &\leq u(\xi) + \int_0^\omega |u'(t)|\,dt \\ &\leq \frac{1}{\gamma^L} \ln \frac{1}{\omega \beta_{l_0}^L} \left(\int_0^\omega \alpha(s)ds + \ln \prod_{i=1}^{q} (1+h_k) \right) + M_1 =: M_2, \\ u(t) &\geq u(\eta) - \int_0^\omega |u'(t)|\,dt \\ &\geq \frac{1}{\gamma^M} \ln \frac{1}{\omega \sum_{j=1}^{l} \beta_j^M} \left(\int_0^\omega \alpha(s)ds + \ln \prod_{i=1}^{q} (1+h_k) \right) - M_1 =: M_3.\end{aligned} \tag{2.19}$$

Put $M = |M_2| + |M_3| + 1$, then $\max_{t\in[0,\omega]} |u(t)| < M$. Evidently, M is not dependent on the choice of λ. Let $\Omega = \{u \in X : \|u\| < M\}$, then Ω verifies the requirement (a) in Lemma 2.1. When $u \in \partial\Omega \cap \operatorname{Ker} L$, u is a constant with $|u| = M$, then

$$QNu = \frac{1}{\omega}\left(\int_0^{\omega}\alpha(t)dt - \sum_{j=1}^{l}\int_0^{\omega}\beta_j(t)\exp(\gamma_j u)dt + \ln\prod_{i=1}^{q}(1+h_i), 0, \dots, 0\right) \neq (0,0,\dots,0). \tag{2.20}$$

For $u \in \Omega \cap \operatorname{Ker} L$,

$$JQNu = \frac{1}{\omega}\left(\int_0^{\omega}\alpha(t)dt - \sum_{j=1}^{l}\int_0^{\omega}\beta_j(t)\exp(\gamma_j u)dt + \ln\prod_{i=1}^{q}(1+h_i)\right), \tag{2.21}$$

where

$$J : \operatorname{Im} Q \longrightarrow \operatorname{Ker} L, (u,0,\dots,0) \longrightarrow u. \tag{2.22}$$

It is easy to prove that $\deg\{JQN, \Omega \cap \operatorname{Ker} L, 0\} \neq 0$. By Lemma 2.1, (2.7) has at least a ω-periodic solution. So (2.1) has at least a positive ω-periodic solution $x^*(t)$.

Next, we consider the stability property of $x^*(t)$. Let $x(t)$ be any solution of system (2.1) with positive initial value $x(0) > 0$, then $x^*(t)$ and $x(t)$ are strictly on $[0,\infty)$. Set Lyapunov function

$$V(t) = |\ln x(t) - \ln x^*(t)|, \quad t \geq 0. \tag{2.23}$$

It is easy to check that $V(t)$ is continuous on $[0,\infty)$. Calculating the upper right derivative of $V(t)$, it follows that when $t \neq t_k$,

$$\begin{aligned} D^+V(t) &= \operatorname{sgn}(\ln x(t) - \ln x^*(t))\left(\frac{\dot{x}(t)}{x(t)} - \frac{\dot{x}^*(t)}{x^*(t)}\right) \\ &= \operatorname{sgn}(x(t) - x^*(t))\left[-\sum_{j=1}^{l}\beta_j(t)(x(t))^{\gamma_j} + \sum_{j=1}^{l}\beta_j(t)(x^*(t))^{\gamma_j}\right]. \end{aligned} \tag{2.24}$$

Note that $\operatorname{sgn}(x(t) - x^*(t)) = \operatorname{sgn}((x(t))^{\gamma_j} - (x^*(t))^{\gamma_j})$ for $j = 1,2,\dots,l$, we have that

$$D^+V(t) \leq -\sum_{j=1}^{l}\beta_j(t)\left|(x(t))^{\gamma_j} - (x^*(t))^{\gamma_j}\right| < 0, \quad t \neq t_k. \tag{2.25}$$

