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Research Article

Solution Estimates for Semilinear Difference-Delay Equations with Continuous Time

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We consider semilinear difference-delay equations with continuous time in a Euclidean space. Estimates are found for the solutions. Such estimates are then applied to obtain the stability and boundedness criteria for solutions.

1. Introduction and notation

To motivate our study, let us consider the amount of money $x(t)$ saved in a bank at time t. At $t = 0$, an initial amount of money x_0 is deposited and it remains the same until interest is added at time $t = 1$. The principle plus interest remains the same after $t = 1$ until new interest is added at time $t = 2$. In general, $x = x(t)$ is a function defined on $[0,\infty)$ and satisfies

$$x(t+1) = (1+r)x(t) \quad (t > 0, x(0) = x_0, r = \text{const}). \tag{1.1}$$

In the case when money is invested in instruments other than bank savings and for $0 \le t < 1$, $x(t)$ can vary due to speculation or other factors. We may then face mathematical models described by

$$x(t+1) = f(x(t)) \quad (t > 0), \tag{1.2}$$

where we may allow $x = x(t)$ to be vector-valued functions. Clearly, the properties of their solutions are worthy of study. Moreover, the importance of continuous time difference equations in applications is very well explained in several books [1–3]. They appear in economics [1], gas dynamics, and propagation models [3]. They appear also as the internal dynamics of nonlinear differential delay systems, when these systems are input-output linearized and controlled (cf. [4]). Results concerning the stability of the origin, in the linear case, can be found in [1] (see Chapter 9 on the equations of neutral type). In [1, Section 9.6], the stability of linear, time-invariant, continuous time difference equations is studied, taking into account small variations that may occur in the delays. In [1, Chapter 9], necessary and sufficient conditions for the asymptotic stability of linear scalar continuous time difference equations, with arbitrary delays, are given. In [5], asymptotic properties of the solutions of scalar difference equations with continuous argument are treated.

To the best of our knowledge, however, stability of general continuous time difference equations has not been extensively investigated in the literature.

For this reason, we will consider the following difference-delay equation with continuous time in a Euclidean space $\mathbb{C}^n$:

$$x(t) = \sum_{k=1}^{m} A_k x(t-h_k) + F(t, x(t-h_1, \dots, x(t-h_m)) + f(t), \quad t > 0, \tag{1.3}$$

where f is a given vector-valued function, a mapping $F : [0,\infty) \times \mathbb{C}^n \to \mathbb{C}^n$ satisfies the conditions pointed below, A_k are n-by-n matrices, and h_k, $k = 1, \dots, m$, are nonnegative numbers. We derive solution estimates to (1.3). These estimates give us explicit stability and boundedness conditions.

2. Preliminaries

Denote by $\|\cdot\|_n$ the Euclidean norm in $\mathbb{C}^n$. As usually, $L^p(\omega)(1 \le p < \infty)$ is the space of functions defined on a set ω with the finite norm

$$|w|_{L^p(\omega)} = \left[\int_\omega ||w(t)||_n^p dt\right]^{1/p}, \tag{2.1}$$

and $M(\omega)$ is the space of bounded measurable functions defined on ω with the finite sup-norm

$$|w|_{C(\omega)} = \sup_{t \in \omega} ||w(t)||_n. \tag{2.2}$$

In addition, put $R_+ = [0,\infty)$ and consider in $\mathbb{C}^n$ the linear equation

$$v(t) = \sum_{k=1}^{m} A_k v(t-h_k) \quad (t > 0), \tag{2.3}$$

where A_k are n-by-n matrices again and

$$0 \le h_k \le 1 \quad (k = 1, \dots, m). \tag{2.4}$$

Take the initial condition

$$v(t) = \phi(t), \quad t \in [-1, 0], \tag{2.5}$$

where ϕ is a given continuous vector-valued function. Let us apply the Laplace transform to problems (2.3), (2.5). Then

$$\tilde{v}(z) = \sum_{k=1}^{m} A_k e^{-h_k z}\tilde{v}(z) + \tilde{\psi}_0(z), \tag{2.6}$$

where $\tilde{v}(z)$ is the Laplace transform to $v(t)$, z is the dual variable, and

$$\tilde{\psi}_0(z) = \sum_{k=1}^{m} A_k e^{-zh_k}\int_{-h_k}^{0} e^{-zt}\phi(t)dt. \tag{2.7}$$

Hence,

$$\tilde{v}(z) = K^{-1}(e^{-z})\tilde{\psi}_0(z), \tag{2.8}$$

where

$$K(w) := I - \sum_{k=1}^{m} A_k w^{h_k} \quad (w \in \mathbb{C}) \tag{2.9}$$

is the characteristic matrix-valued function.

