Hindawi Publishing Corporation
Discrete Dynamics in Nature and Society
Volume 2008, Article ID 263785, 6 pages
doi:10.1155/2008/263785

Research Article

Uniqueness and Multiplicity of Solutions for a Second-Order Discrete Boundary Value Problem with a Parameter

Xi-Lan Liu[1, 2] and Jian-Hua Wu[1]

[1] *College of Mathematics and Information Science, Shaanxi Normal University, Xian, Shaanxi 710062, China*
[2] *Department of Mathematics, Qinghai University for Nationalities, Xining, Qinghai 810007, China*

Correspondence should be addressed to Xi-Lan Liu, doclanliu2002@yahoo.com.cn

Received 15 November 2007; Accepted 10 February 2008

Recommended by Bing-Gen Zhang

This paper is concerned with the existence of unique and multiple solutions to the boundary value problem of a second-order difference equation with a parameter, which is a complement of the work by J. S. Yu and Z. M. Guo in 2006.

1. Introduction and preliminaries

In this paper, we consider the existence, uniqueness, and multiplicity of solutions for a second-order discrete boundary value problem

$$\begin{gathered} p(n+1)u(n+1)+c(n)u(n)+p(n)u(n-1)=\lambda f\big(n,u(n)\big), \quad n\in Z(1,k), \\ u(0)+\alpha u(1)=A, \qquad u(k+1)+\beta u(k)=B, \end{gathered} \tag{1.1}$$

where $\lambda\in\mathbb{R}$ is a parameter. Our technique is based on critical point theory, which is successfully used to deal with the existence of solutions for discrete problems (see [1–9]), especially in [7, 9]. Similarly to [7], we denote by $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{R}$ the sets of all natural numbers, integers, and real numbers, respectively. For $a,b\in\mathbb{Z}$, $Z(a)=\{a,a+1,\dots\}$, $Z(a,b)=\{a,a+1,\dots,b\}$ when $a\le b$. We assume that $p(n)$ is nonzero and real-valued for each $n\in Z(1,k)$, $c(n)$ is real-valued for each $n\in Z(1,k)$, and $f(n,u)$ is real-valued for each $(n,u)\in Z(1,k)\times\mathbb{R}$ and continuous in u. Let $\mathbb{R}^k$ be the real Euclidean space with dimension k. For any $u,v\in\mathbb{R}^k$, $\|u\|$ and (u,v), denote the usual norm and inner product in $\mathbb{R}^k$, respectively.

Consider the functional defined on $\mathbb{R}^k$,

$$J(u) = \frac{1}{2}(Mu, u) + (\eta, u) - \lambda F(u), \quad u = (u(1), u(2), \dots, u(k))^T \in \mathbb{R}^k, \tag{1.2}$$

where $(\cdot)^T$ is the transpose of a vector in $\mathbb{R}^k$,

$$M = \begin{pmatrix} c(1) - \alpha p(1) & p(2) & 0 & \cdots & 0 & 0 \\ p(2) & c(2) & p(3) & \cdots & 0 & 0 \\ 0 & p(3) & c(3) & \cdots & 0 & 0 \\ \cdots & & \cdots & & \cdots & \\ 0 & 0 & 0 & \cdots & c(k-1) & p(k) \\ 0 & 0 & 0 & \cdots & p(k) & c(k) - \beta p(k+1) \end{pmatrix},$$
$$\eta = \begin{pmatrix} p(1)A \\ 0 \\ \vdots \\ 0 \\ p(k+1)B \end{pmatrix}, \tag{1.3}$$
$$F(u) = \sum_{j=1}^{k} \int_0^{u(j)} f(j, s)\,ds.$$

It is easy to see that $J(u)$ is Fréchet differentiable with Fréchet derivative

$$J'(u) = Mu + \eta - \lambda f(u), \tag{1.4}$$

where $f(u) = (f(1, u(1)), f(2, u(2)), \dots, f(k, u(k)))^T$, and there is a one-to-one correspondence between the critical point of functional J and the solution of BVP (1.1). Furthermore, $u = (u(1), u(2), \dots, u(k))^T$ is a critical point of J if and only if $\{u(t)\}_{t=0}^{k+1} = (u(0), u(1), \dots, u(k+1))^T$ is a solution of (1.1), where $u(0) = A - \alpha u(1),\ u(k+1) = B - \beta u(k)$ [7].

Recently, Yu and Guo [7] studied the BVP,

$$\begin{gathered} p(n+1)u(n+1) + c(n)u(n) + p(n)u(n-1) = f(n, u(n)), \\ u(0) + \alpha u(1) = A, \qquad u(k+1) + \beta u(k) = B. \end{gathered} \tag{1.5}$$

They obtained some existence results for (1.5). One of the main results is as follows.