From (2.25), we know that $V(t)$ is decreasing on $t \geq 0$. Since $V \geq 0$, $\lim_{t\to\infty} V(t) = V^* \geq 0$. We will show that $V^* = 0$. Since $x^*(t)$ is a positive periodic solution of (2.1), it follows that $|\ln x(t) - \ln x^*(t)| + |\ln x^*(t)|$ is bounded. From $|\ln x(t)| \leq V + |\ln x^*(t)|$, it follows that $|\ln x(t)|$ is bounded. Thus there is an $M_0 > 0$ such that $|\ln x(t)| < M_0$ and $|\ln x^*(t)| < M_0$. By the mean value theorem, one have

$$\left|(x(t))^{\gamma_j} - (x^*(t))^{\gamma_j}\right| = \gamma_j \exp\{\gamma_j \ln\overline{x}(t)\} V(t), \tag{2.26}$$

where $\bar{x}(t)$ lies between $x(t)$ and $x^*(t)$. Put $m_j^* = \gamma_j \exp\{-M_0\gamma_j\}$, $M_j^* = \gamma_j \exp\{M_0\gamma_j\}$, then

$$m_j^* V(t) \le \left| (x(t))^{\gamma_j} - (x^*(t))^{\gamma_j} \right| \le M_j^* V(t). \tag{2.27}$$

From (2.25) and (2.27), we have $D^+V(t) \le -\beta_{l_0}^L m_{l_0}^* V(t)$, $t \ne t_k$ for $k \in Z^+$. So $\lim_{t\to\infty} V(t) = V^* = 0$. Using again the mean value theorem, one can obtain $\lim_{t\to\infty} |x(t) - x^*(t)| = 0$. From the stability property of $x^*(t)$, one knows that the positive periodic solution of (2.1) is unique. This completes the sufficient part of Lemma 2.2(2). The necessary part is easily obtained from Lemma 2.2(1).

Remark 2.3. In [30], the authors considered the special case $l = 1$ of (2.1). Our result generalizes the result in [30].

Lemma 2.4. *Consider the equation*

$$\begin{aligned} x'(t) &= \alpha(t)x(t), \quad t \ne t_k, \\ x(t_k^+) &= x(t_k) + p, \end{aligned} \tag{2.28}$$

where $\alpha(t)$ is a continuous ω-periodic function, p is a positive constant, and there is an integer $q > 0$ such that $t_{k+q} = t_k + \omega$. Assume that $\int_0^\omega \alpha(t)dt < 0$, then (2.28) has a unique positive, globally asymptotically stable ω-periodic solution.

Remark 2.5. If $\int_0^\omega \alpha(t)dt \ge 0$, then the solution $x(t)$ with any positive initial value satisfies $\lim_{t\to\infty} x(t) = +\infty$.

It is easy to show that the following function

$$x(t) = \begin{cases} x^*(t), & t \in (0,\omega], \\ x(t - j\omega), & t \in (j\omega,(j+1)\omega],\ j \in \{\dots,-2,-1,0,1,\dots\}, \end{cases} \tag{2.29}$$

where

$$x^*(t) = \frac{p\sum_{i=1}^q \exp\left(\int_{t_i}^\omega \alpha(t)dt\right)}{1 - \exp\left(\int_0^\omega \alpha(t)dt\right)} \exp\left(\int_0^t \alpha(s)ds\right) + p\sum_{0<t_i<t} \exp\left(\int_{t_i}^t \alpha(s)ds\right) \tag{2.30}$$

is a unique positive, globally asymptotically stable ω-periodic solution of (2.28).

3. Permanence

In this section, we consider the permanence of (1.3). We assume that $J = [0,\infty)$ and let $t_1 > 0$. Now assume that $x(0) > 0$, $4y(0) > 0$. It is easy to show that $x(t) > 0$, $y(t) > 0$ for $t \ge 0$.

Definition 3.1. System (1.3) is said to be permanent if there are $m, M > 0$(independent of initial value) and a finite t_0 such that, for solution $u = \mathrm{col}(x(t), y(t))$ with initial value $x(0) > 0$, $y(0) > 0$, $m \le x(t)$, $y(t) \le M$ holds for all $t \ge t_0$. Here t_0 may depend on the initial value.

