

Research Article

Precise Rates in Log Laws for NA Sequences

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Let X_1, X_2, \dots be a strictly stationary sequence of negatively associated (NA) random variables with $EX_1 = 0$, set $S_n = X_1 + \dots + X_n$, suppose that $\sigma^2 = EX_1^2 + 2\sum_{n=2}^{\infty} EX_1 X_n > 0$ and $EX_1^2 < \infty$, if $-1 < \alpha \leq 1$; $EX_1^2(\log |X_1|)^\alpha < \infty$, if $\alpha > 1$. We prove $\lim_{\epsilon \downarrow 0} \epsilon^{2\alpha+2} \sum_{n=1}^{\infty} ((\log n)^\alpha / n) P(|S_n| \geq \sigma(\epsilon + \kappa_n) \sqrt{2n \log n}) = 2^{-(\alpha+1)} (\alpha+1)^{-1} E|N|^{2\alpha+2}$, where $\kappa_n = O(1/\log n)$ and N is the standard normal random variable.

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1. Introduction

A finite family of random variables, X_1, X_2, \dots, X_n , is said to be NA if, for every pair of disjoint subsets T_1 and T_2 of $\{1, 2, \dots, n\}$,

$$\text{Cov}(f_1(X_i, i \in T_1), f_2(X_j, j \in T_2)) \leq 0, \quad (1.1)$$

whenever f_1 and f_2 are coordinatewise increasing and the covariance exists. An infinite family is NA if every finite subfamily is NA. This definition was introduced by Alam and Saxena [1] and Joag-Dev and Proschan [2], and has found many applications in percolation theory, multivariate statistical analysis, and reliability theory (see, e.g., Barlow and Proschan [3]).

Let $\{X_n : n \geq 1\}$ be a sequence of NA random variables on some probability space (Ω, \mathcal{F}, P) with mean zero and finite variance. As usual, set $S_0 = 0$, $S_n = X_1 + \dots + X_n$, $n \geq 1$, and write $\sigma_n^2 = ES_n^2$. Under appropriate covariance conditions, many limit theorems have been obtained. For example, the central limit theorem was proved by Newman [4].

THEOREM 1.1. *Let $\{X_n : n \geq 1\}$ be strictly stationary NA sequences with mean zero and*

$$0 < \sigma^2 = EX_1^2 + 2 \sum_{n=2}^{\infty} EX_1 X_n < \infty, \tag{1.2}$$

then

$$\frac{S_n}{(\sigma\sqrt{n})} \xrightarrow{\mathcal{D}} N(0,1), \quad \text{as } n \rightarrow \infty. \tag{1.3}$$

Further results are three series theorems (see, e.g., Matula [5]), probability inequalities (cf. Roussas [6], Shao [7]), weak convergence (see, e.g., Zhang [8]), the complete convergence (cf. Liang and Su [9], Liang [10]), and the law of the iterated logarithm (see, e.g., Shao and Su [11], Zhang [12]), and so forth.

Note that in the above-mentioned limit theorems, the convergence rates of logarithm are little known, the purpose of the present paper is to investigate the precise asymptotics in the law of the logarithm for NA sequences. It is well known that NA sequences can contain independent random variables as special case, many authors have given lots of beautiful results for independent variables. Let us first recall parts of those results, it is very convenient to adopt the following notations: let X_1, X_2, \dots be independent and identically distributed (i.i.d.) nondegenerate random variables with $EX_1 = 0$ and $EX_1^2 = \sigma^2 < \infty$, set $S_n = X_1 + \dots + X_n$, $\log x = \log_e(x \vee e)$. Chow and Lai [13] studied the following results.

