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Research Article

Existence of Positive Solutions for *m*-Point Boundary Value Problems on Time Scales

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We study the one-dimensional p-Laplacian m-point boundary value problem $(\varphi_p(u^{\Delta}(t)))^{\Delta} + a(t)f(t,u(t)) = 0$, $t \in [0,1]_T$, $u(0) = 0$, $u(1) = \sum_{i=1}^{m-2} a_i u(\xi_i)$, where T is a time scale, $\varphi_p(s) = |s|^{p-2}s$, $p > 1$, some new results are obtained for the existence of at least one, two, and three positive solution/solutions of the above problem by using Krasnosel′skll′s fixed point theorem, new fixed point theorem due to Avery and Henderson, as well as Leggett-Williams fixed point theorem. This is probably the first time the existence of positive solutions of one-dimensional p-Laplacian m-point boundary value problem on time scales has been studied.

1. Introduction

With the development of p-Laplacian dynamic equations and theory of time scales, a few authors focused their interest on the study of boundary value problems for p-Laplacian dynamic equations on time scales. The readers are referred to the paper [1–7].

In 2005, He [1] considered the following boundary value problems:

$$\begin{gathered}
(\varphi_p(u^{\Delta}(t)))^{\nabla} + a(t)f(u(t)) = 0, \quad t \in [0,T]_T, \\
u(0) - B_0 u^{\Delta}(\eta) = 0, \qquad u^{\Delta}(T) = 0 \text{ or} \\
u^{\Delta}(0) = 0, \qquad u(T) + B_1 u^{\Delta}(\eta) = 0,
\end{gathered} \tag{1.1}$$

where T is a time scales, $\varphi_p(s) = |s|^{p-2}s$, $p > 1$, $\eta \in (0, \rho(t))_T$. The author showed the existence of at least two positive solutions by way of a new double fixed point theorem.

In 2004, Anderson et al. [2] used the virtue of the fixed point theorem of cone and obtained the existence of at least one solution of the boundary value problem:

$$\begin{gathered}\left(g\left(u^{\Delta}(t)\right)\right)^{\nabla}+c(t)f(u)=0,\quad a<t<b,\\ u(a)-B_0u^{\Delta}(\gamma)=0,\qquad u^{\Delta}(b)=0.\end{gathered}\tag{1.2}$$

In 2007, Geng and Zhu [3] used the Avery-Peterson and another fixed theorem of cone and obtained the existence of three positive solutions of the boundary value problem:

$$\begin{gathered}\left(\varphi_p\left(u^{\Delta}(t)\right)\right)^{\nabla}+a(t)f(u(t))=0,\quad t\in[0,T]_T,\\ u(0)-B_0u^{\Delta}(\eta)=0,\qquad u^{\Delta}(T)=0.\end{gathered}\tag{1.3}$$

Also, in 2007, Sun and Li [4] discussed the existence of at least one, two or three positive solutions of the following boundary value problem:

$$\begin{gathered}\left(\varphi_p\left(u^{\Delta}(t)\right)\right)^{\Delta}+h(t)f(u^{\sigma}(t))=0,\quad t\in[a,b]_T,\\ u(a)-B_0u^{\Delta}(a)=0,\qquad u^{\Delta}(\sigma(b))=0.\end{gathered}\tag{1.4}$$

In this paper, we are concerned with the existence of multiple positive solutions to the m-point boundary value problem for the one dimension p-Laplcaian dynamic equation on time scale T

$$\begin{gathered}\left(\varphi_p\left(u^{\Delta}(t)\right)\right)^{\Delta}+a(t)f(t,u(t))=0,\quad t\in[0,1]_T,\\ u(0)=0,\qquad u(1)=\sum_{i=1}^{m-2}a_iu(\xi_i),\end{gathered}\tag{1.5}$$

where T is a time scale, $\varphi_p(s)=|s|^{p-2}s$, $p>1, 0<\xi_1<\xi_2<\cdots<\xi_{m-2}<1$, $0\le a_i$, $i=1,2,\ldots,m-3$, $a_{m-2}>0$, and

(H_1) $\sum_{i=1}^{m-2}a_i\xi_i<1$;

(H_2) $f\in C_{\mathrm{rd}}([0,1]_T\times[0,\infty),[0,\infty))$;

(H_3) $a\in C_{\mathrm{rd}}([0,1]_T,[0,\infty))$ and there exists $t_0\in(\xi_{m-2},1)$ such that $a(t_0)>0$.

