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Research Article

A Wave-Spectrum Analysis of Urban Population Density: Entropy, Fractal, and Spatial Localization

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The method of spectral analysis is employed to research the spatial dynamics of urban population distribution. First of all, the negative exponential model is derived in a new way by using an entropy-maximizing idea. Then an approximate scaling relation between wave number and spectral density is derived by Fourier transform of the negative exponential model. The theoretical results suggest the locality of urban population activities. So the principle of entropy maximization can be utilized to interpret the locality and localization of urban morphology. The wave-spectrum model is applied to the city in the real world, Hangzhou, China, and spectral exponents can give the dimension values of the fractal lines of urban population profiles. The changing trend of the fractal dimension does reflect the localization of urban population growth and diffusion. This research on spatial dynamics of urban evolvement is significant for modeling spatial complexity and simulating spatial complication of city systems by cellular automata.

1. Introduction

The greatest shortcoming of the human race is our inability to understand the exponential function.

Albert A. Bartlett

Simulating the spatial dynamics of urban population is an interesting but a difficult project. Urban population density can be modeled by two types of functions: one is the exponential function known as Clark's law [1], the other is the power function proposed by Smeed [2]. Geographers used to employ the exponential model to characterize population density of monocentric cities. However, Smeed's model has been favored since fractal cities came to front (see, e.g., [3–5]). In fact, if we reject Smeed's model, we will be unable to interpret the law of allometric growth on urban area and population in theory. On the other hand, if we avoid Clark's model, we will not be able to describe many cities' population density empirically. Geographers are often placed in a dilemma when dealing with spatial dynamics of urban evolution.

In fact, the exponential function implies translational symmetry, while the power function denotes dilation symmetry or scaling symmetry; the exponential function implies simplicity and randomness, while the power function indicates complexity and structure [6, 7]. In fractal geometry, two exponential functions can often construct a power function, while a power function can always be decomposed into two exponential functions [8]. It is difficult for us to understand the exponential function, and it is especially difficult to understand the relation between the exponential distribution and the power-law distribution. A conjecture is that exponential law and power law represent, respectively, two modes of urban evolvement which supplement each other.

Population is one of the two central variables which can be employed to explore the dynamics of cities [9]. However, the underlying rationale of intraurban population growth and diffusion is still a question pending further discussion. Clark's law on urban density can provide a window for us to apprehend the dynamics of urban morphology from the angle of view of population. The negative exponential distribution seems to mean nonfractal structure of urban population, but it can be associated with fractal structure by the Fourier transform. In order to probe the mysteries of fractal cities and the related spatial dynamics, we must research the essentials of negative exponential distribution.

In this paper, the exponential model of urban density will be explored by using the wave-spectral analysis. The significance of studying the classical model is in three aspects. The first is to reveal the locality and localization of urban population evolution, which is very important for simulating spatial complexity of cities through computers. The second is to find a new approach to evaluating a kind of fractal dimension of urban form, which differs from but can make up *box dimension* and *radial dimension*. The third is to understand spatial complexity of urban evolvement in the new perspective. The study of complexity concerns emergence of fractals, localization, strange attractor, symmetry breaking, and so on [10]. Fractal structure and localization can be brought to light to some extent from the negative exponential distribution by means of spectral analysis.

The rest of this paper is structured as follows. Section 2 presents a new derivation of the negative exponential model of urban population density by the entropy-maximizing principle, which is actually one of the fundamental reasons of fractal cities [11]. Based on the exponential function, an approximate power-law relation between wave number and spectral density is derived by Fourier transform. Section 3 provides an empirical analysis, including spectral analysis, correlogram analysis, and information entropy analysis, by applying the theoretical models to the city of Hangzhou, China. The computations lend support to the theoretical inferences given in Section 2. In Section 4, the differences and relationships between the negative exponential distribution and the inverse power-law distribution are discussed to distinguish the concept of *locality* from that of *action at a distance*.

2. Mathematical models

2.1. New derivation of Clark's law

A power law indicating fractal structure of urban systems can be decomposed into two exponential laws [8], and the exponential laws can be derived by using the entropy-maximizing method [12]. This suggests that fractal structure can be interpreted with the principle of entropy maximization, and exponential function is an important bridge between entropy and power law. On the other hand, as complex spatial system, an urban phenomenon can be modeled with different mathematical expressions under different conditions. Urban population density can be described by a number of functions, among which the negative exponential function is always valid in empirical analysis [1, 13, 14]. Since that the exponential law can connect entropy maximization and fractal, we are naturally interested in the cause and effect of the urban exponential distribution. It will be shown that Clark's law comes between entropy-maximizing process and special fractal structure.

As a theoretical study, this paper is focused on a monocentric city, and all the data analyses are based on the idea from statistical average. In this instance, the growth of cities is often regarded as a process of spatiotemporal diffusion [15], which can be abstracted as the following partial differential equation

$$\frac{\partial\rho(x,y,t)}{\partial t} = K\left[\frac{\partial^2\rho(x,y,t)}{\partial x^2} + \frac{\partial^2\rho(x,y,t)}{\partial y^2}\right] - a\rho(x,y,t), \tag{2.1}$$

where a denotes growth/decay coefficient or transfer coefficient, K is called "diffusivity" or diffusion coefficient, and x and y refer to two directions of spatial diffusion. For an isotropic diffusion, one direction (say, x) has no difference from the other direction (say, y), we have $x = y = r$, where r represents the distance from the center of city (where $r = 0$). In order words, we can substitute one-dimension diffusion process for two-dimension process to analyze the isotropic city systems. Now, if ρ does not change with time, namely, if $\partial\rho/\partial t = 0$, then (2.1) reduces to the common differential equation characterizing one-dimension diffusion such as $\mathrm{d}^2\rho(r)/\mathrm{d}r^2 - a\rho(r)/K = 0$ (the initial condition is $\rho_{|r=0} = \rho_0$, while the boundary condition $\rho_{|r\to\infty} = 0$), whose solution is just the exponential function known as Clark's law (see Appendix A). This suggests that the exponential law in fact reflects an instantaneous equilibrium of urban population diffusion.

Assuming that population density $\rho(r)$ at distance r from the city center declines monotonically, Clark [1] proposed an empirical model that can be written as

$$\rho(r) = \rho_0 \exp(-br) = \rho_0 \exp\left(-\frac{r}{r_0}\right), \tag{2.2}$$

where ρ_0 is a constant of proportionality which is supposed to equal the central density, that is, $\rho_0 = \rho(0)$, b denotes a rate at which the effect of distance attenuates, and $r_0 = 1/b$ refers to a characteristic radius of urban population distribution. Thus we have $r_0 = \sqrt{K/a}$. Clark [1] fitted the log transform of (2.1) to more than 20 cities by using linear regression. The results form the solid empirical foundation of the negative exponential law of monocentric urban density.

