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Research Article

A Note on Almost Periodic Points and Minimal Sets in T_1- and T_2-Spaces

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We show that (1) there exist almost periodic orbits in T_2-spaces of which the closures are not minimal sets; (2) there exist minimal sets in locally compact T_1-spaces which are not compact; (3) there exist almost periodic orbits in T_2-spaces of which the closures contain not only almost periodic points. These give answers to the three problems given by Mai and Sun in (2007).

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1. Introduction

Denote by $\mathbb{R}$, $\mathbb{Z}$, $\mathbb{Z}_+$, and $\mathbb{N}$ the sets of real numbers, integers, nonnegative integers, and positive integers, respectively. For any topological space X, denote by $C^0(X)$ the set of all continuous maps of X into itself. For any $f \in C^0(X)$, let $f^0 = id_X$ be the identity map of X, and let $f^i = f f^{i-1}$ be the composition of f and f^{i-1} $(i = 1, 2, 3, \ldots)$. f^n is called the nth *iterate* of f $(n \in \mathbb{Z}_+)$.

The *orbit* of a point $x \in X$ under f, denoted by $O(x, f)$, is the set $\{x, f(x), f^2(x), \ldots\}$. $x \in X$ is called an *almost periodic point* of f and $O(x, f)$ is called an *almost periodic orbit* if for any neighborhood U of x there exists $N \in \mathbb{N}$ such that $\{f^{n+i}(x) : i = 0, 1, \ldots, N\} \cap U \neq \varnothing$ for all $n \in \mathbb{Z}_+$.

A subset W of X is said to be *invariant* or f-*invariant* if $f(W) \subset W$, and W is called a *minimal set* of f if it is nonempty, closed, and f-invariant and if no proper subset of W has these three properties.

The notion of ω-regular space was introduced by Mai and Sun [1].

Definition 1.1 (see [1, Definition 2.1]). A topological space X is called an ω-regular space if for any closed set $W \subset X$, any point $x \in X - W$ and any countable set $A \subset W$, there exist disjoint open sets U and V such that $x \in U$ and $A \subset V$.

Mai and Sun [1] generalized several known results concerning almost periodic points and minimal sets of maps from regular spaces to ω-regular spaces, and obtained the following theorems.

Theorem A (see [1, Theorem 2.3]). *Let X be an ω-regular space, and $f \in C^0(X)$. Then the closure of every almost periodic orbit of f is a minimal set.*

Theorem B (see [1, Theorem 3.8]). *Let X be a locally compact topological space which is either Hausdorff or regular, and $f \in C^0(X)$. Then each minimal set of f is compact.*

Theorem C (see [1, Theorem 4.1]). *Let X be an ω-regular space, and $f \in C^0(X)$. Then all points in the closure of any almost periodic orbit of f are almost periodic.*

Can these theorems be extended to more general topological spaces? [1, Example 2.4 and Remark 4.2] show that Theorems A and C cannot be extended to general T_1-spaces, respectively, and [1, Example 3.9] shows that Theorem B cannot be extended to general locally compact T_0-spaces. However, there remain three problems which have not been solved in [1].

Problem 1.2 (see [1, Problem 2.5]). Can the condition in [1, Theorem 2.3] that X is an ω-regular space be replaced by that X is a T_2-space? In other words, need the closure of an almost periodic orbit in a T_2-space be a minimal set?

Problem 1.3 (see [1, Problem 3.10]). Let X be a locally compact T_1-space. Is each minimal set of any $f \in C^0(X)$ compact?

Problem 1.4 (see [1, Problem 4.4]). Can the condition in [1, Theorem 4.1] that X is an ω-regular space be replaced by that X is a T_2-space? In other words, does the closure of any almost periodic orbit in a T_2-space contain only almost periodic points?

In this note, we study the above three problems, and obtain the following three propositions, which give negative answers to these problems.

Proposition 1.5. *There exist a T_2-space X and a continuous map $f : X \to X$ such that f has an almost periodic orbit of which the closure is not a minimal set.*

Proposition 1.6. *There exist a locally compact T_1-space X and a continuous map $f : X \to X$ such that f has a minimal set which is not compact.*

Proposition 1.7. *There exist a T_2-space X, a continuous map $f : X \to X$, and an almost periodic point x of f such that not all points in the closure $\overline{O(x, f)}$ of the orbit $O(x, f)$ are almost periodic points.*

2. Finer topologies and subspace topologies

In order to prove the above three propositions, we will construct several new T_1- and T_2-spaces by adding some open sets to a known topological space, or construct several new spaces with known spaces being subspaces. For this, in this section we give some lemmas on finer topological spaces and subspaces. These lemmas can be directly derived from the definitions concerned, and the proofs will be omitted.