In this paper, we have organized the paper as follows. In Section 2, we give some lemmas which are needed later. In Section 3, we apply the Krassnoselskiifs [8] fixed point theorem to prove the existence of at least one positive solution to the MBVP(1.5). In Section 4, conditions for the existence of at least two positive solutions to the MBVP (1.5) are discussed by using Avery and Henderson [9] fixed point theorem. In Section 5, to prove the existence of at least three positive solutions to the MBVP (1.5) are discussed by using Leggett and Williams [10] fixed point theorem.

For completeness, we introduce the following concepts and properties on time scales.

A time scale T is a nonempty closed subset of R, assume that T has the topology that it inherits from the standards topology on R.

Definition 1.1. Let T be a time scale, for $t \in T$, one defines the forward jump operator $\sigma : T \to T$ by $\sigma(t) = \inf\{s \in T : s > t\}$, and the backward jump operator $\rho : T \to T$ by $\rho(t) = \sup\{s \in T : s < t\}$, while the graininess function $\mu : T \to [0,\infty)$ is defined by $\mu(t) = \sigma(t) - t$. If $\sigma(t) > t$, one says that is right-scattered, while if $\rho(t) < t$, one says that tis left-scattered. Also, if $t < \sup T$ and $\sigma(t) = t$, then t is called right-dense, and if $t > \inf T$ and $\rho(t) = t$, then t is called left-dense. One also needs below the set T^k as follows: if T has a left-scattered maximum m, then $T^k = T - m$, otherwise $T^k = T$. For instance, if $\sup T = \infty$, then $T^k = T$.

Definition 1.2. Assume $f : T \to R$ is a function and let $t \in T$. Then , one defines $f^{\Delta}(t)$ to be the number (provided it exists) with the property that any given $\varepsilon > 0$, there is a neighborhood U of t such that

$$\left|\left[f(\sigma(t)) - f(s)\right] - f^{\Delta}(t)[\sigma(t) - s]\right| \le \varepsilon\left|\sigma(t) - s\right|, \tag{1.6}$$

for all $s \in U$. One says that f is delta differentiable (or in short: differentiable) on T provided $f^{\Delta}(t)$ exist for all $t \in T$.

If $T = R$, then $f^{\Delta}(t) = f'(t)$, if $T = \mathbb{Z}$, then $f^{\Delta}(t) = \Delta f(t)$.
A function $f : T \to R$.

(i) If f is continuous , then f is rd-continuous.

(ii) The jump operator σ is rd-continuous.

(iii) If f is rd-continuous, then so is f^{σ}.

A function $F : T \to R$ is called an antidervative of $f : T \to \mathbf{R}$, provided $F^{\Delta}(t) = f(t)$ holds for all $t \in T^k$. One defines the definite integral by

$$\int_a^b f(t)\Delta t = F(b) - F(a). \tag{1.7}$$

For all $a, b \in T$. If $f^{\Delta}(t) \ge 0$, then f is nondecreasing.

2. The Preliminary Lemmas

Lemma 2.1 (see [5, 6]). *Assume that (H_1)–(H_3) hold. Then $u(t)$ is a solution of the MBVP* (1.5) *on* $[0,1]_T$ *if and only if*

$$\begin{aligned} u(t) = &-\int_0^t \varphi_q\left(\int_0^s a(\tau) f(\tau, u(\tau))\Delta\tau\right)\Delta s \\ &- t \cdot \frac{\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} \varphi_q\left(\int_0^s a(\tau) f(\tau, u(\tau))\Delta\tau\right)\Delta s}{1 - \sum_{i=1}^{m-2} a_i \xi_i} \\ &+ t \cdot \frac{\int_0^1 \varphi_q\left(\int_0^s a(\tau) f(\tau, u(\tau))\Delta\tau\right)\Delta s}{1 - \sum_{i=1}^{m-2} a_i \xi_i}, \end{aligned} \tag{2.1}$$

where $\varphi_q(s) = |s|^{q-2}s$, $(1/p) + (1/q) = 1$, *and* $q > 1$.