We will call $K(w)$ a stable function if all its characteristic values (i.e., the zeros of $\det K(w)$) lie in the disc $|w| < 1$. In the case of the equation $x(t) = Ax(t-1)$ with a matrix A, the stability means that the spectral radius $r_s(A)$ of A is less than one. For some necessary and sufficient conditions for stability of operators, one may consult the papers [6, 7] and the book [8]. In the sequel, K is a stable function. Let

$$G(t) = \frac{1}{2\pi i}\int_{\operatorname{Re} z > b_0} e^{zt}K^{-1}(e^{-z})dz = \frac{1}{2\pi i}\int_{C} w^{t-1}K^{-1}(w)dw \tag{2.10}$$

be the Green function to (2.3). Here b_0 is a real constant and C is the Jourdan contour surrounding all the characteristic values of $K(\cdot)$. By the inverse Laplace transform, a solution of (2.3) can be represented as

$$v(t) = \frac{1}{2\pi i}\int_{\operatorname{Re} z > b_0} e^{zt}K^{-1}(e^{-z})\psi_0(e^{-z})dz. \tag{2.11}$$

Clearly, if K is stable, then the maximum r_0 of the absolute values of the roots of $\det K$ is less than one and for any $r \in (r_0, 1)$,

$$\sup_{|z|=r} \|K^{-1}(z)\|_n < \infty, \tag{2.12}$$

and thus taking $C = \{z \in \mathbb{C} : |z| = r < 1\}$, we get

$$\|G(t)\|_n \le c_r r^t \quad (t \ge 0;\ c_r = \text{const}). \tag{2.13}$$

Next, consider the nonhomogeneous equation

$$u(t)=\sum_{k=1}^{m}A_k u(t-h_k)+\zeta(t)\quad (t>0) \tag{2.14}$$

with a given $\zeta\in L^2(R_+)$ and the zero initial condition

$$u(t)=0,\quad t\in[-1,0]. \tag{2.15}$$

Again applying the Laplace transform, we arrive at the equality

$$\tilde{u}(z)=K^{-1}(e^{-z})\tilde{\zeta}(z), \tag{2.16}$$

where $\tilde{u}(z)$ is the Laplace transform to $u(t)$ and $\tilde{\zeta}(z)$ is the Laplace transform to $\zeta(t)$. Hence,

$$y(t)=\int_0^t G(t-s)\zeta(s)ds. \tag{2.17}$$

From (2.14) and the Parseval equality, it easily follows that

$$|u|_{L^2(R_+)}\le\theta(K)|\zeta|_{L^2(R_+)}, \tag{2.18}$$

where

$$\theta(K):=\sup_{|z|=1}||K^{-1}(z)||_n. \tag{2.19}$$

3. The main result

Consider (1.3) under the initial condition

$$x(t)=\phi(t),\quad t\in[-1,0]. \tag{3.1}$$

Put $\Omega(\rho):=\{h\in\mathbb{C}^n:\|h\|_n\le\rho\}$ for a positive number $\rho\le\infty$. It is assumed that $f\in L^2(R_+)$ and F continuously map $R_+\times\Omega^m(\rho)$ into $\mathbb{C}^n$ and satisfy the condition

$$||F(t,v_1,\dots,v_m)||_n\le\sum_{k=1}^{m}\nu_k||v_k||_n\quad (v_k\in\Omega(\rho)). \tag{3.2}$$

Here $\nu_k<\infty$ are nonnegative constants. Put

$$\nu:=\nu_1+\cdots+\nu_m. \tag{3.3}$$

Now we are in a position to formulate our main result.

