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Research Article

Permanence for a Generalized Discrete Neural Network System

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We prove that the system of difference equations $x_{n+1}^{(i)} = \lambda_i x_n^{(i)} + f_i(\alpha_i x_n^{(i+1)} - \beta_i x_{n-1}^{(i+1)})$, $i \in \{1,2,\dots,k\}$, $n \in \mathbb{N}$, (we regard that $x_n^{(k+1)} = x_n^{(1)}$) is permanent, provided that $\alpha_i \geq \beta_i$, $\lambda_{i+1} \in [0, \beta_i/\alpha_i)$, $i \in \{1,2,\dots,k\}$, $f_i : \mathbb{R} \to \mathbb{R}$, $i \in \{1,2,\dots,k\}$, are nondecreasing functions bounded from below and such that there are $\delta_i \in (0,1)$ and $M > 0$ such that $f_i(\alpha_i x) \leq \delta_i x$, $i \in \{1,2,\dots,k\}$, for all $x \geq M$. This result considerably extends the results existing in the literature. The above system is an extension of a two-dimensional discrete neural network system.

1. Introduction

Recently, there has been a great interest in studying nonlinear difference equations and systems of nonlinear difference equations, see, for example, [1–31] and the references therein. One of the reasons for this is a necessity for some techniques which can be useful in investigating equations arising in mathematical models describing real life situations in population biology, economics, probability theory, genetics, psychology, sociology, and so forth. Such equations also appear naturally as discrete analogues of differential equations which model various biological and economic systems (see, e.g., [6, 10, 12–16, 18, 27, 28] and the related references therein).

This paper studies the permanence of the following system of difference equations:

$$x_{n+1}^{(1)} = \lambda_1 x_n^{(1)} + f_1\left(\alpha_1 x_n^{(2)} - \beta_1 x_{n-1}^{(2)}\right),$$

$$
\begin{aligned}
x_{n+1}^{(2)} &= \lambda_2 x_n^{(2)} + f_2\left(\alpha_2 x_n^{(3)} - \beta_2 x_{n-1}^{(3)}\right), \\
&\ \vdots \\
x_{n+1}^{(k-1)} &= \lambda_{k-1} x_n^{(k-1)} + f_{k-1}\left(\alpha_{k-1} x_n^{(k)} - \beta_{k-1} x_{n-1}^{(k)}\right), \\
x_{n+1}^{(k)} &= \lambda_k x_n^{(k)} + f_k\left(\alpha_k x_n^{(1)} - \beta_k x_{n-1}^{(1)}\right),
\end{aligned}
\tag{1.1}
$$

$n \in \mathbb{N}$, where $k \in \mathbb{N}$, $\alpha_i, \beta_i > 0$, $\lambda_i \in [0,1)$, and where f_i, $i \in \{1,2,\dots,k\}$, are real functions.

The case $k = 2$, with $\lambda_1 = \lambda_2$ and $f_1 = f_2$, was considered in [5], from which our motivation for this paper stems from. For the one-dimensional case, see [14]. System (1) with $k = 2$ can also be regarded as an extension of the discrete version of a neural network of two neurons with dynamical threshold effects, which has applications in the temporal evolution of sublattice magnetization (see [11]).

We say that system (1) is permanent if there is a compact subset K of $\mathbb{R}^k$ such that every solution of (1) eventually enters K.

Our aim here is to extend considerably the main result of paper [5]. The main result in this paper is the following.

Theorem 1.1. *Consider system (1), where the following conditions are satisfied:*

(a) $\alpha_i \geq \beta_i$, $i \in \{1,2,\dots,k\}$;

(b) $\lambda_{i+1} \in [0, \beta_i/\alpha_i)$, $i \in \{1,2,\dots,k\}$;

(c) $f_i : \mathbb{R} \to \mathbb{R}$, $i \in \{1,2,\dots,k\}$, *are nondecreasing functions bounded from below on* $\mathbb{R}$;

(d) *there are* $\delta_i \in (0,1)$ *and* $M > 0$ *such that for all* $x \geq M$,

$$
f_i(\alpha_i x) \leq \delta_i x, \quad i \in \{1,2,\dots,k\}. \tag{1.2}
$$

Then system (1) is permanent.

In Section 2, we give some auxiliary results which will be used in the proof of Theorem 1.1. Theorem 1.1 is proved in Section 3.