THEOREM 1.2. *Suppose that $\text{Var } X_1 = \sigma^2$ and $\alpha \geq 1$. Then the following are equivalent:*

$$\begin{aligned} & \sum_{n=1}^{\infty} n^{\alpha-2} P(|S_n| \geq \epsilon \sqrt{2n \log n}) < \infty, \quad \forall \epsilon > \sigma\sqrt{\alpha-1}; \\ & \sum_{n=1}^{\infty} n^{\alpha-2} P\left(\max_{1 \leq k \leq n} |S_k| \geq \epsilon \sqrt{2n \log n}\right) < \infty, \quad \forall \epsilon > \sigma\sqrt{\alpha-1}; \\ & \sum_{n=1}^{\infty} n^{\alpha-2} P(|S_n| \geq \epsilon \sqrt{2n \log n}) < \infty, \quad \text{for some } \epsilon > 0; \\ & EX_1 = 0, \quad \frac{E|X_1|^{2\alpha}}{(\log|X_1|)^\alpha} < \infty. \end{aligned} \tag{1.4}$$

Heyde [14] presented an interesting and beautiful result.

THEOREM 1.3. *If $EX_1 = 0$ and $EX_1^2 < \infty$, then*

$$\lim_{\epsilon \downarrow 0} \epsilon^2 \sum_{n=1}^{\infty} P(|S_n| \geq \epsilon n) = EX_1^2. \tag{1.5}$$

This is a precise estimate for the convergence rate of probability series as $\epsilon \downarrow 0$, which has been generalized and extended in several directions. For $\alpha = 1$ in Theorem 1.2, Gut and Spätaru [15] obtained the results as follows.

THEOREM 1.4. Suppose that $EX_1 = 0$ and $EX_1^2 = \sigma^2 < \infty$. Then, for $0 \leq \delta \leq 1$,

$$\lim_{\epsilon \downarrow 0} \epsilon^{2\delta+2} \sum_{n=1}^{\infty} \frac{(\log n)^\delta}{n} P(|S_n| \geq \epsilon \sqrt{n \log n}) = \frac{\sigma^{2\delta+2} E|N|^{2\delta+2}}{\delta+1}, \quad (1.6)$$

where N is a standard normal random variable.

Our starting point is Theorem 1.4, the present work will give the analogue of (1.6) for NA sequences. From now on, we adopt the following notations: let X_1, X_2, \dots be strictly stationary NA sequences with $EX_1 = 0$ and $EX_1^2 < \infty$, $\sigma^2 = EX_1^2 + 2 \sum_{n=2}^{\infty} EX_1 X_n > 0$, and set $S_n = X_1 + \dots + X_n$, $M_n = \max_{1 \leq k \leq n} |S_k|$, write \log for the natural logarithm, $\log x = \log_e(x \vee e)$, $[z]$ denotes the largest integer which is not larger than z , C denotes positive constant, independent of ϵ , it may take different values in each appearance. The paper is organized as follows: we first introduce our main results, after which the proofs of Theorems 2.1 and 2.4 are exposed in Sections 3 and 4, respectively. We now state the main results.

2. Main results

THEOREM 2.1. Let $\kappa_n = O(1/\log n)$, $EX_1^2 < \infty$, if $-1 < \alpha \leq 1$; $EX_1^2 (\log |X_1|)^\alpha < \infty$, if $\alpha > 1$. Then

$$\begin{aligned} \lim_{\epsilon \downarrow 0} \epsilon^{2\alpha+2} \sum_{n=1}^{\infty} \frac{(\log n)^\alpha}{n} P(|S_n| \geq \sigma(\epsilon + \kappa_n) \sqrt{2n \log n}) \\ = 2^{-(\alpha+1)} (\alpha+1)^{-1} E|N|^{2\alpha+2}, \end{aligned} \quad (2.1)$$

where N is a standard normal random variable.