In the real world, urban growth is often not isotropic, but in an average sense, we can regard an anisotropic process as an isotropic process. Just based on this idea, Clark's law is propounded. An urban population density function is actually defined in one-dimension space but it includes information of two-dimension space. Generally speaking, an exponential distribution function can be derived from the entropy maximization principle. Bussiere and Snickers [16] once showed that Clark's model could be derived from Wilson's [17] spatial interaction models (see also Wilson [18]), which is based on the entropy-maximizing principle. In fact, under ideal conditions, Clark's model can be derived in a very simple way from several geographical assumptions by using entropy-maximizing methods. Now, in order to reveal the physical essence of exponential distribution of urban population, a new derivation of (2.2) is given in this subsection. The mathematical deduction is more graceful and compendious than previous derivation presented by Bussiere and Snickers [16], and the process is helpful for exploring the spatial dynamics of urban evolution.

Suppose that the total population in the urban field of a monocentric city is P_t, and the urban growth is considered to be a continuous process in time and space. An urban field is defined as a bounding circle based on the center of the urban cluster, marked by the maximum radius R which contains the whole cluster [3, page 340]. Imagining that the urban map has been digitalized with low resolution, we can "string" $n+1$ pixels, which may be called "cells," by drawing a straight line or a radial from the center of the city to the boundary (see Figure 1). Further, suppose the population in the ith cell along the "line" is ρ_i $(i = 0, 1, 2, \ldots, n)$, and the whole population along the line is P. The variable ρ_i has dual attributes. On the one hand, it denotes the population size within the ith cell along the line, and on the other, it represents just the average population density of the ith ring comprising a number of cells.

Since Clark's law is just the solution to the one-dimension diffusion equation, we can examine one-dimension population distribution based on the idea from statistical average. The postulates of this study can be summarized as follows. (1) A monocentric city has no strict boundary because of scaling invariance of urban form. (2) Population is dense enough in urban field. The next step is to find the functional relationship between density ρ_i and distance r. For this purpose, the entropy-maximizing method is employed. The number of states of the population distributed in all the cells along the radial, W, can be expressed as an ordered division problem

$$W = \binom{P}{\rho_0, \rho_1, \ldots, \rho_n} = \frac{P!}{\prod_{i=0}^{n+1} \rho_i!}. \tag{2.3}$$

We use the state in one-dimension urban space to represent the state in two-dimension urban space in average sense. Then the entropy of population distribution profile, H_e, is given by

$$H_e = \ln W = \ln P! - \sum_{i=0}^{n+1} \ln \rho_i!. \tag{2.4}$$

Suppose that the entropy approaches to maximization. We can define an objective function such as

$$\text{Max}\, H_e = \ln W. \tag{2.5}$$

According to our assumptions stated above, the objective of city evolution is subject to two constraint conditions as follows:

$$\sum_{i=0}^{n} \rho_i = P, \tag{2.6}$$

$$\rho_0 + \sum_{i=1}^{n} 2\pi i \rho_i = P_t. \tag{2.7}$$

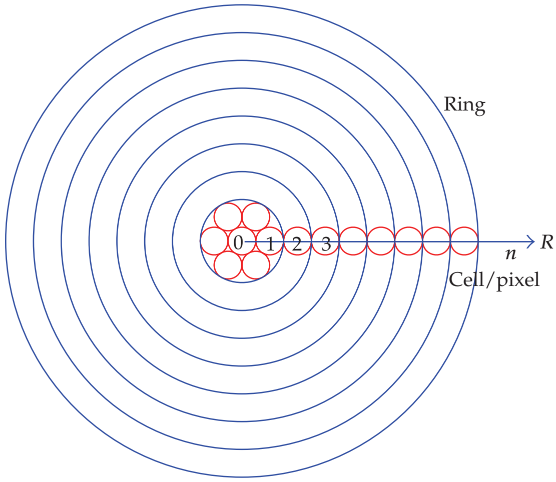


Figure 1: A sketch map of urban field with rings and cells.

Equation (2.6) can be understood easily, but (2.7) need be made clear. Here ρ_0 refers to the population number in the center of the city, and $2\pi i\rho_i$ to the population number in the ith ring that is measured with a circle of cells. In fact, if we measure the distance by the size of the cells, namely, take the diameter of cells as length unit, then i is just the distance from the centroid of the ith cell to the center of the city. That is, when the cells are very very small, the ordinal number i can represent the radius of the ith ring, and $2\pi i$ is the corresponding perimeter.

In this case, if we can find a mathematical expression to describe the relationship between ρ_i and i, the problem will be solved immediately. Thus our question can be turned into the process of finding conditional extremum because that the value of entropy depends on the density of spatial distribution of urban population. A Lagrangian function is constructed as

$$L(\rho_i) = \ln P! - \sum_i \ln \rho_i! + \lambda_1\left(P - \sum_i \rho_i\right) + \lambda_2\left(P_t - \rho_0 - \sum_i 2\pi i\rho_i\right), \tag{2.8}$$

where λ is the Lagrangian multiplier (LM). Theoretically ρ_i and P are both large enough in terms of our postulates. According to the well-known Stirling's formula $N! = (2\pi)^{1/2}N^{N+1/2}e^{-N}$, we have an approximate relation, $\partial \ln N!/\partial N = \ln N$, where N is considerably big. So, differentiating (2.8) partially with respect to ρ_i yields

$$\frac{\partial L(\rho_i)}{\partial \rho_i} = -\ln \rho_i - \lambda_1 - 2\pi\lambda_2 i. \tag{2.9}$$

Considering the condition of extremum $\partial L(\rho_i)/\partial\rho_i = 0$, we have

$$\rho_i = e^{-\lambda_1}e^{-2\pi\lambda_2 i}. \tag{2.10}$$

In theory, we can improve the resolution of digital map unlimitedly, and thus, the cell/pixels become infinitesimal. That is to say, for simplicity, the discrete distance variable represented by i can be replaced with a continuous one represented by r for the time being, that is, $i \to r$, $\rho_i \to \rho(r)$. Inserted with (2.10), the discrete (2.7) in which r is used as a substitute for i can be rewritten as an integration expression

$$\int_0^R 2\pi r\rho(r)\mathrm{d}r = e^{-\lambda_1}\int_0^R 2\pi r e^{-2\pi\lambda_2 r}\,\mathrm{d}r = P_t, \tag{2.11}$$

where R is the radius of urban field and it can be defined by $R = F/2$, here F is the Feret's diameter [3], (see Kaye [19]). Equation (2.11) is the continuous expression replacing (2.7). In keeping with the first postulate, R is large enough. Using integration by decomposition and taking l'Hospital's rule into account, we can find the solution of (2.11) such as (see Appendix B)

$$e^{-\lambda_1} = 2\pi\lambda_2^2 P_t. \tag{2.12}$$

Substituting (2.12) into (2.10) yields

$$\rho(r) = 2\pi\lambda_2^2 P_t e^{-2\pi\lambda_2 r}. \tag{2.13}$$

If $r = 0$ as given, then (2.13) collapses to

$$\rho(0) = 2\pi\lambda_2^2 P_t = \rho_0. \tag{2.14}$$

The characteristic radius of the city, r_0, can be defined by

$$\rho_0 = e^{-\lambda_1} = \frac{P_t}{2\pi r_0^2}. \tag{2.15}$$

Inserting (2.15) into (2.14) gives

$$2\pi\lambda_2 = b = \frac{1}{r_0}. \tag{2.16}$$

Substituting (2.14) and (2.16) into (2.13) immediately yields Clark's law, that is, (2.2). Further, inserting (2.2) into (2.6), we can derive $r_0 = P/\rho_0$. The maximum of entropy can be proved to be $H_{\max} = e\ln r_0$, where e is the base of the natural system of logarithms, having a numerical value of 2.7183 approximately.