Lemma 2.1. *Let $\mathcal{T}$ and $\mathcal{T}'$ be two topologies on a set X. If $\mathcal{T}'$ is finer than $\mathcal{T}$, and $(X,\mathcal{T})$ is a T_i-space, $i = 1,2$, then $(X,\mathcal{T}')$ is also a T_i-space.*

Lemma 2.2. *Let X be a topological space, Y be a subspace of X, and let $f : X \to X$ be a continuous map such that $f(Y) \subset Y$. Then any point $y \in Y$ is an almost periodic point of f if and only if y is an almost periodic point of $f|_Y$.*

From Lemma 2.2, we can obtain immediately.

Lemma 2.3. *Let X be a set, Y be a subset of X, and $f : X \to X$ be a map such that $f(Y) \subset Y$. Suppose that $\mathcal{T}$ and $\mathcal{T}'$ are two topologies on X, $\mathcal{T}|_Y = \mathcal{T}'|_Y$, and f is continuous both for $\mathcal{T}$ and for $\mathcal{T}'$. Then any point $y \in Y$ is an almost periodic point of f for topology $\mathcal{T}$ if and only if y is an almost periodic point of f for topology $\mathcal{T}'$.*

3. Proofs of Propositions 1.5–1.7

Propositions 1.5–1.7 claim that there exist continuous maps of some T_1- or T_2-spaces which have certain special properties. Hence in order to show these propositions, we need to construct maps having these special properties.

Let $S^1 = \{e^{t\mathbf{i}} : t \in \mathbb{R}\}$ be the unit circle in the complex plane $\mathbb{C}$. For any two real numbers $r < s$, write $A(r,s) = \{e^{t\mathbf{i}} : r < t < s\}$. Let $\mathcal{T}_0$ be the usual topology on S^1, and let

$$\mathcal{B}_0 = \{A(r,s) : r \text{ and } s \text{ are real numbers, and } 0 < s - r < \pi\}. \tag{3.1}$$

Then $\mathcal{B}_0$ is a set of open arcs in S^1, which is a basis for the topology $\mathcal{T}_0$.

Let $\theta \in [0,1]$ be a given irrational number, and let $h_\theta : S^1 \to S^1$ be the rotation defined by

$$h_\theta(e^{2\pi\mathbf{i}t}) = e^{2\pi\mathbf{i}(t+\theta)} \quad \text{for any } t \in \mathbb{R}. \tag{3.2}$$

Then under the topology $\mathcal{T}_0$, h_θ is a homeomorphism, S^1 is the unique minimal set of h_θ, and all points in S^1 are almost periodic points of h_θ. Let $w_0 \in S^1$ be a given point, and let

$$W = \{h_\theta^n(w_0) : n \in \mathbb{Z}\}, \qquad Y = S^1 - W. \tag{3.3}$$

Then we have $h_\theta(W) = W$, and $h_\theta(Y) = Y$.

Proof of Proposition 1.5. Let S^1, $\mathcal{B}_0$, $\mathcal{T}_0$, h_θ, W, and Y be as in (3.1)–(3.3). Let $X = S^1$, $f = h_\theta$, and let

$$\mathcal{U}_1 = \{A \cap Y : A \in \mathcal{B}_0\}, \qquad \mathcal{B}_1 = \mathcal{B}_0 \cup \mathcal{U}_1. \tag{3.4}$$

Then for any U, $V \in \mathcal{B}_1$, we have $U \cap V \in \mathcal{B}_1 \cup \{\emptyset\}$. Thus $\mathcal{B}_1$ is a basis for a topology $\mathcal{T}_1$ on X. It follows from $\mathcal{B}_1 \supset \mathcal{B}_0$ that $\mathcal{T}_1 \supset \mathcal{T}_0$. Hence by Lemma 2.1, $(X,\mathcal{T}_1)$ is a T_2-space. Since $\mathcal{T}_1|_W = \mathcal{T}_0|_W$ and $\mathcal{T}_1|_Y = \mathcal{T}_0|_Y$, by Lemma 2.3, every point $x \in X$ is an almost periodic point of f. Because $Y \in \mathcal{T}_1$, that is, Y is an open set, $W = X - Y$ is a closed subset of $(X,\mathcal{T}_1)$. For any $x \in X$, let $\overline{O(x,f)}$ be the closure of the orbit $O(x,f)$ in $(X,\mathcal{T}_1)$. For any $z \in X$ and any $U \in \mathcal{T}_1$ with $z \in U$, there exists an open arc $A \in \mathcal{B}_0$ such that

$$\begin{aligned} z \in A \subset U, &\quad \text{if } z \in W; \\ z \in A \cap Y \subset U, &\quad \text{if } z \in Y. \end{aligned} \tag{3.5}$$

Thus we have

$$\overline{O(x,f)} = \begin{cases} X, & \text{if } x \in Y; \\ W, & \text{if } x \in W. \end{cases} \tag{3.6}$$