COROLLARY 2.2. Under the conditions in Theorem 2.1,

$$\lim_{\epsilon \downarrow 0} \epsilon^{2\alpha+2} \sum_{n=1}^{\infty} \frac{(\log n)^\alpha}{n} P(|S_n| \geq \sigma \epsilon \sqrt{2n \log n}) = \frac{E|N|^{2\alpha+2}}{2^{(\alpha+1)} (\alpha+1)}. \quad (2.2)$$

COROLLARY 2.3. Suppose that $EX_1^2 < \infty$. Then

$$\lim_{\epsilon \downarrow 0} \epsilon^2 \sum_{n=1}^{\infty} n^{-1} P(|S_n| \geq \sigma(\epsilon + \kappa_n) \sqrt{2n \log n}) = \frac{EN^2}{2}. \quad (2.3)$$

THEOREM 2.4. Let $\kappa_n = O(1/\log n)$, $EX_1^2 < \infty$, if $-1 < \alpha \leq 1/2$; $EX_1^2 (\log |X_1|)^\alpha < \infty$, if $\alpha > 1/2$. Then

$$\begin{aligned} \lim_{\epsilon \downarrow 0} \epsilon^{2\alpha+2} \sum_{n=1}^{\infty} \frac{(\log n)^\alpha}{n} P(M_n \geq \sigma(\epsilon + \kappa_n) \sqrt{2n \log n}) \\ = 2^{-\alpha} (\alpha+1)^{-1} E|N|^{2\alpha+2} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^{2\alpha+2}}. \end{aligned} \quad (2.4)$$

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Without loss of generality, throughout the paper, we will suppose that $\sigma^2 = 1$. Let $\Phi(x)$ denote the standard normal distribution function, and put $\Psi(x) = 1 - \Phi(x) + \Phi(-x)$, $x \geq 0$.

3. Proof of Theorem 2.1

In order to prove this result easily, we separate the proof into two propositions, the first one can be formulated as follows.

PROPOSITION 3.1. *Suppose that N be a nondegenerate Gaussian random variable. Then*

$$\begin{aligned} \lim_{\epsilon \downarrow 0} \epsilon^{2\alpha+2} \sum_{n=1}^{\infty} \frac{(\log n)^\alpha}{n} P(|N| \geq (\epsilon + \kappa_n) \sqrt{2 \log n}) \\ = 2^{-(\alpha+1)} (\alpha + 1)^{-1} E|N|^{2(\alpha+1)}. \end{aligned} \quad (3.1)$$

Proof. Noting the definition of κ_n , we first show that

$$\lim_{\epsilon \downarrow 0} \epsilon^{2\alpha+2} \sum_{n=1}^{\infty} \frac{(\log n)^\alpha}{n} P(|N| \geq \epsilon \sqrt{2 \log n}) = 2^{-(\alpha+1)} (\alpha + 1)^{-1} E|N|^{2\alpha+2}. \quad (3.2)$$

By integral formula and transformation, it is enough to show that for any $\alpha > -1$,

$$\begin{aligned} \lim_{\epsilon \downarrow 0} \epsilon^{2\alpha+2} \sum_{n=1}^{\infty} \frac{(\log n)^\alpha}{n} P(|N| \geq \epsilon \sqrt{2 \log n}) \\ = \lim_{\epsilon \downarrow 0} \epsilon^{2\alpha+2} \sum_{n=1}^{\infty} \int_n^{n+1} \frac{(\log x)^\alpha}{x} P(|N| \geq \epsilon \sqrt{2 \log x}) dx \\ = 2^{-\alpha} \int_0^\infty y^{2\alpha+1} P(|N| \geq y) dy \\ = 2^{-(\alpha+1)} (\alpha + 1)^{-1} E|N|^{2\alpha+2}. \end{aligned} \quad (3.3)$$

Write

$$A_n(\epsilon) = \left| P(|N| \geq \epsilon \sqrt{2 \log n}) - P(|N| \geq (\epsilon + \kappa_n) \sqrt{2 \log n}) \right|. \quad (3.4)$$

The proof of (3.1) should be completed, if one could show that

$$\lim_{\epsilon \downarrow 0} \epsilon^{2\alpha+2} \sum_{n=1}^{\infty} \frac{(\log n)^\alpha}{n} A_n(\epsilon) = 0, \quad (3.5)$$

the proof of (3.5) is similar to that of Proposition 2.2 in Huang and Zhang [16].

