Hindawi Publishing Corporation
Discrete Dynamics in Nature and Society
Volume 2008, Article ID 149267, 8 pages
doi:10.1155/2008/149267

Research Article

Permanence of a Discrete Predator-Prey Systems with Beddington-DeAngelis Functional Response and Feedback Controls

Xuepeng Li and Wensheng Yang

School of Mathematics and Computer Science, Fujian Normal University, Fuzhou 350007, China

Correspondence should be addressed to Wensheng Yang, ywensheng@126.com

Received 20 July 2007; Accepted 21 February 2008

Recommended by Leonid Berezansky

We propose a discrete predator-prey systems with Beddington-DeAngelis functional response and feedback controls. By applying the comparison theorem of difference equation, sufficient conditions are obtained for the permanence of the system.

1. Introduction

Zhang and Wang [1] considered the following nonautonomous discrete predator-prey systems with the Beddington-DeAngelis functional response

$$\begin{aligned} x(k+1) &= x(k)\exp\left\{ a(k) - b(k)x(k) - \frac{c(k)y(k)}{\alpha(k)+\beta(k)x(k)+\gamma(k)y(k)} \right\}, \\ y(k+1) &= y(k)\exp\left\{ -d(k) + \frac{f(k)x(k)}{\alpha(k)+\beta(k)x(k)+\gamma(k)y(k)} \right\}. \end{aligned} \tag{1.1}$$

By using a continuation theorem, sufficient criteria are established for the existence of positive periodic solutions of the system (1.1).

As we know, permanence is one of the most important topics on the study of population dynamics. One of the most interesting questions in mathematical biology concerns the survival of species in ecological models. Biologically, when a system of interacting species is persistent in a suitable sense, it means that all the species survive in the long term. It is reasonable to ask for conditions under which the system is permanent. However, Zhang and Wang [1] did not investigate this property of the system (1.1).

As we know, ecosystems in the real world are continuously distributed by unpredictable forces which can result in changes in the biological parameters such as survival rates. Of practical interest in ecology is the question of whether or not an ecosystem can withstand those unpredictable disturbances which persist for a finite period of time. In the language of control variables, we call the disturbance functions as control variables. Already, Gopalsamy and Weng [2] have studied the Logistic growth model with feedback control. To the author knowledge, there is few works dealt with system (1.1) with feedback control.

Therefore, one objective of this paper is to study the following discrete predator-prey systems with Beddington-DeAngelis functional response and feedback controls

$$\begin{aligned} x(k+1) &= x(k)\exp\left\{a(k) - b(k)x(k) - \frac{c(k)y(k)}{\alpha(k)+\beta(k)x(k)+\gamma(k)y(k)} - e_1(k)u_1(k)\right\}, \\ y(k+1) &= y(k)\exp\left\{-d(k) + \frac{f(k)x(k)}{\alpha(k)+\beta(k)x(k)+\gamma(k)y(k)} - e_2(k)u_2(k)\right\}, \\ \Delta u_1(k) &= -\eta_1(k)u_1(k) + q_1(k)x(k), \\ \Delta u_2(k) &= -\eta_2(k)u_2(k) + q_2(k)y(k), \end{aligned} \tag{1.2}$$

where $a(k), b(k), c(k), d(k), f(k), \alpha(k), \beta(k), \gamma(k), e_1(k), e_2(k), \eta_1(k), \eta_2(k), q_1(k)$, and $q_2(k)$ are all bounded nonnegative sequence. For more biological background of system (1.2), one could refer to [1] and the references cited therein.

Throughout this paper, we use the following notations for any bounded sequence $\{a(k)\}$:

$$a^u = \sup_{k\in N} a(k), \qquad a^l = \inf_{k\in N} a(k), \tag{1.3}$$

and assume that $0 < \eta_1^l \le \eta_1^u < 1, 0 < \eta_2^l \le \eta_2^u < 1$.

The aim of this paper is, by further developing the analysis technique of Chen [3], to obtain a set of sufficient conditions which ensure the permanence of the system (1.2).