Theorem 3.1. *Under conditions (3.2) and (2.4), let $f\in L^2(R_+)$ and*

$$\nu\theta(K)<1. \tag{3.4}$$

Then there is a positive constant $c_0 < \infty$, such that the condition

$$c_0\Big(|f|_{L^2(R_+)} + |\phi|_{C(-1,0)}\Big) \le \rho \tag{3.5}$$

implies that a solution x of problems (1.3), (3.1) is in $L^2(R_+)$. Moreover,

$$|x|_{C(R_+)} \le c_0\Big(|\phi|_{C(-1,0)} + |f|_{L^2(R_+)}\Big). \tag{3.6}$$

Proof. First let $\rho = \infty$. From (1.3) and (2.17), it follows that

$$x(t) = v(t) + \int_0^t G(t-s)\big[F(s,x(s-h_1),\ldots,x(t-h_m)) + f(s)\big]ds, \quad t \ge 0, \tag{3.7}$$

where v is a solution of the linear problems (2.3), (2.5). By (2.18),

$$\begin{aligned} &\left|\int_0^t G(t-s)\big[F(s,x(s-h_1),\ldots,x(s-h_m)) + f(s)\big]\right|_{L^2(R_+)} \\ &\quad \le \theta(K)\,\big|F(s,x(s-h_1),\ldots,x(s-h_m)) + f(s)\big|_{L^2(R_+)}. \end{aligned} \tag{3.8}$$

But by (3.2),

$$\begin{aligned} \big|F(s,x(s-h_1),\ldots,x(s-h_m))\big|_{L^2(R_+)} &\le \sum_{k=1}^m \nu_k \big|x(s-h_k)\big|_{L^2(R_+)} \\ &\le \sum_{k=1}^m \nu_k\big(|x|_{L^2(R_+)} + |x|_{L^2(-1,0)}\big) = \nu\big(|x|_{L^2(R_+)} + |\phi|_{L^2(-1,0)}\big). \end{aligned} \tag{3.9}$$

Now from (3.7), it follows that

$$|x|_{L^2(R_+)} \le b_1 + \theta(K)\nu|x|_{L^2(R_+)}, \tag{3.10}$$

where

$$b_1 = |v|_{L^2(R_+)} + \nu\theta(K)|\phi|_{L^2(-1,0)} + \theta(K)|f|_{L^2(R_+)}. \tag{3.11}$$

Since K is stable, (2.11) yields us the inequalities

$$|v|_{C(R_+)} \le \text{const}|\phi|_{C(-1,0)}, \qquad |v|_{L^2(R_+)} \le \text{const}|\phi|_{C(-1,0)}. \tag{3.12}$$

Thus due to (3.4),

$$|x|_{L^2(R_+)} \le \big(1-\nu\theta(K)\big)^{-1} b_1 \le \text{const}\big(|f|_{L^2(R_+)} + |\phi|_{C(-1,0)}\big). \tag{3.13}$$

Furthermore, from (3.7) and the Schwarz inequality now it follows that

$$|x|_{C(R_+)} \le |v|_{C(R_+)} + |G|_{L^2(R_+)}\big(\big|F(s,x(s-h_1),\ldots,x(s-h_m))\big|_{L^2(R_+)} + |f|_{L^2(R_+)}\big). \tag{3.14}$$

But according to (3.2) and (3.13),

$$\begin{aligned} & |F(s,x(s-h_1),\dots,x(s-h_m))|_{L^2(R_+)} \\ & \quad \le \nu(|x|_{L^2(R_+)}+|\phi|_{L^2(-1,0)}) \le \text{const}(|f|_{L^2(R_+)}+|\phi|_{C(-1,0)}). \end{aligned} \tag{3.15}$$

Hence, the required inequality (3.6) follows, provided $\rho=\infty$.

Next, let $\rho<\infty$. On $\mathbb{C}^n$, let us define the function

$$\mu(y)=\begin{cases} 1, & \|y\|_n \le \rho, \\ 0, & \|y\|_n > \rho \end{cases} \tag{3.16}$$

for all $y \in \mathbb{C}^n$. Such a function always exists due to the Urysohn theorem [9, page 15]. Consider the equation

$$\tilde{x}(t)=\sum_{k=1}^{m} A_k\tilde{x}(t-h_k)+\mu(x)F(t,\tilde{x}(t-h_1),\dots,\tilde{x}(t-h_m))+f(t), \quad t\ge 0. \tag{3.17}$$

Since

$$\mu(w(t))\|F(t,w(t-h_1),\dots,w(t-h_m))\|_n \le \sum_{k=1}^{m}\nu_k\|w(t-h_k)\|_n \tag{3.18}$$

for any function $w\in M(0,T)$ with an arbitrary positive $T<\infty$, we can apply our above results to (3.17). Namely, for a solution $\tilde{x}$ of (3.17), estimate (3.6) holds according to the above arguments and (3.18). But $\mu(w(t))F(t,w(t-h_1),\dots,w(t-h_m))$ and $F(t,w(t-h_1),\dots,w(t-h_m))$ coincide for any function w with values in $\Omega(\rho)$. According to condition (3.5), $\tilde{x}(t)\in\Omega(\rho)$, $t\ge 0$, and $x(t)=\tilde{x}(t)$ for $t\ge 0$. This proves the theorem. □

4. Stability of solutions and bounds for $\theta(K)$

We first recall some usual definitions related to the asymptotic properties of solutions.