Before giving the second proposition, the following lemma is necessary. \square

LEMMA 3.2 [17]. Suppose that $\{X_k : k \geq 1\}$ be NA sequences with $EX_k = 0$, $E|X_k|^p < \infty$, for $p \geq 2$. Then, for any $t > p/2$, $x > 0$,

$$P(|S_n| \geq x) \leq \sum_{k=1}^n P\left(|X_k| \geq \frac{x}{t}\right) + 2e^t \left(1 + \frac{x^2}{t \sum_{k=1}^n EX_k^2}\right)^{-t}. \quad (3.6)$$

PROPOSITION 3.3. Suppose that $EX_1^2 < \infty$, if $-1 < \alpha \leq 1$; $EX_1^2(\log|X_1|)^\alpha < \infty$, if $\alpha > 1$. Then

$$\lim_{\epsilon \downarrow 0} \epsilon^{2\alpha+2} \sum_{n=1}^{\infty} \frac{(\log n)^\alpha}{n} \left| P(|S_n| \geq (\epsilon + \kappa_n)\sqrt{2n \log n}) - P(|N| \geq (\epsilon + \kappa_n)\sqrt{2 \log n}) \right| = 0. \quad (3.7)$$

Proof. Set $H(\epsilon) = \lceil \exp(M/\epsilon^2) \rceil$, where $M > 4$, $0 < \epsilon < 1/4$. It is easy to get

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{(\log n)^\alpha}{n} \left| P(|S_n| \geq (\epsilon + \kappa_n)\sqrt{2n \log n}) - P(|N| \geq (\epsilon + \kappa_n)\sqrt{2 \log n}) \right| \\ &= \sum_{n \leq H(\epsilon)} \frac{(\log n)^\alpha}{n} \left| P(|S_n| \geq (\epsilon + \kappa_n)\sqrt{2n \log n}) - P(|N| \geq (\epsilon + \kappa_n)\sqrt{2 \log n}) \right| \\ &+ \sum_{n > H(\epsilon)} \frac{(\log n)^\alpha}{n} \left| P(|S_n| \geq (\epsilon + \kappa_n)\sqrt{2n \log n}) - P(|N| \geq (\epsilon + \kappa_n)\sqrt{2 \log n}) \right| \triangleq I_1 + I_2. \end{aligned} \quad (3.8)$$

We first consider I_1 . Let $\Delta_n = \sup_x |P(|S_n| \geq x\sqrt{n}) - P(|N| \geq x)|$, noting Theorem 1.1, since $\Psi(x)$ is a continuous function, then, for any $x \geq 0$, we have $\lim_{n \rightarrow \infty} \Delta_n = 0$. It follows that

$$\begin{aligned} \epsilon^{2\alpha+2} I_1 &\leq \epsilon^{2\alpha+2} \sum_{n \leq H(\epsilon)} \frac{(\log n)^\alpha}{n} \Delta_n = \epsilon^{2\alpha+2} \sum_{n \leq H(\epsilon)} \frac{(\log n)^\alpha}{n} \Delta_n \\ &\leq C \epsilon^{2\alpha+2} (\log n)^{\alpha+1} \frac{1}{(\log n)^{\alpha+1}} \sum_{n \leq H(\epsilon)} \frac{(\log n)^\alpha}{n} \Delta_n \\ &\leq C M^{\alpha+1} \frac{1}{(\log n)^{\alpha+1}} \sum_{n \leq H(\epsilon)} \frac{(\log n)^\alpha}{n} \Delta_n \rightarrow 0, \quad \text{as } \epsilon \downarrow 0. \end{aligned} \quad (3.9)$$

Note that $(1/(\log n)^{\alpha+1}) \sum_{n \leq H(\epsilon)} ((\log n)^\alpha/n) \Delta_n \rightarrow 0$, $\epsilon \downarrow 0$, then (3.9) holds. Turn to I_2 , one can get

$$\begin{aligned} I_2 &\leq \sum_{n > H(\epsilon)} \frac{(\log n)^\alpha}{n} P(|N| \geq (\epsilon + \kappa_n)\sqrt{2 \log n}) \\ &+ \sum_{n > H(\epsilon)} \frac{(\log n)^\alpha}{n} P(|S_n| \geq (\epsilon + \kappa_n)\sqrt{2n \log n}) \triangleq I_3 + I_4. \end{aligned} \quad (3.10)$$