Lemma 2.2. *Assume that conditions* (H_1)*–*(H_3) *are satisfied, then the solution of the MBVP* (1.5) *on* $[0,1]_T$ *satisfies*

$$u(t) \geq 0, \quad t \in [0,1]_T. \tag{2.2}$$

Lemma 2.3 (see [5]). *If the conditions* (H_1)*–*(H_3) *are satisfied, then*

$$u(t) \geq \gamma \|u\|, \quad t \in \left[\xi_{m-2}, 1\right], \tag{2.3}$$

where

$$\begin{gathered} \|u\| = \sup_{t\in[0,1]_T} \left|u(t)\right|, \\ \gamma = \min\left\{ \frac{a_{m-2}(1-\xi_{m-2})}{1-a_{m-2}\xi_{m-2}},\ a_{m-2}\xi_{m-2},\ \xi_1 \right\}. \end{gathered} \tag{2.4}$$

Lemma 2.4 (see [6]). $\min_{t\in[\xi_{m-2},1]} Au(t) = \min\{Au(1), Au(\xi_{m-2})\}$.

Let E denote the Banach space $C_{\mathrm{rd}}[0,1]_T$ with the norm $\|u\| = \sup_{t\in[0,1]_T}|u(t)|$. Define the cone $P \subset E$, by

$$P = \left\{u \in E \mid u(t) \geq 0,\ t \in \left[\xi_{m-2}, 1\right] \min u(t) \geq \gamma\|u\|,\ u \text{ is concave}\right\}. \tag{2.5}$$

The solutions of MBVP (1.5) are the points of the operator A defined by

$$\begin{aligned} Au(t) = &-\int_0^t \varphi_q\left(\int_0^s a(\tau) f(\tau, u(\tau))\Delta\tau\right)\Delta s \\ &- t\cdot \frac{\sum_{i=1}^{m-2} a_i \int_0^{\xi_i}\varphi_q\left(\int_0^s a(\tau) f(\tau,u(\tau))\Delta\tau\right)\Delta s}{1-\sum_{i=1}^{m-2} a_i\xi_i} \\ &+ t\cdot \frac{\int_0^1 \varphi_q\left(\int_0^s a(\tau) f(\tau,u(\tau))\Delta\tau\right)\Delta s}{1-\sum_{i=1}^{m-2} a_i\xi_i} = u(t). \end{aligned} \tag{2.6}$$

So, $AP \subset P$. It is easy to check that $A : P \to P$ is completely continuous.

3. Existence of at least One Positive Solutions

Theorem 3.1 (see [8]). *Let E be a Banach space, and let $P \subset E$ be a cone. Assume Ω_1 and Ω_2 are open boundary subsets of E with $0 \in \Omega_1$, $\overline{\Omega}_1 \subset \Omega_2$, and let $A : P \cap (\overline{\Omega}_2 \setminus \Omega_1) \to P$ be a completely continuous operator such that either*

(i) $\|Au\| \leq \|u\|$ *for* $u \in P \cap \partial\Omega_1$, $\|Au\| \geq \|u\|$ *for* $u \in P \cap \partial\Omega_2$; *or*

(ii) $\|Au\| \geq \|u\|$ *for* $u \in P \cap \partial\Omega_1$, $\|Au\| \leq \|u\|$ *for* $u \in P \cap \partial\Omega_2$ *hold.*

Then A has a fixed point in $P \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

Theorem 3.2. *Assume conditions* (H_1)–(H_3) *are satisfied. In addition, suppose there exist numbers* $0 < r < R < \infty$ *such that* $f(t,u) \le \varphi_p(m)\varphi_p(r)$, *if* $t \in [0,\sigma(1)]$, $0 \le u \le r$, *and* $f(t,u) \ge \varphi_p(M\gamma)\varphi_p(R)$, *if* $t \in [\xi_{m-2},1]$, $R \le u < \infty$, *where*

$$M = \frac{1-\sum_{i=1}^{m-2}a_i\xi_i}{\gamma\xi_{m-2}\int_{\xi_{m-2}}^{1}\varphi_q(\int_{\xi_{m-2}}^{s}a(\tau)\Delta\tau)\Delta s}, \qquad m = \frac{1-\sum_{i=1}^{m-2}a_i\xi_i}{\int_0^1\varphi_q(\int_0^s a(\tau)\Delta\tau)\Delta s}. \tag{3.1}$$

Then the MBVP (1.5) has at least one positive solution.