Lemma 3.2. *Let $d(t)$ be a continuous ω-periodic function. $\omega > 0$ and $\int_0^{\omega} d(t)dt > 0$, then the following inequality*

$$e^{c(t-s)} \le e^{1+D\omega+\int_s^t d(r)dr}, \quad \text{for } t \ge s \tag{3.1}$$

holds, where constant $c : 0 < c \le \min\{\int_0^{\omega} d(t)dt/\omega,\ 1/\omega\}$ and $D = \max\{|d(t)| : t \in [0,\omega]\}$.

Proof. Put $t = s + n\omega + \mu$, where $n \in \{0,1,2,\dots\}$ and $0 \le \mu < \omega$. Then we have

$$\begin{aligned} e^{1+D\omega+\int_s^t d(r)dr} &= e^{1+D\omega+\int_s^{s+n\omega+\mu} d(r)dr} \\ &\ge e^{1+n\int_0^{\omega} d(r)dr} \ge e^{1+nc\omega} \\ &\ge e^{c(n\omega+\mu)} = e^{c(t-s)}. \end{aligned} \tag{3.2}$$

The proof is complete. □

Theorem 3.3. *The system (1.3) is permanent if and only if*

$$\int_0^{\omega} a(s)ds + \ln\prod_{i=1}^{q}(1+g_i) > 0, \qquad \int_0^{\omega} d(t)dt > 0. \tag{3.3}$$

Proof. Let $\operatorname{col}(x(t), y(t))$ be a solution of (1.3) with $x(0) > 0$, $y(0) > 0$. Since $x(t) > 0$, $y(t) > 0$ for $t \ge 0$; from (1.3), we have that for $t \ne t_k$, $t \ge 0$,

$$x'(t) \le x(t)\left(a(t) - \sum_{j=1}^{m} b_j(t)x^{\lambda_j}(t)\right), \tag{3.4}$$

$$y'(t) \ge -d(t)y(t). \tag{3.5}$$

Now suppose that (3.3) holds and consider the equations

$$\begin{aligned} u'(t) &= u(t)\left(a(t) - \sum_{j=1}^{m} b_j(t)u^{\lambda_j}(t)\right), \qquad t \ne t_k, \\ u(t_k^+) &= (1+g_k)u(t_k), \end{aligned} \tag{3.6}$$

$$\begin{aligned} v'(t) &= -d(t)v(t), \\ v(t_k^+) &= v(t_k) + p, \end{aligned} \qquad t \ne t_k. \tag{3.7}$$

By Lemmas 2.2 and 2.4, (3.6) and (3.7) have unique positive, globally asymptotically stable ω-periodic solutions $\overline{u}(t)$ and $\overline{v}(t)$, respectively. Let $u(t)$ be the solution of (3.6) with initial value $u(0) = x(0)$, let $v(t)$ be the solution of (3.7) with initial value $v(0) = y(0)$. By using the comparison theorem of impulsive differential equations (see [20, Theorem 1.6.1]), it follows that

$$x(t) \le u(t), \quad y(t) \ge v(t), \quad \text{for } t \ge 0. \tag{3.8}$$

The attractivity of $\overline{u}(t)$ and $\overline{v}(t)$ implies that there exists a $T_1 > 0$ such that

$$u(t) \le \overline{u}(t) + 1,\ v(t) \ge \frac{1}{2}\overline{v}(t) \quad \text{for } t \ge T_1. \tag{3.9}$$

It is clear that $x(t)$ is bounded above and $y(t)$ is bounded below by a positive constant.