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Without loss of generality, we can assume that $0 < \epsilon < 1/4$, $M > 4$, then one can get $H(\epsilon) - 1 \geq \sqrt{H(\epsilon)}$. Note, in particular, that the definition of κ_n , for n large enough, we have $|\kappa_n| < \epsilon/4$. Then, for I_3 , it follows that

$$\begin{aligned}
 \epsilon^{2\alpha+2} I_3 &\leq \epsilon^{2\alpha+2} \sum_{n>H(\epsilon)} \frac{(\log n)^\alpha}{n} P(|N| \geq \frac{\epsilon}{2} \sqrt{2 \log n}) \\
 &\leq \epsilon^{2\alpha+2} \int_{H(\epsilon)-1}^{\infty} \frac{(\log x)^\alpha}{x} P(|N| \geq \frac{\epsilon}{2} \sqrt{2 \log x}) dx \\
 &\leq \epsilon^{2\alpha+2} \int_{\sqrt{H(\epsilon)}}^{\infty} \frac{(\log x)^\alpha}{x} P(|N| \geq \frac{\epsilon}{2} \sqrt{2 \log x}) dx \\
 &\leq C \int_{\sqrt{M/4}}^{\infty} y^{2\alpha+1} P(|N| > y) dy \rightarrow 0, \quad \text{as } M \rightarrow \infty,
 \end{aligned} \tag{3.11}$$

uniformly with respect to $0 < \epsilon < 1/4$. We finally estimate I_4 , by Lemma 3.2, which yields, for n large enough,

$$\begin{aligned}
 P(|S_n| \geq (\epsilon + \kappa_n) \sqrt{2n \log n}) &\leq P(|S_n| \geq \frac{\epsilon}{2} \sqrt{2n \log n}) \\
 &\leq nP\left(|X_1| \geq \frac{\epsilon}{2m} \sqrt{2n \log n}\right) + 2e^m \left(1 + \frac{\epsilon^2 \log n}{2mEX_1^2}\right)^{-m} \\
 &\triangleq I_5 + I_6,
 \end{aligned} \tag{3.12}$$

where m is a positive integer to be specified later. Then, observe that $n > H(\epsilon)$ implies $\log n > M/\epsilon^2$, for I_5 , if $-1 < \alpha \leq 1$, the proof is similar to that of Lemma 3.2 [15]; if $\alpha > 1$, applying Fubini's theorem, it turns out that

$$\begin{aligned}
 \sum_{n>H(\epsilon)} \frac{(\log n)^\alpha}{n} I_5 &= \sum_{n>H(\epsilon)} (\log n)^\alpha P\left(|X_1| \geq \frac{\epsilon \sqrt{2n \log n}}{2m}\right) \\
 &\leq C \sum_{n>H(\epsilon)} (\log n)^\alpha \sum_{j=n}^{\infty} P\left(\frac{\sqrt{2Mj}}{2m} \leq |X_1| < \frac{\sqrt{2M(j+1)}}{2m}\right) \\
 &\leq C \sum_{j>H(\epsilon)} P\left(\frac{\sqrt{2Mj}}{2m} \leq |X_1| < \frac{\sqrt{2M(j+1)}}{2m}\right) \sum_{n=H(\epsilon)}^j (\log n)^\alpha \\
 &\leq C \sum_{j>H(\epsilon)} j(\log j)^\alpha P\left(j \leq \frac{2m^2 X_1^2}{M} < j+1\right) \leq \frac{CEX_1^2 (\log |X_1|)^\alpha}{M}.
 \end{aligned} \tag{3.13}$$