Entropy maximization suggests the most probable distribution on some conditions. The negative exponential distribution of urban population density is not inevitable, but it is the most probable state for at least the monocentric city. This kind of distribution suggests a special fractal profile, which can be brought to light by Fourier analysis and scaling wave-spectrum relation.

2.2. Wave-spectrum function of urban density

The negative exponential model of urban density is in essence a special spatial correlation function. A power-law relation between wave number and spectrum density can be derived from the exponential function. Considering the interaction between the cells along the radial in Figure 1, we can construct a density-density correlation function as follows:

$$C(r) = \int_{-\infty}^{\infty} \rho(x)\rho(x+r)\mathrm{d}x = 2\rho_0^2 \int_0^{\infty} e^{-2x/r_0 - r/r_0}\,\mathrm{d}x, \tag{2.17}$$

where $\rho(x)$ denotes the population density of cell X at distance x from the city center, $\rho(x+r)$ refers to the population density of another cell at distance r from X. Given $x = 0$, it follows that one cell becomes the center of the city, and the spatial correlation function collapses to an exponential function

$$C(r) = \rho(0)\rho(r) = \rho_0^2 e^{-r/r_0}. \tag{2.18}$$

If the data are so normalized that $\rho_0 = 1$, we have $C(r) = \rho(r)$, and thus (2.18) is equivalent to (2.2). In this case, Clark's law is just a special density-density correlation function, which indicates spatial correlating action between the city center and the location at distance r from the center. The distance parameter, r_0, is relative to the spatial correlation length. A larger value of the characteristic radius (r_0) suggests a longer correlation distance.

Note that the autocorrelation function and the energy spectrum can be converted to each other through Fourier's cosine transform:

$$S(k) = \int_{-\infty}^{\infty} C(r)e^{-j2\pi kr}\,\mathrm{d}r = 2\int_0^{\infty} C(r)\cos 2\pi kr\,\mathrm{d}r, \tag{2.19}$$

where $j = \sqrt{-1}$ is the unit of complex number, k denotes wave number, that is, the reciprocal of the wavelength, $S(k)$ represents corresponding energy spectral density. The concept of *energy spectrum* comes from engineering mathematics. The product of Fourier transform of a function and its conjugate bears an analogy with the mathematical form of energy in physics [20]. In the light of the symmetry of correlation function, the Fourier transform of (2.18) can be given in the form

$$F(k) = \rho_0^2 \int_{-\infty}^{\infty} e^{-r/r_0} e^{-j2\pi kr}\,\mathrm{d}r = \frac{2r_0\rho_0^2}{1 + j2\pi kr_0}. \tag{2.20}$$

As $2r_0\rho_0^2$ is large enough, we have

$$\left[\frac{1}{2r_0\rho_0^2}\right]^2 \longrightarrow 0. \tag{2.21}$$

Thus the energy spectral density can be gained according to energy integral such as [21]

$$S(k) = |F(k)|^2 = \frac{(2r_0\rho_0^2)^2}{1 + r_0^2(2\pi k)^2} \approx \frac{1}{(\pi k)^2/(\rho_0^4)} \propto k^{-2}. \tag{2.22}$$

In practice, the length of sample path, L, is generally limited, therefore the *wave spectrum* density $W(k) = |G(k)|^2/L$ is always employed to substitute for the *energy spectrum* density $S(k)$. Then (2.22) can be rewritten as [22]

$$W(k) \propto k^{-2}. \tag{2.23}$$

Equation (2.23) is an approximate expression based on ideal conditions, and it can be generalized to the following scaling relation:

$$W(k) \propto k^{-\beta}, \tag{2.24}$$

where β is called "*spectral exponent*" which usually ranges from 0 to 3. When β value is near 1, (2.24) indicates what is called $1/\beta$ noise (see, e.g., [23, 24]). In fact, the spectral exponent is associated with a fractal dimension of urban population profiles.

For a time series or spatial series, if the relation between spectral density and frequency or wave number follows the scaling law defined by (2.24), a fractal structure can be revealed. It has been demonstrated that, for $d_E = 1$ dimension variables, the connection between β and D is given by [25–27]

$$D = d_E + \frac{3-\beta}{2} = \frac{5-\beta}{2} = 2 - H, \tag{2.25}$$

where d_E refers to the dimension of Euclidean space. Accordingly, $\beta = 5 - 2D$, where D is the fractal dimension of urban population profiles ($d_E < D < d_E + 1$), and H denotes the Hurst exponent ($0 \le H \le 1$). Further, the autocorrelation coefficients of the rate of changes can be derived from the fractional Brownian motion as in [25]

$$C_\Delta(r) = \frac{\langle -\rho(r-1)\rho(r+1)\rangle}{\langle \rho(r)^2 \rangle} = 2^{2H-1} - 1. \tag{2.26}$$

This is a special density-density correlation function, which can be understood by means of the knowledge of time series analysis. Many methods of analyzing times series, including autocorrelation analysis, autoregression analysis, and spectral analysis, can be employed to deal with spatial series [28]. If $D = 1.5$ or $\beta = 2$, then we have $H = 1/2$, and thus $C_\Delta = 0$. In this case, the ith cells act directly on and only on the $(i \pm 1)$th cells, and do not act on the $(i \pm 2)$th cells or more. If $D < 1.5$ or $\beta > 2$, then we have $H > 1/2$, and thus $C_\Delta > 0$. In this case, for $u > 1$, the ith cell can act directly on the $(i \pm u)$th cells positively. If $D > 1.5$ or $\beta < 2$, then we have $H < 1/2$, and thus $C_\Delta < 0$. In this case, the ith cell will act directly on the $(i \pm u)$th cells negatively.

For the negative exponential function of urban population density, the expected result of spectral exponent is $\beta \approx 2$, thus the fractal dimension is $D \approx 3/2 = 1.5$, and the Hurst exponent is $H = 2 - D \approx 0.5$, which gives the autocorrelation coefficient $C_\Delta \approx 0$. That suggests a spatial locality of city systems. In physics, the *principle of locality* coming from Einstein [29] is that distant objects cannot have direct influence on each another. In other words, an object is influenced directly only by its immediate surroundings. The fact of spatial autocorrelation coefficient $C_\Delta \to 0$ implies that a population cell tends to interact only on the immediate cells.