Next, we show that $y(t)$ is bounded above. Let $\overline{x} = \sup_{t\ge 0} x(t)$, then $0 < \overline{x} < \infty$. From the second equation of (3.3), we have that when $t \ne t_k$,

$$\begin{aligned} y'(t) &\le y(t)\left(-d(t) + \frac{e(t)\overline{x}^2}{\overline{x}^2 + Ay^2(t)}\right) \\ &\le -d(t)y(t) + \frac{\overline{e}\overline{x}^2}{\overline{x}^2 + Ay^2(t)}y(t), \end{aligned} \tag{3.10}$$

where $\overline{e} = \max_{t\in R} e(t)$. Since the function $g(x) = Bx/(C + Ax^2)$ $(A > 0,\ B > 0,\ C > 0)$ has maximal for $x > 0$, there is $M_4 > 0$ such that when $t \ne t_k$,

$$y'(t) + d(t)y(t) \le M_4. \tag{3.11}$$

For any $t \in (t_j, t_{j+1}]$, we have from Lemma 3.2 that

$$\begin{aligned} y(t) &\le y(t_0)\exp\left(-\int_0^t d(t)dt\right) + M_4\int_0^t \exp\left(\int_t^s d(r)dr\right)ds + \sum_{t_0<t_k<t} p\exp\left(\int_t^{t_k} d(r)dr\right) \\ &\le y(t_0)e^{1+D\omega-ct} + Me^{1+D\omega}\int_0^t e^{c(s-t)}ds + pe^{1+D\omega}\sum_{k=1}^{j} e^{c(t_k-t)} \\ &\le y(t_0)e^{1+D\omega} + \frac{Me^{1+D\omega}}{c}(1-e^{-ct}) + pe^{1+D\omega}\sum_{k=1}^{j} e^{c(1-k)\alpha} \\ &\le y(t_0)e^{1+D\omega} + \frac{Me^{1+D\omega}}{c} + \frac{pe^{1+D\omega}}{1-e^{-c\alpha}}, \end{aligned} \tag{3.12}$$

where $\alpha = \min_{k\ge 1}(t_k - t_{k-1}) > 0$ and constant $c : 0 < c \le \min\{\int_0^\omega d(t)dt/\omega,\ 1/\omega\}$ and $D = \max\{|d(t)| : t \in [0,\omega]\}$. So $y(t)$ is bounded above.

Let $\underline{y} = \min\{y(t),\ t \ge 0\}$, $\overline{y} = \max\{y(t),\ t \ge 0\}$, then $\underline{y} > 0$, $\overline{y} < \infty$. From (1.3), we have that for $t \ne t_k$, $t \ge 0$,

$$\begin{aligned} x'(t) &\ge x(t)\left(a(t) - \sum_{j=1}^{m} b_j(t)x^{\lambda_j}(t) - \frac{b_0(t)x(t)\overline{y}}{x^2(t) + A\underline{y}^2}\right) \\ &\ge x(t)\left(a(t) - \sum_{j=1}^{m} b_j(t)x^{\lambda_j}(t) - \frac{b_0(t)\overline{y}}{A\underline{y}^2}x(t)\right). \end{aligned} \tag{3.13}$$

Consider the equation

$$\begin{aligned} z'(t) &= z(t)\left(a(t) - \sum_{j=1}^{m} b_j(t) z^{\lambda_j}(t) - \frac{b_0(t)\overline{y}}{A\underline{y}^2} z(t)\right), \quad t \neq t_k, \\ z(t_k^+) &= (1+g_k) z(t_k). \end{aligned} \tag{3.14}$$

By Lemma 2.2, (3.14) has a unique positive, globally asymptotically stable ω-periodic solution $\overline{z}(t)$. Let $z(t)$ be the solution of (3.14) with initial value $z(0) = x(0)$. By using the comparison theorem of impulsive differential equations, it follows that

$$x(t) \geq z(t) \quad \text{for } t \geq 0. \tag{3.15}$$

From the attractivity of $\overline{z}(t)$, there exists $T_2 > 0$ such that $z(t) \geq (1/2)\overline{z}(t)$ for $t \geq T_2$. It is clear that $x(t)$ is bounded below by a positive constant. This completes the sufficient part of Theorem 3.3.