Theorem 1.1 (A_0). *Suppose that $f(n, z)$ satisfies the following assumption:*

(f3) *there exist constants $a_1 > 0$, $a_2 > 0$, $R > 0$, and $\beta > 2$ such that*

$$\sum_{j=1}^{k} \int_0^{u(j)} f(j, s)\,ds \ge a_1 \|u\|^{\beta} - a_2 \quad \text{for } u = (u(1), u(2), \dots, u(k))^T \in \mathbb{R}^k,\ \|u\| \ge R. \tag{1.6}$$

Then BVP (1.5) *has at least one solution.*

Equation (1.6) shows that $F(u) = \sum_{j=1}^{k}\int_0^{u(j)} f(j,s)ds > 0$ for $\|u\|$ large enough. Since $F(u)$ may be negative, then the conclusion of Theorem 1.1 cannot be drawn, which motivates us to consider (1.1). Note that if we take $\lambda = -1$ in (1.1), then $F(u) < 0$. Under the similar condition to (1.6) when $\lambda < 0$, we not only obtain the existence of solutions for (1.1), but also the multiplicity.

Let E be a Banach space with a direct sum decomposition $E = X \oplus Y$. The functional $I \in C^1(E,\mathbb{R})$ has a local linking at 0 if for some $\rho > 0, I(u) \le 0, u \in X$, $\|u\| \le \rho$, and $I(u) \ge 0, u \in Y$, $\|u\| \le \rho$. The functional $I \in C^1$ $(E,\mathbb{R})$ is said to satisfy the (PS) condition if any sequence $x_n \subset X$ for which $I(x_n)$ is bounded, and $I'(x_n) \to 0$ $(n \to \infty)$ possesses a convergent subsequence in E.

Theorem A (see[10]). *Let E be a Banach space. Suppose that $I \in C^1(E,\mathbb{R})$ satisfies the (PS) condition and has a local linking at* 0. *Assume that I is bounded below and $\inf_E I < 0$. Then I has at least two nontrivial critical points.*

Theorem B (see[6, 11]). *Let X be a real Banach space, $I \in C^1(X,\mathbb{R})$ with I even, bounded from below, and satisfying (PS) condition. Suppose $I(0) = 0$. There is a set $K \subset X$ such that K is homeomorphic to a unit sphere S^{n-1} in $\mathbb{R}^n$ $(n \in \mathbb{N})$ by an odd map, and $\sup_K I < 0$. Then I possesses at least n distinct pairs of critical points.*

2. Main results

Following conditions will be useful to prove our main results.

(H1) There exist numbers $\alpha_1 > 2$ and $a_1 > 0$ such that

$$F(u) \ge a_1\|u\|^{\alpha_1} \quad \text{for } u = \big(u(1), u(2), \dots, u(k)\big)^T \in \mathbb{R}^k. \tag{2.1}$$

(H2) $\lim_{u\to 0}(F(u)/\|u\|^2) = 0$ for $u = (u(1), u(2), \dots, u(k))^T \in \mathbb{R}^k$.

Theorem 2.1. *Suppose that M is positive definite, $f(j,0) = 0$, $f(j,u)u < 0$ for $j \in Z(1,k)$ and $u \neq 0$, and that $p(1)A = p(k+1)B = 0$. Then* (1.1) *has only trivial solution for $\lambda > 0$.*

Proof. Note that

$$\begin{aligned}\big(J'(u),u\big) &= (Mu,u) + (\eta,u) - \lambda\big(f(u),u\big)\\ &\ge \lambda_1\|u\|^2 - \lambda\sum_{j=1}^{k} f\big(j,u(j)\big)u(j)\\ &\ge \lambda_1\|u\|^2 > 0\end{aligned} \tag{2.2}$$

for $u \neq 0$, where λ_1 is the least eigenvalue of M, which means that the Nahari manifold is empty. Thus (1.1) has only trivial solution. □

Theorem 2.2. *Suppose that (H1) and (H2) hold, $p(1)A = p(k+1)B = 0$ and M is neither positive definite nor negative definite. Then* (1.1) *has at least two nontrivial solutions for $\lambda < 0$.*

Proof. We will prove that the functional $J(u)$ satisfies all conditions of Theorem A by two steps.