We say that system (1.2) is permanent if there are positive constants M and m such that for each positive solution $(x(k), y(k), u_1(k), u_2(k))$ of system (1.2) satisfies

$$\begin{aligned} m &\le \liminf_{k\to+\infty} x(k) \le \limsup_{k\to+\infty} x(k) \le M, \\ m &\le \liminf_{k\to+\infty} y(k) \le \limsup_{k\to+\infty} y(k) \le M, \\ m &\le \liminf_{k\to+\infty} u_i(k) \le \limsup_{k\to+\infty} u_i(k) \le M, \quad i = 1,2. \end{aligned} \tag{1.4}$$

For biological reasons, we only consider solution $(x(k), y(k), u_1(k), u_2(k))$ with $x(0) > 0; y(0) > 0; u_i(0) > 0, i = 1, 2$. Then system (1.2) has a positive solution $(x(k), y(k), u_1(k), u_2(k))$ passing through $(x(0), y(0), u_1(0), u_2(0))$.

2. Permanence

In this section, we establish a permanence result for system (1.2).

First, let us consider the first order difference equation

$$y(n+1) = Ay(n) + B, \quad n = 1, 2, \ldots, \tag{2.1}$$

where A, B are positive constants. Following Lemma 2.1 is a direct corollary of Theorem 6.2 of L. Wang and M. Q. Wang [4, page 125].

Lemma 2.1. *Assume that* $|A| < 1$, *for any initial value* $y(0)$, *there exists a unique solution* $y(n)$ *of* (2.1) *which can be expressed as follows:*

$$y(n) = A^n\left(y(0) - y^*\right) + y^*, \tag{2.2}$$

where $y^* = B/(1-A)$. *Thus, for any solution* $\{y(n)\}$ *of system* (2.1),

$$\lim_{n\to+\infty} y(n) = y^*. \tag{2.3}$$

Following Comparison Theorem of difference equation is Theorem 2.1 of [4, page 241].

Lemma 2.2. *Let* $k \in N_{k_0}^+ = \{k_0, k_0+1, \dots, k_0+l, \dots\}, r \geq 0$. *For any fixed* $k, g(k,r)$ *is a nondecreasing function with respect to* r, *and for* $k \geq k_0$, *the following inequalities hold:*

$$y(k+1) \leq g(k, y(k)), \qquad u(k+1) \geq g(k, u(k)). \tag{2.4}$$

If $y(k_0) \leq u(k_0)$, *then* $y(k) \leq u(k)$ *for all* $k \geq k_0$.

Now let us consider the following single species discrete model:

$$N(k+1) = N(k)\exp\{a(k) - b(k)N(k)\}, \tag{2.5}$$

where $\{a(k)\}$ and $\{b(k)\}$ are strictly positive sequences of real numbers defined for $k \in N = \{0, 1, 2, \dots\}$ and $0 < a^l \leq a^u, 0 < b^l \leq b^u$. Similarly to the proof of [5, Propositions 1 and 3], we can obtain the following.

Lemma 2.3. *Any solution of system* (2.5) *with initial condition* $N(0) > 0$ *satisfies*

$$m \leq \lim_{k\to+\infty}\inf N(k) \leq \lim_{k\to+\infty}\sup N(k) \leq M, \tag{2.6}$$

where

$$M = \frac{1}{b^l}\exp\{a^u - 1\}, \qquad m = \frac{a^l}{b^u}\exp\{a^l - b^u M\}. \tag{2.7}$$

Lemma 2.4 (see [6]). *Let* $x(n)$ *and* $b(n)$ *be nonnegative sequences defined on* N *and* $c \geq 0$ *is a constant. If*

$$x(n) \leq c + \sum_{s=0}^{n-1} b(s)x(s), \quad \text{for } n \in N. \tag{2.8}$$

Then

$$x(n) \le c\prod_{s=0}^{n-1}[1+b(s)], \quad \textit{for } n \in N. \tag{2.9}$$

Proposition 2.5. *Assume that*

$$-d^l + \frac{f^u}{\beta^l} > 0 \tag{2.10}$$

holds, then

$$\begin{aligned} &\lim_{k\to+\infty}\sup x(k) \le M_1, \\ &\lim_{k\to+\infty}\sup y(k) \le M_2, \\ &\lim_{k\to+\infty}\sup u_i(k) \le W_i, \quad i = 1,2, \end{aligned} \tag{2.11}$$

where

$$\begin{aligned} M_1 &= \frac{1}{b^l}\exp\{a^u - 1\}, \\ M_2 &= \exp\left\{2\left(-d^l + \frac{f^u}{\beta^l}\right)\right\}, \\ W_i &= \frac{q_i^u M_i}{\eta_i^l}, \quad i = 1,2. \end{aligned} \tag{2.12}$$