Throughout the paper, we often use the notation $u_l^{(i)} = u_l^{(j)}$ (or $x_l^{(i)} = x_l^{(j)}$, etc.) if $i = j$ $(\operatorname{mod} k)$.

2. Auxiliary results

The following system plays an important role in the proof of the main result of this paper:

$$
\begin{aligned}
u_{n+1}^{(1)} &= f_1\left(\alpha_1 u_n^{(2)}\right), \\
u_{n+1}^{(2)} &= f_2\left(\alpha_2 u_n^{(3)}\right), \\
&\ \vdots \\
u_{n+1}^{(k-1)} &= f_{k-1}\left(\alpha_{k-1} u_n^{(k)}\right), \\
u_{n+1}^{(k)} &= f_k\left(\alpha_k u_n^{(1)}\right).
\end{aligned}
\tag{2.1}
$$

Lemma 2.1. *Assume that $f_i : \mathbb{R} \to \mathbb{R}$, $i \in \{1,2,\dots,k\}$, are nondecreasing functions and that there are $\delta_i \in (0,1)$ and $M > 0$ such that for all $x \geq M$,*

$$f_i(\alpha_i x) \leq \delta_i x, \quad i \in \{1,2,\dots,k\}. \tag{2.2}$$

Then every solution of system (2.1) is eventually bounded from above (independently of initial conditions).

Proof. We prove that there exist $n_1 \in \mathbb{N}$ such that $u_{n_1}^{(i)} < M$ for each $i \in \{1,\dots,k\}$. First, we prove that there is an $n_0 \in \mathbb{N}$ such that $u_{n_0}^{(1)} < M$. Assume to the contrary that $u_n^{(1)} \geq M > 0$ for every $n \in \mathbb{N}$. Then, we have

$$\begin{aligned} u_{n+1}^{(k)} &= f_k\big(\alpha_k u_n^{(1)}\big) \leq \delta_k u_n^{(1)} < u_n^{(1)}, \\ u_{n+2}^{(k-1)} &= f_{k-1}\big(\alpha_{k-1} u_{n+1}^{(k)}\big) \leq f_{k-1}\big(\alpha_{k-1} u_n^{(1)}\big) \leq \delta_{k-1} u_n^{(1)} < u_n^{(1)}, \\ &\vdots \\ u_{n+k-1}^{(2)} &= f_2\big(\alpha_2 u_{n+k-2}^{(3)}\big) \leq f_2\big(\alpha_2 u_n^{(1)}\big) \leq \delta_2 u_n^{(1)} < u_n^{(1)}, \\ u_{n+k}^{(1)} &= f_1\big(\alpha_1 u_{n+k-1}^{(2)}\big) \leq f_1\big(\alpha_1 u_n^{(1)}\big) \leq \delta_1 u_n^{(1)}, \end{aligned} \tag{2.3}$$

which implies that

$$u_{n+k}^{(1)} \leq \delta_1 u_n^{(1)}. \tag{2.4}$$

Since this inequality holds for every $n \in \mathbb{N}$, it follows that

$$u_{n+km}^{(1)} \leq \delta_1^m u_n^{(1)} \tag{2.5}$$

for every $m \in \mathbb{N}$. Letting $m \to \infty$ in (2.5) and using the fact that $\delta_1 \in (0,1)$, we obtain that for each fixed $n \in \mathbb{N}$, $\limsup_{m\to\infty} u_{n+km}^{(1)} \leq 0$, which is a contradiction.

Now, we prove that there exists an $m_d \in \mathbb{N}$ such that $u_{n_0+km_d}^{(i)} < M$ for $i \in \{1,2,\dots,k\}$. Assume now that $u_{n_0+km}^{(2)} \geq M$ for every $m \in \mathbb{N}$. Then, similar to above, it can be shown that

$$0 < M \leq u_{n_0+km}^{(2)} \leq \delta_2^m u_{n_0}^{(2)} \tag{2.6}$$

for every $m \in \mathbb{N}$, which together with the fact $\delta_2 \in (0,1)$ leads to a contradiction. Hence, there is an m_1 such that $u_{n_0+km_1}^{(2)} < M$.

On the other hand, by the monotonicity of the functions f_i, $i \in \{1,\dots,k\}$, similar to (2.3), we have that $u_{n_0}^{(1)} < M$, implies $u_{n_0+km}^{(1)} < M$, for every $m \in \mathbb{N}$, in particular for $m = m_1$. Hence, $u_{n_0+km_1}^{(i)} < M$, $i \in \{1,2\}$. Repeating this procedure $k-2$ times, the claim follows.