3. Empirical analysis

3.1. Study area and data resource

The city of Hangzhou is taken as an example to verify the wave-spectrum relation of urban population density and related theory. Hangzhou is the capital of Zhejiang province, China. The urban density data in 1964, 1982, 1990, and 2000 come from census, which is processed by Feng [14]. The census tract data are based on *jiedao*, which bears an analogy with the UK enumeration districts [30], or the US subdistricts [13], while the system of *jiedao* has an analogy with the urban zonal system in Western literature (see [3, page 325]). In the demographic sense, a *jiedao* is a census tract. The data are processed by means of spatial weighed average. The length of sample path is 26, and the maximum urban radius is 15.3 km. The method of processing data is illuminated in detail by Feng [14]. Some necessary explanations on sampling and data processing are made as follows.

(1) Sampling area. The data cover the metropolitan area (MA) of Hangzhou, which is greater than the urbanized area (UA). The spatial scopes of sampling in four years are same in order to make it sure that the parameters from 1964 through 2000 are comparable. Because of scaling invariance of urban form [3], we take no account of the borderline between the urban and rural areas.

(2) Calculation method. A series of concentric rings are drawn around city center in proportional spacing (see Figure 2). The ratio of each partial zone to the whole area between two rings is taken as the weight of computing urban population density. A region between two adjacent rings can be named a circular belt, which will be numbered as $p = 0, 1, \ldots, n$, where n is the number of circular belt. The zones can be numbered as $q = 1, 2, \ldots, m$, where m is the number of zones. Let S_{pq} be the common area of the pth circular belt and the qth zone, that is,

$$S_{pq} = B_p \bigcap Z_q, \tag{3.1}$$

where B_p represents the pth belt, Z_q denotes the qth zone, both of them are measured with area; therefore, S_{ij} is the area of the intersection of B_p and Z_q. Defining a weighted coefficient w_{pq} as

$$w_{pq} = \frac{S_{pq}}{S_p} = \frac{S_{pq}}{\sum_q S_{pq}} = \frac{S_{pq}}{\pi\left(r_{p+1}^2 - r_p^2\right)}, \tag{3.2}$$

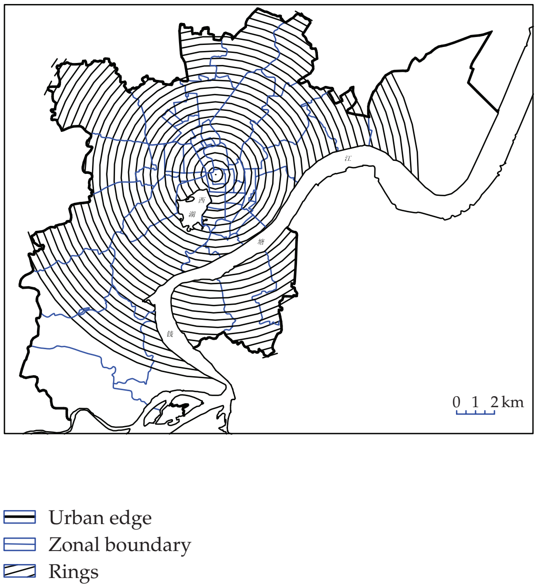


Figure 2: Study area and zonal system in the Hangzhou metropolis (from Feng [14]).

we have

$$\overline{\rho}_p = \sum_{q=1}^{m} w_{pq}\rho_q = \frac{1}{S_p}\sum_{q=1}^{m} S_{pq}\rho_q, \tag{3.3}$$

in which ρ_q is the population density of the qth zone (*jiedao*), which can be known from the census datum. Thus

$$S_p = \sum_{q=1}^{m} S_{pq} = \pi\left(r_{p+1}^2 - r_p^2\right), \tag{3.4}$$

where r_p refers to the radius of the pth ring. It is evident that $\overline{\rho}_p$ denotes the average density of the pth circular belt. The weighted arithmetic average can lessen the influence of zone's scope on the estimated results of population density as much as possible.

(3) Spatial scale. The radius difference between rings is 0.6 km, less than r_0, that is, the average distance of urban population activity. The parameter values of r_0 can be estimated with Clark's model, namely, (2.2).

3.2. Data processing method and results

The population density of Hangzhou city will be analyzed from three angles of view: spectral analysis, correlogram analysis, and information entropy analysis. Accordingly, we will compute wave-spectral density, autocorrelation function, and information entropy. The procedure of wave spectrum analysis based on fast Fourier transform (FFT) can be summed up as five steps.

Table 1: Wave number and spectral density of Hangzhou urban population density: 1964–2000.

Wave number (k)	Spectral density $W(k)$			
	1964	1982	1990	2000
0	470847029	687015768	969469086	1494703196
0.03125	266735175	376124165	518629122	675108949
0.0625	163038091	231101694	283630373	294766651
0.09375	80198737	109872611	124617149	102574154
0.125	34091648	46787205	52487145	31878318
0.15625	11713224	17423260	24322424	24756703
0.1875	10527175	14968825	21343643	29845517
0.21875	12592259	18232629	22844713	24457983
0.25	11068578	17010714	15856897	11715162
0.28125	10257519	15964575	12434438	6237997
0.3125	10488070	16550655	14519617	8907607
0.34375	9395933	14802082	14191199	10727756
0.375	7431240	12355945	11425873	10425179
0.40625	6006155	9418308	8389117	9420298
0.4375	4840628	7937045	6060730	6493748
0.46875	5667641	8597673	6329055	4789005
0.5	6772368	9776351	7245194	4698112

Step 1 (sample path extension). The symmetrical rule of the FFT's recursive algorithm requires the length of time series to be an integer power of 2, that is, $L = 2^z$ $(z = 1, 2, 3, \ldots)$. However, there are 26 data points in our spatial sample path ($n = 26 \approx 2^{4.7}$). A process called "zero-padding" can be used to bring the number up to the next power of 2. In this case, the best way is to add 6 zeros at the end of the data series to bring the number to 32 (i.e., $L = 2^5$).

Step 2 (FFT of spatial series). Performing the FFT on the extended population density data of Hangzhou city yields a complex data series $F(k)$. The processing method is so accessible that MS Excel can give the results conveniently.

Step 3 (spectrum density calculation). The formula is such as $W(k) = |F(k)|^2/L = |F(k)|^2/32$. It is not difficult for us to compute the spectral density based on the FFT results (see Table 1). The spectral density is just the product of FFT result and its conjugate divided by the extended sample path length ($L = 32$).

Step 4 (making wave-spectra plots). As soon as the population density is transformed into spectral density, a plot reflecting the relation between wave number and spectral density can be given easily (see Figure 3). Let the circular belts be numbered as $p = 0, 1, 2, \ldots, L-1$, where $L = 2^5 = 32$. Thus the wave number will be defined by $k = p/L$.

Step 5 (modeling the wave-spectra relations). If the wave-spectra plots displayed in Figure 3 show some attenuation trend, we can fit the data of Table 1 to (2.24). A least square computation will give the spectral exponent β, from which we can estimate the fractal dimension D and Hurst exponent H by means of (2.25).