Now suppose that the system (1.3) is permanent. If $\int_0^\omega d(t)dt \leq 0$, we obtain from Remark 2.5 that the solution $v(t)$ with initial value $v(0) = y(0)$ of (3.7) satisfies $\lim_{t\to\infty} v(t) = +\infty$. The comparison theorem of an impulsive differential equation implies $\lim_{t\to\infty} y(t) = +\infty$, which is a contradiction. So $\int_0^\omega d(t)dt > 0$.

Let $\int_0^\omega a(s)ds + \ln\prod_{i=1}^{q}(1+g_i) \leq 0$. Since $x(t) \geq C_* > 0$ for $t \geq 0$, from (1.3), we have for $t \neq t_k$, $t \geq 0$,

$$x'(t) \leq x(t)\left(a(t) - b_{i_0}(t) C_*^{\lambda_{i_0}} - \sum_{j=1}^{i_0-1} b_j(t) x^{\lambda_j}(t) - \sum_{j=i_0+1}^{m} b_j(t) x^{\lambda_j}(t)\right). \tag{3.16}$$

Consider the equation

$$\begin{aligned} z'(t) &= z(t)\left(a(t) - b_{i_0}(t) C_*^{\lambda_{i_0}} - \sum_{j=1}^{i_0-1} b_j(t) z^{\lambda_j}(t) - \sum_{j=i_0+1}^{m} b_j(t) z^{\lambda_j}(t)\right), \quad t \neq t_k, \\ z(t_k^+) &= (1+g_k) z(t_k). \end{aligned} \tag{3.17}$$

Clearly, $\int_0^\omega (a(s) - b_{i_0}(t) C_*^{\lambda_{i_0}})ds + \ln\prod_{i=1}^{q}(1+g_i) < 0$. Lemma 2.2(1) implies the solution $z(t)$ with initial value $z(0) = x(0)$ of (3.17) satisfies $\lim_{t\to\infty} z(t) = 0$. From the comparison theorem of impulsive differential equations, one easily obtain that $\lim_{t\to\infty} x(t) = 0$, which is a contradiction. This completes the proof of Theorem 3.3. □

Theorem 3.4. *Suppose that* $\int_0^\omega d(t)dt > 0$ *and*

$$\int_0^\omega a(s)\,ds + \ln\prod_{i=1}^{q}(1+g_i) < 0 \tag{3.18}$$

hold, then the solution $(x(t), y(t))$ *of system (1.3) with any positive initial value satisfies*

$$(x(t), y(t)) \longrightarrow (0, y^*(t)) \quad \text{as } t \longrightarrow \infty, \tag{3.19}$$

where $y^*(t)$ is a unique positive ω-periodic solution of the following equation:

$$\begin{aligned} y'(t) &= -d(t)y(t), \quad t \neq t_k, \\ y(t_k^+) &= y(t_k) + p. \end{aligned} \tag{3.20}$$

4. Positive periodic solution

In this section, we investigate the existence and attractivity of the periodic solution of system (1.3). We say that a positive solution of system (1.3) is globally asymptotically stable if it attracts all the other positive solutions of the system.

Theorem 3.3 shows that when the assumption (3.3) holds, any solution of system (1.3) with a positive initial value ultimately enters the region $\Omega = \{(x,y) | C_* \leq x, y \leq C^*\}$, where C_* and C^* are positive constants. By applying the Brouwer's fixed point theorem, we have the following result.

Theorem 4.1. *Assume (3.3) holds, then system (1.3) has at least one positive ω-positive solution which lies in Ω.*

In order to investigate the uniqueness and stability property of the positive ω-periodic solution, we recall some results on abstract persistence of autonomous semiflows.

Let (X,d) be a Banach space with metric d. Suppose that $T(t): X \to X, t \geq 0$ is a C^0 semiflow on X, that is, $T(0) = E$, $T(t+s) = T(t)T(s)$ for $t,s \geq 0$, and $T(t)x$ is continuous in t and x. $T(t)$ is said to be point dissipative in X if there is a bounded nonempty set B in X such that for any $x \in X$, there is a $t_0 = t_0(x,B) > 0$ such that $T(t)x \in B$ for $t \geq t_0$.