Using the fact $u_{n_1}^{(i)} < M$, $i \in \{1,\dots,k\}$, and (2.1), we have that

$$u_{n_1+1}^{(i)} = f_i\big(\alpha_i u_{n_1}^{(i+1)}\big) \leq f_i(\alpha_i M) \leq \delta_i M < M \tag{2.7}$$

for each $i \in \{1,\dots,k\}$. By induction, it follows that

$$u_{n+1}^{(i)} = f_i(\alpha_i u_n^{(i+1)}) \le f_i(\alpha_i M) \le \delta_i M < M, \tag{2.8}$$

for every $n \ge n_1$ and for each $i \in \{1,\dots,k\}$, finishing the proof of the lemma. □

The following lemma is a dual result to Lemma 2.1 and its proof is omitted.

Lemma 2.2. *Assume that $f_i : \mathbb{R} \to \mathbb{R}$, $i \in \{1,2,\dots,k\}$, are nondecreasing functions and that there are $\delta_i \in (0,1)$ and $M > 0$ such that for all $x \le -M$,*

$$f_i(\alpha_i x) \ge \delta_i x, \quad i \in \{1,2,\dots,k\}. \tag{2.9}$$

Then every solution of system (2.1) is eventually bounded from below (independent of initial conditions).

Lemma 2.3. *Assume that $f_i : \mathbb{R} \to \mathbb{R}$, $i \in \{1,2,\dots,k\}$, are nondecreasing functions, and $(x_n^{(1)},\dots,x_n^{(k)})$ is a nonnegative solution of the following system of difference inequalities:*

$$\begin{aligned}
x_{n+1}^{(1)} &\le \lambda_1 x_n^{(1)} + f_1(\alpha_1 x_n^{(2)} - \beta_1 x_{n-1}^{(2)}), \\
x_{n+1}^{(2)} &\le \lambda_2 x_n^{(2)} + f_2(\alpha_2 x_n^{(3)} - \beta_2 x_{n-1}^{(3)}), \\
&\vdots \\
x_{n+1}^{(k-1)} &\le \lambda_{k-1} x_n^{(k-1)} + f_{k-1}(\alpha_{k-1} x_n^{(k)} - \beta_{k-1} x_{n-1}^{(k)}), \\
x_{n+1}^{(k)} &\le \lambda_k x_n^{(k)} + f_k(\alpha_k x_n^{(1)} - \beta_k x_{n-1}^{(1)}),
\end{aligned} \tag{2.10}$$

with initial conditions $(x_0^{(1)},\dots,x_0^{(k)})$ and $(x_1^{(1)},\dots,x_1^{(k)})$. Further, assume that $(u_n^{(1)},\dots,u_n^{(k)})$ is the solution of (2.1) with the following initial conditions:

$$\begin{aligned}
\alpha_1 u_1^{(2)} &= \alpha_1 x_1^{(2)} - \beta_1 x_0^{(2)}, \\
\alpha_2 u_1^{(3)} &= \alpha_2 x_1^{(3)} - \beta_2 x_0^{(3)}, \\
&\vdots \\
\alpha_{k-1} u_1^{(k)} &= \alpha_{k-1} x_1^{(k)} - \beta_{k-1} x_0^{(k)}, \\
\alpha_k u_1^{(1)} &= \alpha_k x_1^{(1)} - \beta_k x_0^{(1)},
\end{aligned} \tag{2.11}$$

and that

$$\lambda_{i+1} \le \frac{\beta_i}{\alpha_i}, \quad i \in \{1,\dots,k\}. \tag{2.12}$$

Then for every $n \in \mathbb{N}$ and each $i \in \{1,\dots,k\}$, the following inequalities hold true:

$$\alpha_i x_n^{(i+1)} \le \lambda_{i+1}^{n-1} \beta_i x_0^{(i+1)} + \sum_{j=1}^{n} \lambda_{i+1}^{n-j} \alpha_i u_j^{(i+1)}. \tag{2.13}$$

Proof. First, note that (2.11) can be written in the form

$$\alpha_i x_1^{(i+1)} = \alpha_i u_1^{(i+1)} + \beta_i x_0^{(i+1)}, \quad i \in \{1,\dots,k\}, \tag{2.14}$$

which shows that (2.13) holds for $n = 1$.