Furthermore, one can easily get $\limsup_{\epsilon \downarrow 0} \epsilon^{2\alpha+2} \sum_{n>H(\epsilon)} (\log n)^\alpha I_5/n = 0$. We finally estimate I_6 , by the arbitrariness of $m(> 1)$, one can obviously choose an appropriate positive

integer m , such that $m > \alpha + 1$. Then we have

$$\begin{aligned}
 \sum_{n>H(\epsilon)} \frac{(\log n)^\alpha}{n} I_6 &\leq C \sum_{n>H(\epsilon)} \frac{(\log n)^\alpha}{n} \left(\frac{\epsilon^2 \log n}{2mEX_1^2} \right)^{-m} \\
 &\leq C \sum_{n>H(\epsilon)} \frac{(\log n)^\alpha}{n} (\epsilon^2 \log n)^{-m} \\
 &\leq C\epsilon^{-2m} \int_{H(\epsilon)-1}^{\infty} \left(\frac{(\log x)^{\alpha-m}}{x} \right) dx \\
 &\leq C\epsilon^{-2m} (\log(H(\epsilon)))^{\alpha+1-m} \\
 &\leq C\epsilon^{-2\alpha-2} M^{\alpha+1-m},
 \end{aligned} \tag{3.14}$$

it is easy to get $\lim_{M \rightarrow \infty} \epsilon^{2\alpha+2} \sum_{n>H(\epsilon)} (\log n)^\alpha I_6/n = 0$, uniformly with respect to $0 < \epsilon < 1/4$. Thus the proof of Proposition 3.3 is completed. \square

Proof of Theorem 2.1. Combining Propositions 3.1 and 3.3, one can complete the proof of this theorem immediately. \square

4. Proof of Theorem 2.4

The following propositions will simplify the proof of Theorem 2.4, which are stated as follows.

PROPOSITION 4.1. *Suppose that $\{W(t) : t \geq 0\}$ be a standard Wiener process (Brownian motion). Then*

$$\begin{aligned}
 \lim_{\epsilon \downarrow 0} \epsilon^{2\alpha+2} \sum_{n=1}^{\infty} \frac{(\log n)^\alpha}{n} P\left(\sup_{0 \leq s \leq 1} |W(s)| \geq (\epsilon + \kappa_n) \sqrt{2 \log n} \right) \\
 = 2^{-\alpha} (\alpha + 1)^{-1} E|N|^{2\alpha+2} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^{2\alpha+2}}.
 \end{aligned} \tag{4.1}$$

Proof. Noting the result of Billingsley [18],

$$\begin{aligned}
 P\left(\sup_{0 \leq s \leq 1} |W(s)| \geq x \right) &= 1 - \sum_{k=-\infty}^{\infty} (-1)^k P((2k-1)x \leq N \leq (2k+1)x) \\
 &= 4 \sum_{k=0}^{\infty} (-1)^k P(N \geq (2k+1)x) = 2 \sum_{k=0}^{\infty} (-1)^k P(|N| \geq (2k+1)x),
 \end{aligned} \tag{4.2}$$

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where N is the standard normal random variable. Then, according to Proposition 3.1, one can complete the proof easily. \square

LEMMA 4.2 [7]. *Suppose that $\{X_n : n \geq 1\}$ be strictly stationary NA sequences, $EX_1 = \mu$, $0 < \text{Var} X_1 = \sigma^2 < \infty$, and $B^2 = EX_1^2 + 2 \sum_{n=2}^{\infty} EX_1 X_n > 0$, set $S_m = \sum_{k=1}^m X_k$, write*

$$W_n(t) = \frac{1}{B\sqrt{n}}(S_m + (nt - m)X_{m+1} - nt\mu), \quad m \leq n < m+1, 0 \leq t \leq T. \quad (4.3)$$

Then

$$W_n(t) \xrightarrow{\mathcal{D}} W(t) \quad \text{in } C[0, T], \quad (4.4)$$

where $W(t)$ is the standard Wiener process and $C[0, T]$ is the usual C space on $[0, T]$.