Step 1. J is bounded from below and satisfies (PS) condition. Let $\lambda_{-l}, \lambda_{-l+1}, \dots, \lambda_{-1}, \lambda_1, \lambda_2, \dots, \lambda_m$ denote all the eigenvalues of M, where $\lambda_{-l} \le \lambda_{-l+1} \le \cdots \le \lambda_{-1} < 0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_m$ and $l + m = k$. For any $j \in Z(-l,-1) \bigcup Z(1,m)$, set ξ_j to be an eigenvector of M corresponding to the eigenvalue λ_j, $j = -l, -l+1, \dots, -1, 1, 2, \dots, m$, such that

$$(\xi_i, \xi_j) = \begin{cases} 0 & \text{for } i \neq j, \\ 1 & \text{for } i = j. \end{cases} \tag{2.3}$$

Let X and Y be subspaces of $\mathbb{R}^k$ defined by

$$\begin{aligned} X &= \left\{ x \in \mathbb{R}^k : x = \sum_{j=1}^{m} x_j \xi_j,\ x_j \in \mathbb{R},\ j \in Z(1,m) \right\}, \\ Y &= \left\{ y \in \mathbb{R}^k : y = \sum_{j=-l}^{-1} y_j \xi_j,\ y_j \in \mathbb{R},\ j \in Z(-l,-1) \right\}, \end{aligned} \tag{2.4}$$

respectively. Then $\mathbb{R}^k$ has the direct sum decomposition $\mathbb{R}^k = X \oplus Y$. In view of (H1), we have

$$\begin{aligned} J(u) &\ge \frac{1}{2}\lambda_{-l}\|u\|^2 - \lambda a_1 \|u\|^{\alpha_1} \\ &\ge -\frac{1}{2}\left|\lambda_{-l}\right|(1 - 2/\alpha_1)\left(\left|\lambda_{-l}\right| / (-\alpha_1 \lambda a_1)\right)^{2/(\alpha_1 - 2)}. \end{aligned} \tag{2.5}$$

The second inequality follows from the elementary inequality $-ax^2 + bx^q \ge -a(1-2/q)((2a)/(qb))^{2/(q-2)}$, where $a > 0,\ b > 0,\ x > 0$, and $q > 2$, which can be easily obtained by the fact that the function $h(x) = -ax^2 + bx^q\ (a, b > 0,\ x \ge 0)$ attains its minimum at $((2a)/(qb))^{1/(q-2)}$. Thus $J(u)$ is bounded from below.

Equation (2.5) shows that $J(u)$ is coercive, so we can obtain that any (PS) sequence must be bounded in $\mathbb{R}^k$ and, by a standard argument, has a convergent subsequence.

Step 2. J has a local linking at 0. Indeed, by (H2) for given $\lambda < 0$ and sufficiently small $\varepsilon > 0$ such that $-\lambda\varepsilon + (1/2)\lambda_{-1} < 0$, there exists $r > 0$ small enough such that for $\|u\| < r$,

$$F(u) < \varepsilon \|u\|^2 \tag{2.6}$$

holds. Then for $y \in Y$ such that $0 < \|y\| < r$, we have

$$\begin{aligned} J(y) &\le \frac{1}{2}\lambda_{-1}\|y\|^2 - \lambda\varepsilon\|y\|^2 \\ &= \left(\frac{1}{2}\lambda_{-1} - \lambda\varepsilon\right)\|y\|^2 < 0. \end{aligned} \tag{2.7}$$

On the other hand, for $x \in X$ with $\|x\| < r$, we have

$$J(x) \ge \frac{1}{2}\lambda_1 \|x\|^2 - \lambda a_1 \|x\|^{\alpha_1} \ge 0. \tag{2.8}$$

The application of Theorem A finishes our proof. □

Remark 2.3. By the above proof, we see that replacing (H1) with (f3), and by adding the condition that $F(u) \geq 0$ for $u \in \mathbb{R}^k$, Theorem 2.2 still holds, where (f3) is the same as in Theorem 1.1. Indeed, we have

$$J(u) \geq \begin{cases} \frac{1}{2}\lambda_{-l}R^2 & \text{for } \|u\| \leq R, \\ -\frac{1}{2}\left|\lambda_{-l}\right|(1-2/\beta)\left(\left|\lambda_{-l}\right|/(-\beta\lambda a_1)\right)^{2/(\beta-2)} + \lambda a_2 & \text{for } \|u\| > R, \end{cases} \tag{2.9}$$

which means that $J(u)$ is bounded from below. The fact that $J(u)$ has local linking at 0 may be verified similarly.

If we further impose some condition on $f(u)$ and matrix M, then the following result can be derived.