From this and (2.10), we have that

$$\begin{aligned} \alpha_i x_2^{(i+1)} &\le \alpha_i \left(\lambda_{i+1} x_1^{(i+1)} + f_{i+1}\left(\alpha_{i+1} x_1^{(i+2)} - \beta_{i+1} x_0^{(i+2)}\right)\right) \\ &= \lambda_{i+1}\left(\alpha_i u_1^{(i+1)} + \beta_i x_0^{(i+1)}\right) + \alpha_i f_{i+1}\left(\alpha_{i+1} u_1^{(i+2)}\right) \\ &= \lambda_{i+1}\beta_i x_0^{(i+1)} + \lambda_{i+1}\alpha_i u_1^{(i+1)} + \alpha_i u_2^{(i+1)}, \end{aligned} \tag{2.15}$$

from which it follows that (2.13) holds for $n = 2$. Now, we use the method of induction. Assume that (2.13) holds for some $n \in \mathbb{N}$. We have that

$$\begin{aligned} \alpha_i x_{n+1}^{(i+1)} &\le \alpha_i \left(\lambda_{i+1} x_n^{(i+1)} + f_{i+1}\left(\alpha_{i+1} x_n^{(i+2)} - \beta_{i+1} x_{n-1}^{(i+2)}\right)\right) \\ &\le \lambda_{i+1}\left(\lambda_{i+1}^{n-1}\beta_i x_0^{(i+1)} + \sum_{j=1}^{n} \lambda_{i+1}^{n-j}\alpha_i u_j^{(i+1)}\right) + \alpha_i f_{i+1}\left(\alpha_{i+1} x_n^{(i+2)} - \beta_{i+1} x_{n-1}^{(i+2)}\right) \\ &= \lambda_{i+1}^{n}\beta_i x_0^{(i+1)} + \sum_{j=1}^{n} \lambda_i^{n+1-j}\alpha_i u_j^{(i+1)} + \alpha_i f_{i+1}(\alpha_{i+1} x_n^{(i+2)} - \beta_{i+1} x_{n-1}^{(i+2)}). \end{aligned} \tag{2.16}$$

Now, we prove that

$$f_{i+1}\left(\alpha_{i+1} x_n^{(i+2)} - \beta_{i+1} x_{n-1}^{(i+2)}\right) \le u_{n+1}^{(i+1)}, \tag{2.17}$$

from which the result follows.

From (2.10) and by condition (2.12), we have that

$$\begin{aligned} \alpha_{i+1} x_n^{(i+2)} - \beta_{i+1} x_{n-1}^{(i+2)} &\le (\alpha_{i+1}\lambda_{i+2} - \beta_{i+1}) x_{n-1}^{(i+2)} + \alpha_{i+1} f_{i+2}\left(\alpha_{i+2} x_{n-1}^{(i+3)} - \beta_{i+2} x_{n-2}^{(i+3)}\right) \\ &\le \alpha_{i+1} f_{i+2}\left(\alpha_{i+2} x_{n-1}^{(i+3)} - \beta_{i+2} x_{n-2}^{(i+3)}\right). \end{aligned} \tag{2.18}$$

From this and by the monotonicity of f, it follows that

$$f_{i+1}\left(\alpha_{i+1} x_n^{(i+2)} - \beta_{i+1} x_{n-1}^{(i+2)}\right) \le f_{i+1}\left(\alpha_{i+1} f_{i+2}\left(\alpha_{i+2} x_{n-1}^{(i+3)} - \beta_{i+2} x_{n-2}^{(i+3)}\right)\right). \tag{2.19}$$

Repeating this procedure, we obtain

$$\begin{aligned} & f_{i+1}\left(\alpha_{i+1} x_n^{(i+2)} - \beta_{i+1} x_{n-1}^{(i+2)}\right) \\ & \quad \le f_{i+1}\left(\alpha_{i+1} f_{i+2} \cdots \alpha_{i+n-1} f_{i+n}\left(\alpha_{n+i} x_1^{(n+i+1)} - \beta_{n+i} x_0^{(n+i+1)}\right) \cdots\right). \end{aligned} \tag{2.20}$$