(i) The zero solution of the equation

$$x(t)=\sum_{k=1}^{m} A_k x(t-h_k)+F(t,x(t-h_1,\dots,x(t-h_m))), \quad t>0, \tag{4.1}$$

is said to be stable if, for any $\varepsilon>0$, there exists $\delta>0$, such that the inequality $|\phi|_{C(-1,0)}\le\delta$ implies $\|x(t)\|\le\varepsilon$, $t>0$, for a solution $x(t)$ of problems (4.1), (3.1).

(ii) The zero solution of (4.1) is said to be attractive if there exists $\delta>0$, such that the inequality $|\phi|_{C(-1,0)}\le\delta$ implies $x(t)\to 0$ as $t\to\infty$ for a solution $x(t)$ of problems (3.1), (4.1).

(iii) The zero solution of (4.1) is said to be asymptotically stable if it is both stable and attractive. It is globally asymptotically stable if one can take $\delta=\infty$.

Theorem 3.1 implies the following result.

Corollary 4.1. *Under the hypothesis of Theorem 1.1, the zero solution of (4.1) is asymptotically stable. Moreover, it is globally asymptotically stable if (3.2) holds with $\rho=\infty$.*

Let us say that (4.1) is a *quasilinear equation* if

$$\lim_{v_1,\dots,v_m\to 0}\frac{||F(t,v_1,\dots,v_m)||_n}{\sum_{k=1}^m||v_k||_n}=0 \tag{4.2}$$

uniformly in $t\ge 0$.

THEOREM 4.2 (stability in the linear approximation). *Let (4.1) be a quasilinear equation with a stable characteristic function K. Then the zero solution to (4.1) is asymptotically stable.*

Proof. In view of (4.2), condition (3.2) holds for all sufficiently small $\rho>0$, and $\nu\to 0$ as $\rho\to 0$. So we can take ρ sufficiently small in such a way that condition (3.4) holds. By Corollary 4.1, we may then obtain the desired stability. □

Let us derive estimates for $\theta(K)$. Let $N(A)$ be the Frobenius (Hilbert-Schmidt) norm of an $n\times n$ matrix A, that is,

$$N^2(A)=\text{Trace } AA^*. \tag{4.3}$$

Here A^* is the adjoint. As it was proved in [10, Chapter 2],

$$||A^{-1}||_n\le\frac{N^{n-1}(A)}{|\det A|(n-1)^{(n-1)/2}}, \tag{4.4}$$

provided A is invertible. Hence,

$$||K^{-1}(z)||_n\le\frac{N^{n-1}(K(z))}{|\det K(z)|(n-1)^{(n-1)/2}}. \tag{4.5}$$

But

$$N^{n-1}(K(z))=N\left(I-\sum_{k=1}^m A_k z^{h_k}\right)\le\sqrt{n}+\sum_{k=1}^m N(A_k)\quad(|z|=1). \tag{4.6}$$

Thus

$$\theta(K)\le\widetilde{\theta}(K):=\frac{\left(\sqrt{n}+\sum_{k=1}^m N(A_k)\right)^{n-1}}{\inf_{|z|=1}|\det K(z)|}. \tag{4.7}$$

So in Theorems 3.1 and 4.2, one can replace $\theta(K)$ by $\widetilde{\theta}(K)$. Note also that any eigenvalue $\lambda_j(K(z))$ of $K(z)$ with fixed K is equal to $1-\lambda_j(B(z))$, where $\lambda_j(B(z))$ is the eigenvalue of

$$B(z):=\sum_{k=1}^m A_k z^{h_k}. \tag{4.8}$$

So if in some norm $\|\cdot\|_a$ the inequality

$$\gamma_0:=\sum_{k=1}^m||A_k||_a<1 \tag{4.9}$$

holds, then $\inf_{|z|=1}|\det K(z)| \geq (1-\gamma_0)^n$. About localization of the characteristic values of matrix-valued functions, see, for instance, [11, 12] and references therein. For lower estimates for determinants, consult [13].

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