Proof. Let $s(k) = (x(k), y(k), u_1(k), u_2(k))$ be any positive solution of system (1.2); from (1.2), we have

$$x(k+1) \le x(k)\exp\{a(k) - b(k)x(k)\}. \tag{2.13}$$

By applying Lemmas 2.2 and 2.3, it immediately follows that

$$\lim_{k\to+\infty}\sup x(k) \le \frac{1}{b^l}\exp\{a^u - 1\} := M_1. \tag{2.14}$$

From the second equation of the system (1.2), we can obtain

$$\begin{aligned} y(k+1) &\le y(k)\exp\left\{-d(k) + \frac{f(k)}{\beta(k)}\right\} \\ &\le y(k)\exp\left\{-d^l + \frac{f^u}{\beta^l}\right\}. \end{aligned} \tag{2.15}$$

Let $y(k) = \exp\{u(k)\}$, then

$$\begin{aligned} u(k+1) &\le u(k) + \left(-d^l + \frac{f^u}{\beta^l}\right) \\ &= \sum_{s=0}^{k} b(s)u(s) + \left(-d^l + \frac{f^u}{\beta^l}\right), \end{aligned} \tag{2.16}$$

where

$$b(s)=\begin{cases}0, & 0\le s\le k-1,\\ 1, & s=k.\end{cases} \tag{2.17}$$

Condition (2.10) shows that Lemma 2.4 could be applied to (2.16), and so by applying Lemma 2.4, it immediately follows that

$$u(k+1)\le 2\left(-d^l+\frac{f^u}{\beta^l}\right). \tag{2.18}$$

This is

$$\limsup_{k\to+\infty} y(k)\le \exp\left\{2\left(-d^l+\frac{f^u}{\beta^l}\right)\right\}:=M_2. \tag{2.19}$$

For any positive constant ε small enough, it follows from (2.14) and (2.19) that there exists enough large K_0 such that

$$x(k)\le M_1+\varepsilon,\quad y(k)\le M_2+\varepsilon,\quad \forall\, k\ge K_0. \tag{2.20}$$

From the third and fourth equations of the system (1.2) and (2.20), we can obtain

$$\begin{aligned}\Delta u_1(k)&\le -\eta_1(k)u_1(k)+q_1(k)(M_1+\varepsilon),\\ \Delta u_2(k)&\le -\eta_2(k)u_2(k)+q_2(k)(M_2+\varepsilon).\end{aligned} \tag{2.21}$$

So

$$\begin{aligned}u_1(k+1)&\le (1-\eta_1^l)u_1(k)+q_1^u(M_1+\varepsilon),\\ u_2(k+1)&\le (1-\eta_2^l)u_2(k)+q_2^u(M_2+\varepsilon).\end{aligned} \tag{2.22}$$

By applying Lemmas 2.1 and 2.2, it immediately follows that

$$\begin{aligned}\limsup_{k\to+\infty} u_1(k)&\le \frac{q_1^u(M_1+\varepsilon)}{\eta_1^l},\\ \limsup_{k\to+\infty} u_2(k)&\le \frac{q_2^u(M_2+\varepsilon)}{\eta_2^l}.\end{aligned} \tag{2.23}$$

Setting $\varepsilon\to 0$ in the above inequality leads to

$$\begin{aligned}\limsup_{k\to+\infty} u_1(k)&\le \frac{q_1^u M_1}{\eta_1^l},\\ \limsup_{k\to+\infty} u_2(k)&\le \frac{q_2^u M_2}{\eta_2^l}.\end{aligned} \tag{2.24}$$

This completes the proof of Proposition 2.5. □

Now we are in the position of stating the permanence of the system (1.2).

Theorem 2.6. *In addition to* (2.10), *assume further that*

$$a^l - \frac{c^u}{\gamma^l} - e_1^u W_1 > 0,$$
$$-d^u + f^l m_1 - e_2^u W_2 > 0, \tag{2.25}$$

then system (1.2) *is permanent, where*

$$m_1 = \frac{a^l - c^u/\gamma^l - e_1^u W_1}{b^u} \exp\left\{ a^l - \frac{c^u}{\gamma^l} - e_1^u W_1 - b^u M_1 \right\}. \tag{2.26}$$

Proof. By applying Proposition 2.5, we see that to end the proof of Theorem 2.6, it is enough to show that under the conditions of Theorem 2.6,

$$\liminf_{k\to+\infty} x(k) \geq m_1,$$
$$\liminf_{k\to+\infty} y(k) \geq m_2, \tag{2.27}$$
$$\liminf_{k\to+\infty} u_i(k) \geq w_i, \quad i = 1, 2.$$