Now, note that system (2.1) is defined by

$$u_{n+1}^{(i+1)} = f_{i+1}\left(\alpha_{i+1} u_n^{(i+2)}\right), \tag{2.21}$$

which implies

$$u_{n+1}^{(i+1)} = f_{i+1}\Big(\alpha_{i+1} f_{i+2} \cdots \alpha_{i+n-1} f_{i+n}\Big(\alpha_{n+i} u_1^{(n+i+1)}\Big) \cdots\Big). \tag{2.22}$$

By (2.11), we have that

$$\alpha_{n+i} u_1^{(n+i+1)} = \alpha_{n+i} x_1^{(n+i+1)} - \beta_{n+i} x_0^{(n+i+1)}. \tag{2.23}$$

Thus, (2.20) and (2.22) imply (2.17) as desired. □

Similar to Lemma 2.3, the following (dual) lemma can be proved.

Lemma 2.4. *Assume that $f_i : \mathbb{R} \to \mathbb{R}$, $i \in \{1,2,\dots,k\}$, are nondecreasing functions, and $(x_n^{(1)},\dots,x_n^{(k)})$ is a nonpositive solution of the following system of difference inequalities:*

$$\begin{aligned} x_{n+1}^{(1)} &\geq \lambda_1 x_n^{(1)} + f_1\Big(\alpha_1 x_n^{(2)} - \beta_1 x_{n-1}^{(2)}\Big), \\ x_{n+1}^{(2)} &\geq \lambda_2 x_n^{(2)} + f_2\Big(\alpha_2 x_n^{(3)} - \beta_2 x_{n-1}^{(3)}\Big), \\ &\vdots \\ x_{n+1}^{(k-1)} &\geq \lambda_{k-1} x_n^{(k-1)} + f_{k-1}\Big(\alpha_{k-1} x_n^{(k)} - \beta_{k-1} x_{n-1}^{(k)}\Big), \\ x_{n+1}^{(k)} &\geq \lambda_k x_n^{(k)} + f_k\Big(\alpha_k x_n^{(1)} - \beta_k x_{n-1}^{(1)}\Big), \end{aligned} \tag{2.24}$$

with initial conditions $(x_0^{(1)},\dots,x_0^{(k)})$ and $(x_1^{(1)},\dots,x_1^{(k)})$. Further, assume that $(u_n^{(1)},\dots,u_n^{(k)})$ is the solution of (2.1) satisfying (2.11) and (2.12).

Then for every $n \in \mathbb{N}$ and each $i \in \{1,\dots,k\}$, the following inequalities hold true:

$$\alpha_i x_n^{(i+1)} \geq \lambda_{i+1}^{n-1} \beta_i x_0^{(i+1)} + \sum_{j=1}^{n} \lambda_{i+1}^{n-j} \alpha_i u_j^{(i+1)}. \tag{2.25}$$

3. Proof of the main result

In this section, we prove the main result of this paper, Theorem 1.1.

Proof of Theorem 1.1. Let

$$z_n^{(i)} = f_i\big(\alpha_i x_n^{(i+1)} - \beta_i x_{n-1}^{(i+1)}\big), \quad i \in \{1,\dots,k\}. \tag{3.1}$$

By standard arguments, from (1), we have that

$$x_n^{(i)} = \lambda_i^{n-1} x_1^{(i)} + \sum_{j=1}^{n-1} \lambda_i^{n-j-1} z_j^{(i)}. \tag{3.2}$$

Assume that m_0 is a lower bound for all f_i, $i \in \{1,\dots,k\}$. Without loss of generality, we may assume that m_0 is a negative number. Then from (3.2), we have that

$$x_n^{(i)} \geq \lambda_i^{n-1} x_1^{(i)} + m_0 \frac{1-\lambda_i^{n-1}}{1-\lambda_i}, \tag{3.3}$$

for each $i \in \{1,\dots,k\}$, and consequently the following estimates eventually hold:

$$x_n^{(i)} \geq \min_{i=1,\dots,k} \left\{ \frac{m_0}{1-\lambda_i} - 1 \right\} := L, \tag{3.4}$$

for $i \in \{1,\dots,k\}$, for example, for $n \geq n_1$.