LEMMA 4.3 [11]. *Let $\{X_n : n \geq 1\}$ be a sequence of NA random variable with mean zero and finite second moments. Set $S_n = X_1 + \dots + X_n$ and $B_n^2 = \sum_{k=1}^n EX_k^2$. Then for all $x > 0$, $a > 0$, and $0 < \beta < 1$,*

$$P\left(\max_{1 \leq k \leq n} |S_k| \geq x\right) \leq 2P\left(\max_{1 \leq k \leq n} |X_k| \geq a\right) + \frac{2}{1-\beta} \exp\left(-\frac{\beta x^2}{2(ax + B_n^2)}\right). \quad (4.5)$$

PROPOSITION 4.4. *Suppose that $EX_1^2 < \infty$, if $-1 < \alpha \leq 1/2$; $EX_1^2(\log |X_1|)^\alpha < \infty$, if $\alpha > 1/2$. Then*

$$\begin{aligned} \lim_{\epsilon \downarrow 0} \epsilon^{2\alpha+2} \sum_{n \geq 1} \frac{(\log n)^\alpha}{n} & \left| P\left(M_n \geq (\epsilon + \kappa_n)\sqrt{2n \log n}\right) \right. \\ & \left. - P\left(\sup_{0 \leq s \leq 1} |W(s)| \geq (\epsilon + \kappa_n)\sqrt{2 \log n}\right) \right| = 0. \end{aligned} \quad (4.6)$$

Proof. Let $H(\epsilon)$ be as above, it follows that

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{(\log n)^\alpha}{n} \left| P\left(M_n \geq (\epsilon + \kappa_n)\sqrt{2n \log n}\right) - P\left(\sup_{0 \leq s \leq 1} |W(s)| \geq (\epsilon + \kappa_n)\sqrt{2 \log n}\right) \right| \\ &= \sum_{n \leq H(\epsilon)} \frac{(\log n)^\alpha}{n} \left| P\left(M_n \geq (\epsilon + \kappa_n)\sqrt{2n \log n}\right) - P\left(\sup_{0 \leq s \leq 1} |W(s)| \geq (\epsilon + \kappa_n)\sqrt{2 \log n}\right) \right| \\ &+ \sum_{n > H(\epsilon)} \frac{(\log n)^\alpha}{n} \left| P\left(M_n \geq (\epsilon + \kappa_n)\sqrt{2n \log n}\right) - P\left(\sup_{0 \leq s \leq 1} |W(s)| \geq (\epsilon + \kappa_n)\sqrt{2 \log n}\right) \right| \\ &\triangleq I'_1 + I'_2. \end{aligned} \quad (4.7)$$

Noting Lemma 4.2, we have $M_n/\sqrt{n} \xrightarrow{\mathcal{Q}} \sup_{0 \leq t \leq 1} |W(t)|$, as $n \rightarrow \infty$. Similar to Theorem 2.1, one can get $\lim_{\epsilon \downarrow 0} \epsilon^{2\alpha+2} I'_1 = 0$. We now estimate I'_2 , it turns out that

$$\begin{aligned} I'_2 &\leq \sum_{n>H(\epsilon)} \frac{(\log n)^\alpha}{n} P\left(\sup_{0 \leq s \leq 1} |W(s)| \geq (\epsilon + \kappa_n)\sqrt{2\log n}\right) \\ &\quad + \sum_{n>H(\epsilon)} \frac{(\log n)^\alpha}{n} P\left(\max_{1 \leq k \leq n} |S_k| \geq (\epsilon + \kappa_n)\sqrt{2n\log n}\right) \triangleq I'_3 + I'_4. \end{aligned} \quad (4.8)$$