From Proposition 2.5, for all $\varepsilon > 0$, there exists a $K_1 > 0, K_1 \in N$, for all $k > K_1$,

$$x(k) \leq M_1 + \varepsilon, \quad y(k) \leq M_2 + \varepsilon; \quad u_i(k) \leq W_i + \varepsilon, \quad i = 1, 2. \tag{2.28}$$

From the first equation of systems (1.2) and (2.28), we have

$$x(k+1) \geq x(k) \exp\left\{ a(k) - b(k)x(k) - \frac{c(k)}{\gamma(k)} - e_1(k)(W_1 + \varepsilon) \right\},$$
$$= x(k) \exp\left\{ a(k) - \frac{c(k)}{\gamma(k)} - e_1(k)(W_1 + \varepsilon) - b(k)x(k) \right\} \tag{2.29}$$

for all $k > K_1$.

Condition (2.25) shows that Lemmas 2.2 and 2.3 could be applied to (2.29), and so by applying Lemmas 2.2 and 2.3 to (2.29), it immediately follows that

$$\liminf_{k\to+\infty} x(k) \geq \frac{a^l - c^u/\gamma^l - e_1^u(W_1 + \varepsilon)}{b^u} \exp\left\{ a^l - \frac{c^u}{\gamma^l} - e_1^u(W_1 + \varepsilon) - b^u M_1 \right\}. \tag{2.30}$$

Setting $\varepsilon \to 0$ in (2.30) leads to

$$\liminf_{k\to+\infty} x(k) \geq \frac{a^l - c^u/\gamma^l - e_1^u W_1}{b^u} \exp\left\{ a^l - \frac{c^u}{\gamma^l} - e_1^u W_1 - b^u M_1 \right\} := m_1. \tag{2.31}$$

Then, for any positive constant ε small enough, from (2.31) we know that there exists an enough large $K_2 > K_1$ such that

$$x(k) \geq m_1 - \varepsilon, \quad \forall k \geq k_2. \tag{2.32}$$

From the second equation of systems (1.2), (2.28), and (2.32), we have

$$\begin{aligned} y(k+1) &\geq y(k)\exp\left\{-d(k)+\frac{f(k)}{\beta(k)}-\frac{f(k)}{\beta(k)}\left(\frac{\alpha(k)+\gamma(k)y(k)}{\alpha(k)+\beta(k)x(k)+\gamma(k)y(k)}\right)-e_2(k)u_2(k)\right\} \\ &\geq y(k)\exp\left\{-d(k)+\frac{f(k)}{\beta(k)}-\frac{f(k)}{\beta(k)}\left(\frac{\alpha(k)}{\alpha(k)+\beta(k)(m_1-\varepsilon)}\right)\right. \\ &\qquad \left. -\frac{f(k)}{\beta(k)}\left(\frac{\gamma(k)y(k)}{\alpha(k)+\beta(k)(m_1-\varepsilon)}\right)-e_2(k)(W_2+\varepsilon)\right\} \\ &\geq y(k)\exp\left\{-d(k)+f(k)(m_1-\varepsilon)-e_2(k)(W_2+\varepsilon)-\frac{f(k)\gamma(k)}{\beta(k)[\alpha(k)+\beta(k)(m_1-\varepsilon)]}y(k)\right\} \end{aligned} \tag{2.33}$$

for all $k > K_2$.

Condition (2.25) shows that Lemmas 2.2 and 2.3 could be applied to (2.33), and so by applying Lemmas 2.2 and 2.3 to (2.33), it immediately follows that

$$\begin{aligned} \liminf_{k\to+\infty} y(k) \geq{}& \frac{\beta^l[\alpha^l+\beta^l(m_1-\varepsilon)][-d^u+f^l(m_1-\varepsilon)-e_2^u(W_2+\varepsilon)]}{f^u\gamma^u} \\ &\times \exp\left\{-d^u+f^l(m_1-\varepsilon)-e_2^u(W_2+\varepsilon)-\frac{f^u\gamma^u}{\beta^l[\alpha^l+\beta^l(m_1-\varepsilon)]}M_2\right\}. \end{aligned} \tag{2.34}$$

Setting $\varepsilon \to 0$ in (2.34) leads to

$$\begin{aligned} \liminf_{k\to+\infty} y(k) \geq{}& \frac{\beta^l(\alpha^l+\beta^l m_1)(-d^u+f^l m_1-e_2^u W_2)}{f^u\gamma^u} \\ &\times \exp\left\{-d^u+f^l m_1-e_2^u W_2-\frac{f^u\gamma^u}{\beta^l(\alpha^l+\beta^l m_1)}M_2\right\} := m_2. \end{aligned} \tag{2.35}$$