From this we see that, in the T_2-space $(X, \mathcal{T}_1)$, W is the unique minimal set of f, and for any $x \in Y$, the closure $\overline{O(x,f)}$ of the almost periodic orbit $O(x,f)$ is not a minimal set. Proposition 1.5 is proven. □

Proof of Proposition 1.6. Let S^1, $\mathcal{B}_0$ and h_θ be as in (3.1)-(3.2). Take

$$X = S^1 \times \{0,1\}, \qquad \mathcal{B}_{20} = \{A \times \{0\} : A \in \mathcal{B}_0\}. \tag{3.7}$$

For any $A \in \mathcal{B}_0$ and any $x \in A$, write

$$U(A,x) = ((A \times \{0\}) \cup \{(x,1)\}) - \{(x,0)\}. \tag{3.8}$$

Let

$$\mathcal{B}_{21} = \{U(A,x) : A \in \mathcal{B}_0,\ x \in A\}, \qquad \mathcal{B}_2 = \mathcal{B}_{20} \cup \mathcal{B}_{21}. \tag{3.9}$$

Then $\mathcal{B}_2$ is a basis for a topology $\mathcal{T}_2$ on X, and the topological space $(X, \mathcal{T}_2)$ is a T_1-space. For any $A \subset \mathcal{B}_0$, let $\overline{A}$ be the closure of A in $(S^1, \mathcal{T}_0)$. Then $\overline{A}$ is a closed arc, which is homeomorphic to the compact interval $[0,1]$. Obviously, for any $A \in \mathcal{B}_0$ and any $x \in A$, $((\overline{A} \times \{0\}) \cup \{(x,1)\}) - \{(x,0)\}$ (resp., $\overline{A} \times \{0\}$) is a compact neighborhood of the point $(x,1)$ (resp. $(x,0)$) in $(X, \mathcal{T}_2)$, which is homeomorphic to the subspace $\overline{A}$ of $(S^1, \mathcal{T}_0)$. Thus $(X, \mathcal{T}_2)$ is a locally compact space. Define a map $f : X \to X$ by

$$f(x,i) = (h_\theta(x), 0), \quad \forall (x,i) \in X = S^1 \times \{0,1\}. \tag{3.10}$$

Then f is continuous for the topology $\mathcal{T}_2$. It is easy to see that every point $(x,i) \in X$ is an almost periodic point of f, and the closure $\overline{O((x,i),f)}$ of the orbit $O((x,i),f)$ in $(X, \mathcal{T}_2)$ is always the whole space X. Hence X is the unique minimal set of f. Since $\mathcal{B}_2$ is an open cover of X which has no finite subcover even has no countable subcover, the minimal set X is not compact. Proposition 1.6 is proven. □

Proof of Proposition 1.7. Let S^1, $\mathcal{B}_0$, $\mathcal{T}_0$, h_θ, W, and Y be as in (3.1)–(3.3). Take $X = S^1$ and $f = h_\theta$. Let

$$\mathcal{B}_{31} = \{A \cap W : A \in \mathcal{B}_0\}, \tag{3.11}$$

$$\mathcal{B}_{32} = \{(A \cap W) \cup \{y\} : A \in \mathcal{B}_0,\ y \in A \cap Y\}, \tag{3.12}$$

and let $\mathcal{B}_3 = \mathcal{B}_{31} \cup \mathcal{B}_{32}$. Then $U \cap V \in \mathcal{B}_3 \cup \{\varnothing\}$ for any $U, V \in \mathcal{B}_3$, and hence, $\mathcal{B}_3$ is a basis for a topology $\mathcal{T}_3$ on X. Clearly, under this topology $\mathcal{T}_3$, we have the following.

Claim 1. X is a T_2-space, and $f : X \to X$ is continuous.

Claim 2. No point $y \in Y$ is a recurrent point of f, and hence, no point $y \in Y$ is an almost periodic point of f.

Noting $\mathcal{T}_3|_W = \mathcal{T}_0|_W$, by Lemma 2.3, we have the following.

Claim 3. Every point $w \in W$ is an almost periodic point of f.

For any $X_0 \subset X$, let $\overline{X}_0$ be the closure of X_0 in the space $(X, \mathcal{T}_3)$. Then we have

Claim 4. $\overline{O(w,f)} = X$ for any $w \in W$, and $\overline{O(y,f)} = O(y,f)$ for any $y \in Y$.

It follows from Claims 3, 4, and 2 that, for any $x \in W$, not all points in the closure $\overline{O(x,f)}$ of the almost periodic orbit $O(x,f)$ are almost periodic points. Proposition 1.7 is proven. □

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References

[1] J.-H. Mai and W.-H. Sun, "Almost periodic points and minimal sets in ω-regular spaces," *Topology and Its Applications*, vol. 154, no. 15, pp. 2873–2879, 2007.