Observe that $P(\sup_{0 \leq s \leq 1} |W(s)| \geq x) \leq 2P(|N| \geq x)$, see [18]. Similar to Theorem 2.1, we have $\lim_{\epsilon \downarrow 0} \epsilon^{2\alpha+2} I'_3 = 0$. We then consider I'_4 , as a matter of fact, by Lemma 4.3, take $x = \epsilon\sqrt{2n\log n}/2$, $a = (2n\epsilon\sqrt{\log n})^{1/2}$. For n large enough, one could get

$$\begin{aligned} P\left(\max_{1 \leq k \leq n} |S_k| \geq (\epsilon + \kappa_n)\sqrt{2n\log n}\right) &\leq P\left(\max_{1 \leq k \leq n} |S_k| \geq \frac{\epsilon}{2}\sqrt{2n\log n}\right) \\ &\leq 2nP\left(|X_1| \geq (2n\epsilon\sqrt{\log n})^{1/2}\right) + \frac{2}{1-\beta} \exp\left(-\frac{\beta\epsilon^2 \log n}{8((\epsilon\sqrt{\log n})^{3/2} + 1)}\right) \triangleq I'_5 + I'_6. \end{aligned} \quad (4.9)$$

Without loss of generality, we can assume $0 < \epsilon < 1/4$, $M > 16$, Notice that $n > H(\epsilon)$ if and only if $\log n > M/\epsilon^2$, then for I'_6 , we have

$$\begin{aligned} \epsilon^{2\alpha+2} \sum_{n>H(\epsilon)} \frac{(\log n)^\alpha I'_6}{n} &\leq C\epsilon^{2\alpha+2} \sum_{n>H(\epsilon)} (\log n)^\alpha \exp\left(\frac{-\beta(\epsilon\sqrt{\log n})^{1/2}/9}{n}\right) \\ &\leq C\epsilon^{2\alpha+2} \int_{H(\epsilon)-1}^{\infty} (\log x)^\alpha \exp\left(\frac{-\beta(\epsilon\sqrt{\log x})^{1/2}}{9}\right) \frac{dx}{x} \\ &\leq C\epsilon^{2\alpha+2} \int_{\sqrt{H(\epsilon)}}^{\infty} (\log x)^\alpha \exp\left(\frac{-\beta(\epsilon\sqrt{\log x})^{1/2}}{9}\right) \frac{dx}{x} \\ &\leq C\epsilon^{2\alpha+2} \int_{\beta^{\frac{4}{M/9}}}^{\infty} \epsilon^{-2\alpha-2} y^{4\alpha+3} \exp(-y) dy \\ &\leq C \int_{\beta^{\frac{4}{M/9}}}^{\infty} y^{4\alpha+3} \exp(-y) dy \rightarrow 0, \quad \text{as } M \rightarrow \infty. \end{aligned} \quad (4.10)$$

We then estimate I'_5 . If $-1 < \alpha \leq 1/2$, the following result is obvious according to $EX_1^2 < \infty$; if $\alpha > 1/2$, by Fubini's theorem, it follows that

$$\begin{aligned}
 \sum_{n>H(\epsilon)} \frac{(\log n)^\alpha I'_5}{n} &= \sum_{n>H(\epsilon)} (\log n)^\alpha P\left(|X_1| \geq (2n\epsilon\sqrt{\log n})^{1/2}\right) \\
 &\leq C \sum_{n>H(\epsilon)} (\log n)^\alpha \sum_{k=n}^{\infty} P\left(\sqrt{k} \leq \frac{|X_1|}{\sqrt[4]{4M}} < \sqrt{k+1}\right) \\
 &\leq C \sum_{k>H(\epsilon)} P\left(\sqrt{k} \leq \frac{|X_1|}{\sqrt[4]{4M}} < \sqrt{k+1}\right) \sum_{n=H(\epsilon)}^k (\log n)^\alpha \quad (4.11) \\
 &\leq C \sum_{k>H(\epsilon)} k(\log k)^\alpha P\left(\sqrt{k} \leq \frac{|X_1|}{\sqrt[4]{4M}} < \sqrt{k+1}\right) \\
 &\leq CEX_1^2 \frac{(\log |X_1|)^\alpha}{\sqrt[4]{4M}} < \infty. \quad \square
 \end{aligned}$$

Proof of Theorem 2.4. The proof follows from Propositions 4.1 and 4.4. □

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