Without loss of generality, we may assume that $\varepsilon < (1/2)\min\{m_1, m_2\}$. For any positive constant ε small enough, it follows from (2.31) and (2.35) that there exists enough large $K_3 > K_2$ such that

$$x(k) \geq m_1-\varepsilon, \quad y(k) \geq m_2-\varepsilon, \quad \forall\, k \geq K_3. \tag{2.36}$$

From the third and fourth equations of the system, (1.2) and (2.36), we can obtain that

$$\begin{aligned} \Delta u_1(k) &\geq -\eta_1(k)u_1(k)+q_1(k)(m_1-\varepsilon), \\ \Delta u_2(k) &\geq -\eta_2(k)u_2(k)+q_2(k)(m_2-\varepsilon). \end{aligned} \tag{2.37}$$

So

$$\begin{aligned} u_1(k+1) &\geq (1-\eta_1^u)u_1(k)+q_1^l(m_1-\varepsilon), \\ u_2(k+1) &\geq (1-\eta_2^u)u_2(k)+q_2^l(m_2-\varepsilon). \end{aligned} \tag{2.38}$$

By applying Lemmas 2.1 and 2.2, it immediately follows that

$$\liminf_{k\to+\infty} u_1(k) \geq \frac{q_1^l(m_1-\varepsilon)}{\eta_1^u},$$
$$\liminf_{k\to+\infty} u_2(k) \geq \frac{q_2^l(m_2-\varepsilon)}{\eta_2^u}. \tag{2.39}$$

Setting $\varepsilon \to 0$ in the above inequality leads to

$$\liminf_{k\to+\infty} u_1(k) \geq \frac{q_1^l m_1}{\eta_1^u} := w_1,$$
$$\liminf_{k\to+\infty} u_2(k) \geq \frac{q_2^l m_2}{\eta_2^u} := w_2. \tag{2.40}$$

This completes the proof of Theorem 2.6. □

To check the conditions of Theorem 2.6, we give an example. We consider the following discrete predator-prey systems with Beddington-DeAngelis functional response and feedback controls

$$\begin{aligned} x(k+1) &= x(k)\exp\left\{1 - x(k) - \frac{0.8y(k)}{1+0.2x(k)+2y(k)} - 0.001u_1(k)\right\}, \\ y(k+1) &= y(k)\exp\left\{-0.01 + \frac{0.1x(k)}{1+0.2x(k)+2y(k)} - 0.001u_2(k)\right\}, \\ \Delta u_1(k) &= -0.8u_1(k) + x(k), \\ \Delta u_2(k) &= -0.5u_2(k) + y(k). \end{aligned} \tag{2.41}$$

One could easily obtain that the conditions of Theorem 2.6 are satisfied. Hence, by Theorem 2.6, we see that system (2.41) is permanent.

Acknowledgment

This work is supported by the Foundation of Education Department of Fujian Province (Grant no. JA05204), and the Foundation of Science and Technology Department of Fujian Province (Grant no. 2005K027).

References

[1] J. Zhang and J. Wang, "Periodic solutions for discrete predator-prey systems with the Beddington-DeAngelis functional response," *Applied Mathematics Letters*, vol. 19, no. 12, pp. 1361–1366, 2006.

[2] K. Gopalsamy and P.-X. Weng, "Feedback regulation of logistic growth," *International Journal of Mathematics and Mathematical Sciences*, vol. 16, no. 1, pp. 177–192, 1993.

[3] F. Chen, "Permanence of a discrete n-species food-chain system with time delays," *Applied Mathematics and Computation*, vol. 185, no. 1, pp. 719–726, 2007.

[4] L. Wang and M. Q. Wang, *Ordinary Difference Equation*, Xinjiang University Press, China, 1991.

[5] F. Chen, "Permanence and global attractivity of a discrete multispecies Lotka-Volterra competition predator-prey systems," *Applied Mathematics and Computation*, vol. 182, no. 1, pp. 3–12, 2006.

[6] Y. Takeuchi, *Global Dynamical Properties of Lotka-Volterra Systems*, World Scientific, River Edge, NJ, USA, 1996.

[7] F. Chen, "Permanence and global attractivity of a discrete multispecies Lotka-Volterra competition predator-prey systems," *Applied Mathematics and Computation*, vol. 182, no. 1, pp. 3–12, 2006.