All the population density data of Hangzhou city in four years satisfy the power law in the form stipulated by (2.24) to a great extent. A least squares calculation utilizing the data

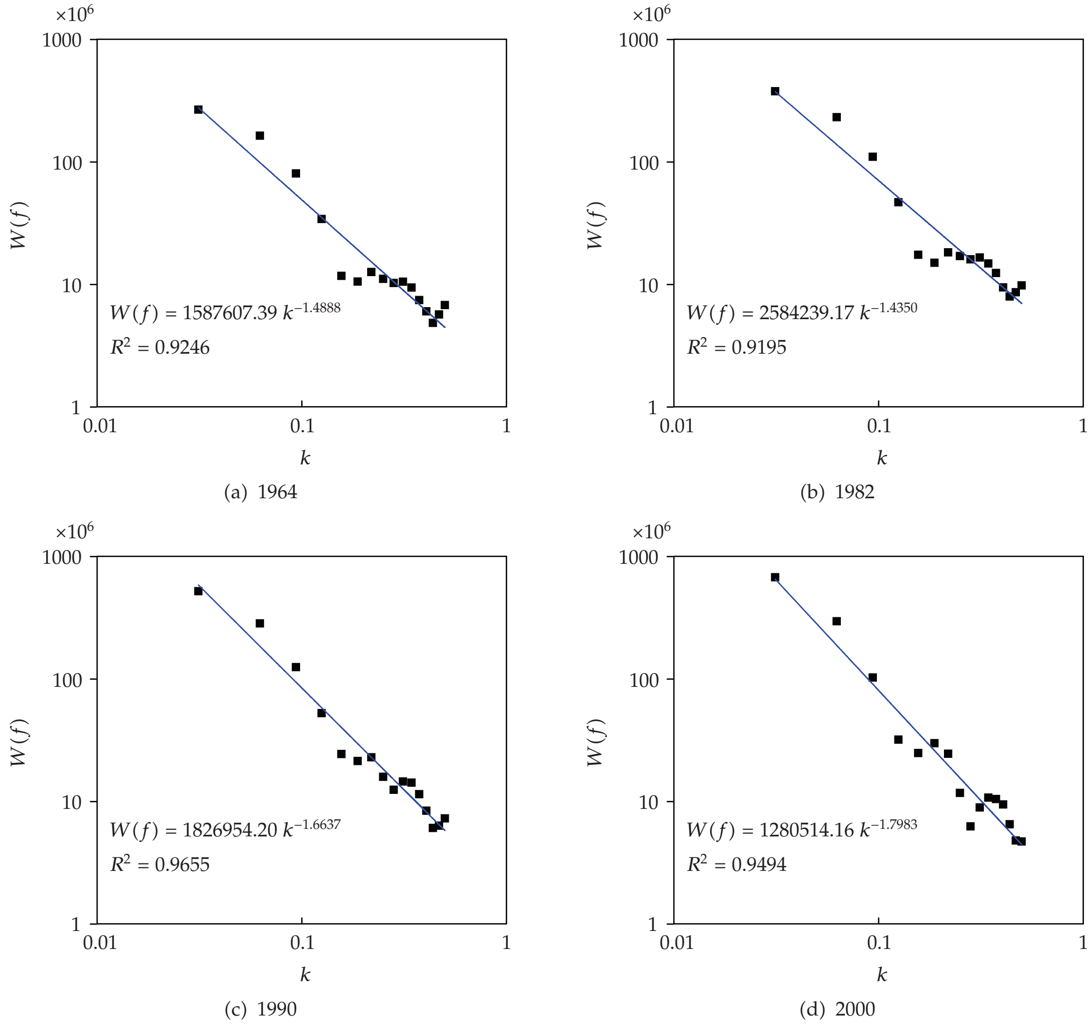


Figure 3: Wave-spectra plots of Hangzhou urban population density distribution: 1964–2000.

of Table 1 gives four spectral exponent β values. The fractal dimension D, Hurst exponent H, and the autocorrelation coefficient C_Δ can be evaluated consequently (see Table 2). From 1964 to 2000, the spectral exponent values become closer and closer to $\beta = 2$, the fractal dimension values become closer and closer to $D = 3/2$, and the Hurst exponent values become closer and closer to $H = 1/2$. All of these suggest a phenomenon of localization of urban population evolution: a population cell is inclined towards acting directly on the immediate cells, and not on the alternate cells, that is, nonimmediate cells. What is more, the wave-spectrum relations and spectral exponent values indicating $1/f$ noise [23] remind us of the self-organized criticality (SOC) of urban evolution [31–33].

It should be made clear that the fractal dimension used here is different from those employed to characterize two-dimension urban form such as *box dimension* and *radial dimension* ([34], White and Engelen [35]). Generally speaking, we need three kinds of fractal dimensions at least to characterize the city form with fractal structure. The first is the *box*

Table 2: Estimated values of model parameters and related statistics of Hangzhou urban density. The characteristic radius (r_0) values are estimated by means of least squares computation based on (2.2), and using r_0 values, we can compute the maximum entropy with the formula $H_{\max} = e \ln r_0$. The unit of entropy is "nat."

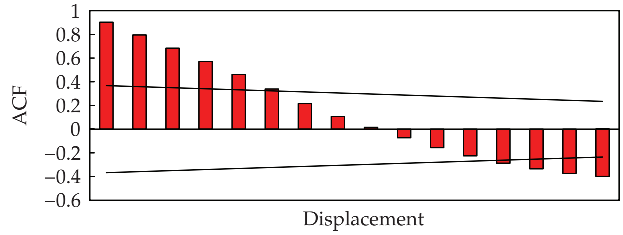
Year	Characteristic radius (r_0)	Spectral exponent (β)	Goodness of fit (R^2)	Fractal dimension (D)	Hurst exponent (H)	Autocorrelation coefficient (C_Δ)	Maximum entropy ($H_{\max}$)
1964	3.564	1.489	0.925	1.756	0.245	−0.298	3.455
1982	3.671	1.435	0.920	1.783	0.218	−0.324	3.535
1990	3.628	1.664	0.966	1.668	0.332	−0.208	3.503
2000	3.946	1.798	0.949	1.601	0.399	−0.130	3.731

Table 3: Autocorrelation function (ACF) and partial autocorrelation function (PACF) values of Hangzhou's population density: 1964–2000. As the sample path is not too long, only the first five values are really significant ($26^{1/2} \approx 5$). In time or spatial series analysis, we can judge the nature of series by standard error or by Box-Ljung statistic including Q-statistic and corresponding significance. Generally, it is easier and more visual to use the two standard-error bands shown in the histograms.