Now, we prove that $x_n^{(i)}$, $i \in \{1,\dots,k\}$, are bounded above. Let

$$y_n^{(i)} = x_{n+n_1}^{(i)} - L, \quad i \in \{1,\dots,k\}. \tag{3.5}$$

Note that such defined $y_n^{(i)}$ are nonnegative numbers for every $n \in \mathbb{N}_0$ and $i \in \{1,\dots,k\}$. Also, we have

$$\begin{aligned} y_{n+1}^{(i)} &= \lambda_i x_{n+n_1}^{(i)} + f_i\left(\alpha_i x_{n+n_1}^{(i+1)} - \beta_i x_{n+n_1-1}^{(i+1)}\right) - L \\ &= \lambda_i y_n^{(i)} + f_i\left(\alpha_i x_{n+n_1}^{(i+1)} - \beta_i x_{n+n_1-1}^{(i+1)}\right) - L(1-\lambda_i). \end{aligned} \tag{3.6}$$

In view of condition (a), we have that

$$\alpha_i x_{n+n_1}^{(i+1)} - \beta_i x_{n+n_1-1}^{(i+1)} = \alpha_i y_n^{(i+1)} - \beta_i y_{n-1}^{(i+1)} + L(\alpha_i - \beta_i) \leq \alpha_i y_n^{(i+1)} - \beta_i y_{n-1}^{(i+1)}. \tag{3.7}$$

From (3.6) and (3.7) and condition (c), it follows that

$$y_{n+1}^{(i)} \leq \lambda_i y_n^{(i)} + g_i\left(\alpha_i y_n^{(i+1)} - \beta_i y_{n-1}^{(i+1)}\right), \tag{3.8}$$

where $g_i(x) = f_i(x) - L(1-\lambda_i)$, $i \in \{1,\dots,k\}$.

It is easy to see that if $x \geq \max\{M, -2L(1-\lambda_1)/(1-\delta_1),\dots,-2L(1-\lambda_k)/(1-\delta_k)\}$, then

$$g_i(\alpha_i x) \leq \frac{1+\delta_i}{2} x, \quad i \in \{1,\dots,k\}. \tag{3.9}$$

Now, consider the system

$$v_{n+1}^{(i)} = g_i\left(\alpha_i v_n^{(i+1)}\right), \quad i \in \{1,\dots,k\}, \tag{3.10}$$

with the initial conditions

$$v_1^{(i+1)} = \frac{\alpha_i y_1^{(i+1)} - \beta_i y_0^{(i+1)}}{\alpha_i}, \quad i \in \{1,\dots,k\}. \tag{3.11}$$

By employing Lemma 2.1, it follows that there is a positive constant M_0 and an $n_1 \in \mathbb{N}$ such that $v_n^{(i)} \leq M_0$ for every $n \geq n_1$ and $i \in \{1,\dots,k\}$. On the other hand, by Lemma 2.3, we have that

$$\begin{aligned} \alpha_i y_n^{(i+1)} &\leq \lambda_{i+1}^{n-1} \beta_i x_0^{(i+1)} + \sum_{j=1}^{n} \lambda_{i+1}^{n-j} \alpha_i v_j^{(i+1)} \\ &\leq \lambda_{i+1}^{n-1} \beta_i x_0^{(i+1)} + \lambda_{i+1}^{n-n_1+1} \sum_{j=1}^{n_1-1} \lambda_{i+1}^{n_1-j-1} \alpha_i v_j^{(i+1)} + \frac{\alpha_i M_0}{1-\lambda_{i+1}}, \end{aligned} \tag{3.12}$$

from which it follows that eventually

$$y_n^{(i+1)} \leq \frac{M_0}{1-\lambda_{i+1}} + 1, \tag{3.13}$$

$i \in \{1,\dots,k\}$, and the result is proven. □

By using Lemmas 2.2 and 2.4, similar to the proof of Theorem 1.1 the following theorem can be proved.

THEOREM 3.1. *Consider system (1), where the following conditions are satisfied:*

(a) $\alpha_i \geq \beta_i$, $i \in \{1,2,\dots,k\}$;

(b) $\lambda_{i+1} \in [0, \beta_i/\alpha_i)$, $i \in \{1,2,\dots,k\}$;

(c) $f_i : \mathbb{R} \to \mathbb{R}$, $i \in \{1,2,\dots,k\}$, *are nondecreasing functions bounded from above on* $\mathbb{R}$;

(d) *there are* $\delta_i \in (0,1)$ *and* $M_2 > 0$ *such that for all* $x \leq -M_2$,

$$f_i(\alpha_i x) \geq \delta_i x, \quad i \in \{1,2,\dots,k\}. \tag{3.14}$$

Then system (1) is permanent.

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