Definition 4.2. Assume that $X = X_0 \cup \partial X$ and $X_0 \cap \partial X_0 = \emptyset$ with X_0 being open in X. The semiflow $T(t): X \to X$ is said to be of uniform persistence with respect to $(X_0, \partial X_0)$ if there exists an $\eta > 0$ such that for any $x \in X_0$, $\liminf_{t\to\infty} d(T(t)x, \partial X_0) \geq \eta$.

The following lemma can be found in [32].

Lemma 4.3. *Let $S: X \to X$ be a continuous map with $S(X_0) \subset X_0$. Assume that*
(1) *S is point dissipative,*
(2) *S is compact,*
(3) *S is uniformly persistent with respect to $(X_0, \partial X_0)$.*

Then there exists a global attractor A_0 for S in X_0 relative to strongly bounded sets in X_0, and S has a coexistence sate $x_0 \in A_0$.

Theorem 4.4. *Assume (3.3) holds, then the ω-positive solution of system (1.3) is globally asymptotically stable.*

Proof. To apply Lemma 4.3, we consider the properties of the solution operator $T(t)$ of the following ordinary differential equations with impulse

$$\begin{aligned} z'(t) &= F(t, z(t)), \quad t \neq t_k, \\ z(t_k^+) - z(t_k) &= I_k(z(t_k)), \quad k \in Z^+, \\ z(0) &= \phi, \end{aligned} \tag{4.1}$$

where $F \in C([0,\infty) \times R^2, R^2)$, $\phi \in R^2$, $F(t+\omega,u) = F(t,u)$, $I_k \in C(R^2,R^2)$ and $\omega > 0$. There exists an integer $q > 0$ such that $I_{k+q} = I_k$, $t_{k+q} = t_k + \omega$.

The solution operator $T(t)$ of system (4.1) can be expressed as

$$T(t)z = ze^{-\lambda t} + \int_0^t e^{-\lambda(t-s)}[F(s,T(s)z) + \lambda T(s)z]ds + \sum_{0<t_k<t} e^{-\lambda(t-t_k)} I_k(T(t_k)z), \tag{4.2}$$

where λ is a positive constant. We show that $T(0) = E$, $T(t+\omega) = T(t)T(\omega)$. Obviously, $T(0) = E$. When $t \neq t_k$, letting

$$u(s) = \begin{cases} T(s)z, & 0 \le s \le \omega, \\ T(s-\omega)T(\omega)z, & \omega \le s \le t+\omega, \end{cases} \tag{4.3}$$

we can verify that $u(s)$, $s \in [0, t+\omega]$ is a solution of (4.1) with an initial value $u(0) = z$. By the uniqueness theorem, we have $T(t+\omega)z = u(t+\omega) = T(t)T(\omega)z$.

When $t = t_k$, we have

$$T(t_k^+ + \omega)z = T(t_k+\omega)z + I_k(T(t_k+\omega)z) = T(t_k)T(\omega)z + I_k(T(t_k)T(\omega)z) = T(t_k^+)T(\omega)z. \tag{4.4}$$

Let $S = T(\omega)$, $S^2 = S \circ S = T(\omega) \circ T(\omega) = T(2\omega)$. Since $T(t)$ is a completely continuous operator (see [19]), so is S. Put

$$\begin{gathered} X_i^+ = \{z_i : z_i \in R,\ z_i \ge 0\}, \quad i = 1,2, \qquad X_{i0}^+ = \{z_i : z_i \in R,\ z_i > 0\}, \quad i = 1,2, \\ X = X_1^+ \times X_2^+, \qquad X_0 = X_{10}^+ \times X_{20}^+, \qquad \partial X_0 = X/X_0. \end{gathered} \tag{4.5}$$

When (3.3) holds, system (1.3) is permanent, thus S satisfies (1) and (3) in Lemma 4.3. By Lemma 4.3, S admits a global attractor. The proof is complete. □

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