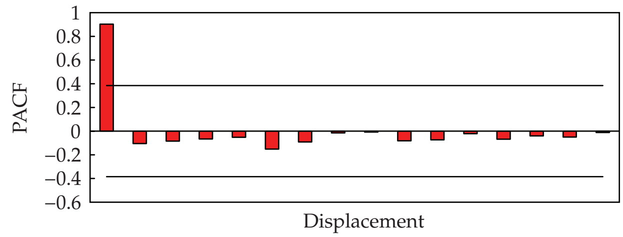
Distance (r)	Lag (l)	1964		1982		1990		2000	
		ACF	PACF	ACF	PACF	ACF	PACF	ACF	PACF
0.9	1	0.882	0.882	0.878	0.878	0.892	0.892	0.903	0.903
1.5	2	0.757	−0.093	0.753	−0.075	0.773	−0.110	0.796	−0.105
2.1	3	0.626	−0.099	0.622	−0.099	0.656	−0.058	0.683	−0.084
2.7	4	0.496	−0.073	0.486	−0.099	0.532	−0.107	0.571	−0.066
3.3	5	0.365	−0.090	0.359	−0.057	0.410	−0.065	0.462	−0.052
3.9	6	0.253	−0.014	0.246	−0.028	0.292	−0.072	0.339	−0.152
4.5	7	0.142	−0.084	0.142	−0.050	0.177	−0.071	0.216	−0.091

dimension D_b, which can be estimated by the box-counting method [36]; the second is *radial dimension* D_f, which is defined by the area-radius scaling [3, 5]; and the third is the dimension of fractal lines [26, 27], the author of this paper calls it *profile dimension* D_s when it is applied to urban morphology. The third type of dimension can be estimated easily through the wave-spectrum relation (see Appendix C).

Spectral analysis and correlation analysis represent different sides of the same coin in theory, while empirically correlation analysis and spectral analysis supplement each other. Therefore, a correlogram analysis of Hangzhou urban density should be made to consolidate the results of wave-spectra analysis. A spatial autocorrelation function can be based on the relationship between $\rho(r)$ and $\rho(r + l)$, where l refers to displacement analogous to time lag in time series analysis. Part of the autocorrelation function (ACF) and partial autocorrelation function (PACF) values is listed in Table 3, and the results in 2000 are shown in Figure 4. The ACF attenuates gradually when the displacement becomes long and displays some damped oscillation. What interests us is the PACF, which cuts off at a displacement of 1. That is, partial autocorrelation coefficients (PACCs) are not significantly different from 0 when the displacement $l > 1$ (see Table 3 and Figure 4(b)). The cutoff of PACF at a displacement of 1 suggests a possible locality in spatial activities of urban population: a population cell acts directly on and only on the proximate population cells, not on alternate cells. Evidently, the correlogram analysis lends further support to the conclusion drawn from the wave-spectrum analysis.



(a) Spatial ACF histogram



(b) Spatial PACF histogram

Figure 4: Histograms of spatial ACF and PACF of Hangzhou's population density (2000). The two lines in the histograms are called "two standard-error bands," according to which we can know whether or not there is significant difference between ACF or PACF values and zero.

There exists a mathematical relation between fractal dimension and information entropy. In a sense, Hausdorff dimension can be proved to be equivalent to Shannon's information entropy [37]. It is hard to evaluate the population distribution entropy by using (2.4), we can only estimate the maximum entropy by using the formula $H_{\max} = e \ln r_0$, which is based on one-dimensional *continuous* measure (see Table 2). However, it is easy to calculate the one-dimensional *discrete* information entropy of population profile along the radial (see Figure 1). Defining a probability such as

$$P_i = \frac{\rho_i}{\sum_{i=0}^{n} \rho_i}, \tag{3.5}$$

where variables ρ and n fulfill the same roles as in (2.3) or (2.4), then we have an information entropy

$$H_e = -\sum_{i=0}^{n} P_i \ln P_i, \tag{3.6}$$

in which H_e refers to the Shannon's entropy. The results of spatial entropy for Hangzhou's population distribution in four years are as follows: $H_e = 2.459$ nat in 1964, $H_e = 2.484$ nat in 1982, $H_e = 2.549$ nat in 1990, and $H_e = 2.725$ nat in 2000. The maximum information entropy based on discrete measure is $H_{\mathrm{m}} = \ln(26) = 3.258$ nat. The redundancy Z measuring the ratio of actual entropy to the maximum entropy and subtracting this ratio from 1 can be applied to spatial entropy statistics [38]. Using the formula $Z = 1 - H/H_{\mathrm{m}}$, the redundancy is computed as follows: $Z_{1964} = 0.245$, $Z_{1982} = 0.237$, $Z_{1990} = 0.218$, $Z_{2000} = 0.164$. The information entropy values become larger and larger, and the redundancy values approach the minimum value 0.

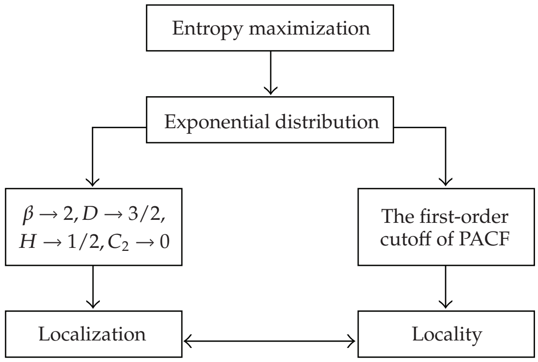


Figure 5: Entropy maximization suggesting localization of urban population distribution.

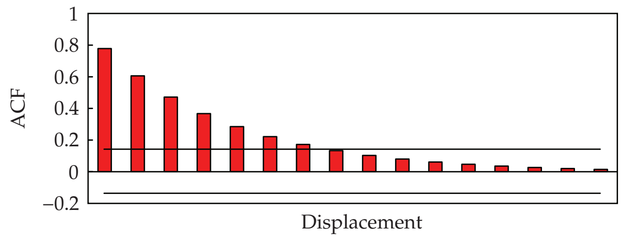
This trend gives further weight to the viewpoint that the dynamics of urban population evolvement in Hangzhou is actually a process of entropy maximization.

As stated above, information entropy maximization implies negative exponential distribution of urban population density, and the exponential distribution denotes spectral exponent $\beta = 2$ and thus fractal dimension $D = 1.5$. On the other hand, the PACF based on the exponential function shows a cutoff at a displacement of 1. All of these suggest a localization tendency of urban population distribution in Hangzhou. The reasoning process from entropy maximization to localization of spatial distributions of urban population is illustrated as Figure 5. In physics, localization is a phenomenon according to which the stationary quantum states of electrons in an extended system are localized due to disorder [39]. As for cities, localization can be defined as follows: a system of nonlocality changes gradually to that of locality.

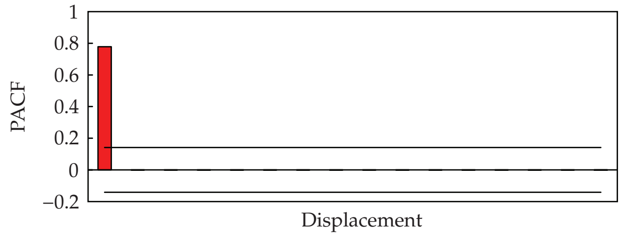
4. Questions and discussion

As indicated above, urban population density can be modeled by different functions under different conditions. The diversity or variability of urban models suggests asymmetry or symmetry breaking of geographical systems, which thus suggests spatial complexity of city systems and complication of urban evolution. Besides the negative exponential function, the inverse power function is also very important in modeling urban form. The relations between the exponential function and the power function were expounded by Batty and Kim [40]. Two questions will be discussed and answered here. The first is the difference between the negative exponential distribution and the inverse power-law distribution where the spatial dynamics is concerned, and the second is the locality and localization of urban population evolution.