Theorem 2.4. *Suppose (H1) and (H2) hold, $p(1)A = p(k+1)B = 0$, $f(u)$ is odd in u, that is, $f(j,-u) = -f(j,u)$ for $(j,u) \in Z(1,k)\times\mathbb{R}$, and that M is neither positive definite nor negative definite and has l distinct negative eigenvalues. Then* (1.1) *has at least l distinct pairs of solutions for $\lambda < 0$.*

Proof. By the proof of Theorem 2.2, $J(u)$ is bounded from below and satisfies (PS) condition. In addition, $J(0) = 0$, $J(u)$ is even. Consider the subset K of $\mathbb{R}^k$:

$$K = \left\{ y \in Y : y = \sum_{j=-l}^{-1} y_j\xi_j,\ \sum_{j=-l}^{-1} y_j^2 = \rho^2,\ y_j \in \mathbb{R},\ j \in Z(-l,-1) \right\}, \tag{2.10}$$

where ρ is a positive number small enough to be determined later, Y is defined by Theorem 2.2 similarly. Define the mapping $T : K \to S^{l-1}$ by

$$T(y) = T\left(y = \sum_{j=-l}^{-1} y_j\xi_j \right) = \left(-\frac{y_{-l}}{\rho}, -\frac{y_{-l+1}}{\rho}, \dots, -\frac{y_{-1}}{\rho} \right), \tag{2.11}$$

where S^{l-1} is a unit sphere in $\mathbb{R}^l$. Then T is a homeomorphism between K and S^{l-1}, and K is a subset of the finite dimensional space Y equipped with the Euclidian norm. We can choose $\rho > 0$ and $\varepsilon > 0$ small enough such that $(1/2)\lambda_{-1} - \lambda\varepsilon < 0$ for $y \in K$, and then we have

$$J(y) \leq \frac{1}{2}\lambda_{-1}\|y\|^2 - \lambda F(u) \leq \frac{1}{2}\lambda_{-1}\rho^2 - \lambda\varepsilon\rho^2 \leq \left(\frac{1}{2}\lambda_{-1} - \lambda\varepsilon\right)\rho^2. \tag{2.12}$$

For above $\rho > 0$, we have

$$\sup_K J < 0, \tag{2.13}$$

which together with Theorem B concludes the proof. □

Remark 2.5. The condition that $J(u)$ is bounded from below is crucial to prove both Theorems 2.2 and 2.4. As in Remark 2.3, if we replace (H1) by (f3) and the condition that $F(u) \geq 0$ for $u \in \mathbb{R}^k$, then Theorem 2.4 is also true.

Acknowledgments

The authors would like to thank Professor B. G. Zhang and referee's valuable suggestions and comments. This research was partially supported by the NNSF (10571115) of China, China Postdoctoral Science Foundation (20070421106), the Key Project of Chinese Ministry of Education, and the Foundation of Chinese Nationalities Committee (07QH04).

References

[1] Z. Guo and J. Yu, "Existence of periodic and subharmonic solutions for second-order superlinear difference equations," *Science in China Series A*, vol. 46, no. 4, pp. 506–515, 2003.
[2] Z. Guo and J. Yu, "The existence of periodic and subharmonic solutions of subquadratic second order difference equations," *Journal of the London Mathematical Society*, vol. 68, no. 2, pp. 419–430, 2003.
[3] J. Yu, Z. Guo, and X. Zou, "Periodic solutions of second order self-adjoint difference equations," *Journal of the London Mathematical Society*, vol. 71, no. 1, pp. 146–160, 2005.
[4] Z. Guo and J. Yu, "Periodic and subharmonic solutions for superquadratic discrete Hamiltonian systems," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 55, no. 7-8, pp. 969–983, 2003.
[5] L. Jiang and Z. Zhou, "Existence of nontrivial solutions for discrete nonlinear two point boundary value problems," *Applied Mathematics and Computation*, vol. 180, no. 1, pp. 318–329, 2006.
[6] P. H. Rabinowitz, *Minimax Methods in Critical Point Theory with Applications for Differential Equations*, CBMS Regional Conference Series in Mathematics, American Mathematical Society, Providence, RI, USA, 1984.
[7] J. Yu and Z. Guo, "On boundary value problems for a discrete generalized Emden–Fowler equation," *Journal of Differential Equations*, vol. 231, no. 1, pp. 18–31, 2006.
[8] J. Yu and Z. Guo, "Boundary value problems of discrete generalized Emden–Fowler equation," *Science in China Series A*, vol. 49, no. 10, pp. 1303–1314, 2006.
[9] G. Zhang and S. S. Cheng, "Existence of solutions for a nonlinear system with a parameter," *Journal of Mathematical Analysis and Applications*, vol. 314, no. 1, pp. 311–319, 2006.
[10] H. Brezis and L. Nirenberg, "Remarks on finding critical points," *Communications on Pure and Applied Mathematics*, vol. 44, no. 8-9, pp. 939–963, 1991.
[11] T. Gyulov and S. Tersian, "Existence of trivial and nontrivial solutions of a fourth-order ordinary differential equation," *Electronic Journal of Differential Equations*, no. 41, pp. 1–14, 2004.