The negative exponential function known as Clark's model and the inverse power function known as Smeed's model are two types of special spatial correlation functions. Exponential correlation function implies simple structure, while power-law correlation function suggests complex dynamics. If and only if a system falls into the self-organized critical state, the spatial correlation will follow a power law. Otherwise it follows an exponential law [22, 23]. In urban studies, the exponential function is always used to characterize urban population density, while the power function can be used to model urban land use density. Research into the relation between exponential law and power law is instructive for us to explore deeply the spatial dynamics of urban morphology.



(a) ACF histogram



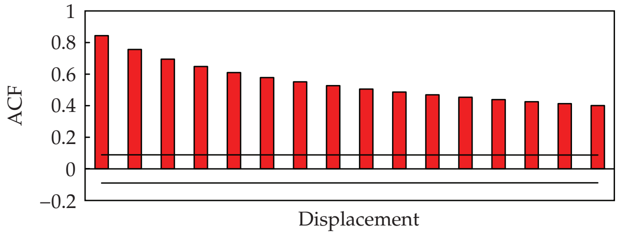
(b) PACF histogram

Figure 6: Histograms of spatial ACF and PACF based on the exponential distribution.

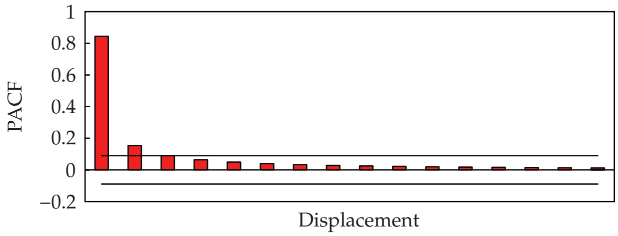
The negative exponential distribution indicates locality in theory, while the power law implies action at a distance. This viewpoint can be verified by the correlogram analysis based on simple simulation computation. Both the ACF and PACF can provide a summary of a time or spatial series' dynamics [41]. The ACF based on the standard exponential distribution displays a gradual one-sided damping (see Figure 6(a)), while the PACF of the exponential distribution cuts off at a displacement of 1—the partial autocorrelations drop abruptly to 0 beyond displacement 1 (see Figure 6(b)). The PACF seems to suggest a property of locality associated with the exponential distribution of urban density. As for Hangzhou city, the PACF is consistent with the result based on the standard exponential distribution, but the ACF differs in the damping way just because that the exponential distribution in the real world is not often very standard. In other words, the urban population dynamics of Hangzhou from 1964 to 2000 is only gradual localization without proper locality.

It is revealing to compare the correlogram of the exponential distribution with that of the power-law distribution. The ACF and PACF based on the standard power-law function differ from those based on the standard exponential function in an important way. The ACF of the power-law distribution displays a slow one-sided damping (see Figure 7(a)), while the corresponding PACF displays rapid one-sided damping without cutoff (see Figure 7(b)). In short, both the ACF and PACF of the power-law distribution are trailing, and this phenomenon reminds us of the action at a distance of spatial activities.

The differences of ACF and PACF between the exponential distribution and the power-law distribution are obvious and interesting. The ACF of the power-law distribution decays more slowly than that of the exponential distribution. In particular, the PACF of the power-law distribution is trailing, while the PACF of the exponential distribution cuts it off at the displacement of 1. The former suggests an action at a distance, while the latter reminds us of locality of spatial interaction (see Figure 8). The similarities and differences between the correlograms of the exponential distribution and that of the power-law distribution are

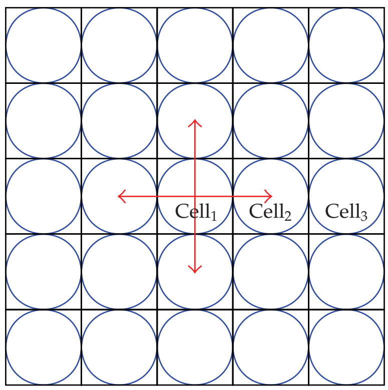


(a) ACF histogram

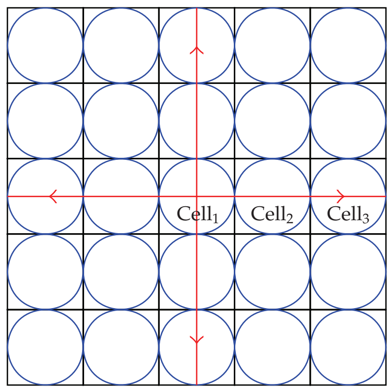


(b) PACF histogram

Figure 7: Histograms of spatial ACF and PACF based on the power-law distribution.



(a) Locality



(b) Action at a distance

Figure 8: Sketch maps of locality and action at a distance of urban dynamics. In (a) indicative of locality, Cell_1 only acts on Cell_2, not on Cell_3, while in (b) indicating action at a distance, Cell_1 not only acts on Cell_2, but also on Cell_3, Cell_4, and so on.

tabulated as follows (see Table 4). The correlograms of population density distributions of Hangzhou are more similar to those of the exponential distribution than those of the power-law distribution.

As indicated above, the exponential distribution has a characteristic length, r_0, which indicates simple geometrical patterns, while the power-law distribution has no characteristic length, which indicates complex patterns associated with fractal form and structure. Revealing the relationship between locality and action at a distance of urban evolution is very important for modeling spatial complexity by using cellular automata (CA). The original CA model possesses locality. In urban simulation, the CA's locality is gradually replaced by action at a distance [35, 42, 43]. For urban-land dynamics, the CA model with

Table 4: Autocorrelation function (ACF) and partial autocorrelation function (PACF) values.

Distribution	Function	Correlogram	Suggestion
Exponential distribution	ACF	Tailing: gradual one-sided damping	
	PACF	Cutoff at a displacement of 1	Locality
Power-law distribution	ACF	Tailing: slow one-sided damping	
	PACF	Tailing: rapid one-sided damping	Action at a distance
Urban density of Hangzhou city	ACF	Damped oscillation	
	PACF	Cutoff at a displacement of 1	Locality

action at a distance is suitable, but for urban population dynamics, the things may be more complicated because that urban population models are not one and only.

The scaling wave-spectrum relation and fractal properties of urban density suggest a dual character of urban evolution. On the one hand, the growth of cities look like particle motion, which can be simulated by means of CA technique, including diffusion-limited aggregation (DLA) and dielectric breakdown model (DBM), and so forth [3]. On the other hand, the statistical average of urban population distribution reminds us of the wave motion, or a ripple spreading from the center to the periphery. A city seems to be a set of dynamic particles indicating chaos or disorder distributed on the ripple indicative of order. In fact, intuitively, the spatial complexity displayed by city seems to express a struggle between order and chaos. An urban model of ripple-particle duality should be proposed to address temporal-spatial evolution of cities. As space is limited, the related questions will be made clear in the future work.

5. Conclusions

The study of this paper may be of revelation for modeling spatial complexity and simulating the urban growth and form. Geographers used to rely heavily on the rules associated with *action at a distance*, but neglect the rules based on the *locality* of urban population activity. However, urban spatial dynamics seems to be the unity of opposites of *locality* and *action at a distance*. The keys of comprehending this paper rest with three aspects. (1) Density is a zero-dimension measure, but urban density function is defined in one-dimension space, from which we can learn the information of two-dimension space. (2) Urban density models are in essence spatial correlation function, which can be converted into *energy spectrum* by Fourier transform and vice versa. Energy spectral density divided by sample path length is the wave spectral density. (3) If the relation between wave-spectrum density and wave number shows scaling invariance, fractal dimension can be estimated indirectly through the spectral exponent. The main points of the paper can be summarized as follows.

Firstly, one of the important physical mechanisms of urban growth and population diffusion is information entropy maximization indicating spatial optimization. From the viewpoint of statistical average, urban population density distributions of monocentric cities always satisfy the negative exponential function, which can be derived by using entropy-maximizing methods. Entropy maximization actually implies minimum cost when benefit is certain, or maximum benefit when cost is determinate. In other words, entropy maximizing in human systems suggests a process of optimization. Urban population density tends to evolve into an optimum distribution through self-organization.

Secondly, the negative exponential distribution implies locality or localization of urban population activities. Entropy maximization interprets the negative exponential distribution, and the scaling wave-spectrum relation coming from the negative exponential function predicts a locality of urban population activities in theory. In terms of the empirical evidences, the wave spectral analysis shows a localization process of urban population evolution, while the spatial autocorrelation analysis associated with wave spectral analysis demonstrates a locality of spatial interaction of population cells.

Thirdly, urban evolution seems to possess a dual nature of locality and action at a distance. The concept of locality should be as important as the idea of action at a distance for urban modeling and simulation. Locality is to urban population what action at a distance is to urban land use. The former relates to the negative exponential distribution, while the latter to the inverse power-law distribution. The power law indicates fractal structure, and the exponential law can be connected with fractals by Fourier transform. A conjecture is that if a city evolves into a self-organized critical state, the negative exponential distribution may change to the inverse power distribution.

Appendices

A. How to derive (2.2) from (2.1)

Substituting polar coordinates for Cartesian coordinates, we can also derive the negative exponential function from the diffusion model. Let us consider a Laplacian equation such as

$$\nabla^2\rho = \frac{\partial^2\rho}{\partial x^2} + \frac{\partial^2\rho}{\partial y^2} - \frac{a}{K}\rho = 0, \tag{A.1}$$

where ∇^2 is the Laplacian operator, other notations fulfill the same roles as in (2.1). For the anisotropic diffusion in two-dimension space, the relation between Cartesian coordinates and polar coordinate is $x = r\cos\theta$ and $y = r\sin\theta$. Thus (A.1) can be converted into

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\rho}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\rho}{\partial\theta^2} - \frac{a}{K}\rho = 0, \tag{A.2}$$

in which $r = (x^2 + y^2)^{1/2}$ refers to polar radius and θ to polar angle. However, if we examine the isotropic diffusion in one-dimension space, we will have $\theta = 0$, then $x = r\cos(0) = r$, $y = r\sin(0) = 0$, thus (A.1) in which y is of inexistence can be changed to

$$\nabla^2\rho = \frac{\partial^2\rho}{\partial r^2} - \frac{a}{K}\rho = 0. \tag{A.3}$$

The initial condition is $\rho_{|r=0} = \rho_0$, while the boundary condition is $\rho_{|r\to\infty} = 0$. A special solution to (A.3) is just Clark's model, namely (2.2) in the text.

B. How to derive (2.12) **from** (2.11)

We can derive (2.12) from (2.11) as follows. According to the l'Hospital's rule, when $r = R \to \infty$, we have

$$\lim_{r\to\infty} re^{-2\pi\lambda_2 r} = \lim_{r\to\infty} \frac{1}{2\pi\lambda_2 e^{2\pi\lambda_2 r}} = 0. \tag{B.1}$$

Using integration by decomposition yields

$$\begin{aligned}\int_0^R re^{-2\pi\lambda_2 r}\mathrm{d}r &= -\frac{1}{2\pi\lambda_2}\int_0^R r\,\mathrm{d}e^{-2\pi\lambda_2 r}\\ &= -\frac{1}{2\pi\lambda_2}\left\{\left[re^{-2\pi\lambda_2 r}\right]_0^R - \int_0^R e^{-2\pi\lambda_2 r}\mathrm{d}r\right\}\\ &= -\frac{1}{(2\pi\lambda_2)^2}\left[e^{-2\pi\lambda_2 r}\right]_0^R\\ &= \left(\frac{1}{2\pi\lambda_2}\right)^2.\end{aligned} \tag{B.2}$$

Please note that the $R \to \infty$ in (B.2). Therefore, we get

$$2\pi e^{-\lambda_1}\int_0^R re^{-2\pi\lambda_2 r}\mathrm{d}r = \frac{e^{-\lambda_1}}{2\pi\lambda_2^2} = P_t, \tag{B.3}$$

which is just equivalent to (2.12) in the text.

C. Box dimension, radial dimension, and profile dimension

Suppose that a three-axis coordinate system is constructed by x (latitude), y (longitude), and z (altitude). We use the three-axis coordinate to describe the Euclidean space in which a city exists. Then the box dimension D_b and the radial dimension D_f are defined in the space described by axes x and y, while the profile dimension D_s is defined in the space described by axes x and z, or by axes y and z. This paper is mainly involved with the profile dimension D_s, which is derived from the fractional Brownian motion (fBm) and dimensional analysis. In the course of urban development, the values of box dimension and radial dimension always increase over time. However, the profile dimension values of urban density decreases with the lapse of time, approaching to 1.5.

Actually, radial dimension D_f can reflect the information of the three-dimension space in the sense of average. The author has derived a relation between the radial dimension and profile dimension of urban morphology by using Fourier transform. The result is $D_f + D_s = 7/2$, where D_f refers to the *radial dimension*, and D_s to the *profile dimension*. According to the fractal dimension equation, the radial dimension of Hangzhou's population, D_f, can be estimated as follows: $D_f = 1.744$ in 1964, $D_f = 1.717$ in 1982, $D_f = 1.832$ in 1990, and $D_f = 1.899$ in 2000. This kind of fractal dimension value increases with the passage of time and approaches to $d = 2$. The related problems will be discussed in detail in the companion paper "Exploring fractal parameters of urban growth and form with wave-spectrum analysis" (forthcoming).

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