Proof. Define the cone P as in (2.5), define a completely continuous integral operator $A : P \to P$ by

$$\begin{aligned} Au(t) = &-\int_0^t \varphi_q\left(\int_0^s a(\tau)f(\tau,u(\tau))\Delta\tau\right)\Delta s \\ &- t\cdot\frac{\sum_{i=1}^{m-2}a_i\int_0^{\xi_i}\varphi_q(\int_0^s a(\tau)f(\tau,u(\tau))\Delta\tau)\Delta s}{1-\sum_{i=1}^{m-2}a_i\xi_i} \\ &+ t\cdot\frac{\int_0^1\varphi_q(\int_0^s a(\tau)f(\tau,u(\tau))\Delta\tau)\Delta s}{1-\sum_{i=1}^{m-2}a_i\xi_i}. \end{aligned} \tag{3.2}$$

From (H_1)–(H_3), Lemmas 2.1 and 2.2, $AP \subset P$. If $u \in P$ with $\|u\| = r$, then we get

$$\begin{aligned} Au(t) = &-\int_0^t \varphi_q\left(\int_0^s a(\tau)f(\tau,u(\tau))\Delta\tau\right)\Delta s \\ &- t\cdot\frac{\sum_{i=1}^{m-2}a_i\int_0^{\xi_i}\varphi_q(\int_0^s a(\tau)f(\tau,u(\tau))\Delta\tau)\Delta s}{1-\sum_{i=1}^{m-2}a_i\xi_i} \\ &+ t\cdot\frac{\int_0^1\varphi_q(\int_0^s a(\tau)f(\tau,u(\tau))\Delta\tau)\Delta s}{1-\sum_{i=1}^{m-2}a_i\xi_i} \\ \le{}& t\cdot\frac{\int_0^1\varphi_q(\int_0^s a(\tau)f(\tau,u(\tau))\Delta\tau)\Delta s}{1-\sum_{i=1}^{m-2}a_i\xi_i} \\ \le{}& \varphi_q(\varphi_p(m)\varphi_p(r))\cdot\frac{\int_0^1\varphi_q(\int_0^s a(\tau)\Delta\tau)\Delta s}{1-\sum_{i=1}^{m-2}a_i\xi_i} \\ \le{}& rm\cdot\frac{\int_0^1\varphi_q(\int_0^s a(\tau)\Delta\tau)\Delta s}{1-\sum_{i=1}^{m-2}a_i\xi_i} = r = \|u\|. \end{aligned} \tag{3.3}$$

This implies that $\|Au\| \le \|u\|$. So, if we set $\Omega_1 = \{u \in C_{\mathrm{rd}}([0,1]) \mid \|u\| < r\}$, then $\|Au\| \le \|u\|$, for $u \in P \cap \partial\Omega_1$.

Let us now set $\Omega_2 = \{u \in C_{\mathrm{rd}}([0,1]) \mid \|u\| < R\}$.

Then for $u \in P$ with $\|u\| < R$, by Lemma 2.4 we have $u(t) \geq \gamma\|u\|$, $t \in [\xi_{m-2}, 1]$. Therefore, we have

$$\begin{aligned}
\|Au(t)\| &\geq Au(\xi_{m-2}) \\
&= -\int_0^{\xi_{m-2}} \varphi_q\left(\int_0^s a(\tau) f(\tau, u(\tau))\Delta\tau\right)\Delta s \\
&\quad - \xi_{m-2}\frac{\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} \varphi_q(\int_0^s a(\tau) f(\tau, u(\tau))\Delta\tau)\Delta s}{1-\sum_{i=1}^{m-2} a_i\xi_i} \\
&\quad + \xi_{m-2}\cdot\frac{\int_0^1 \varphi_q(\int_0^s a(\tau) f(\tau, u(\tau))\Delta\tau)\Delta s}{1-\sum_{i=1}^{m-2} a_i\xi_i} \\
&= \frac{\xi_{m-2}\int_0^1 \varphi_q(\int_0^s a(\tau) f(\tau, u(\tau))\Delta\tau)\Delta s - \int_0^{\xi_{m-2}} \varphi_q(\int_0^s a(\tau) f(\tau, u(\tau))\Delta\tau)\Delta s}{1-\sum_{i=1}^{m-2} a_i\xi_i} \\
&\quad + \frac{1}{1-\sum_{i=1}^{m-2} a_i\xi_i}\sum_{i=1}^{m-2} a_i\left(\xi_i\int_0^{\xi_{m-2}} \varphi_q\left(\int_0^s a(\tau) f(\tau, u(\tau))\Delta\tau\right)\Delta s\right. \\
&\qquad\qquad \left. - \xi_{m-2}\int_0^{\xi_i} \varphi_q\left(\int_0^s a(\tau) f(\tau, u(\tau))\Delta\tau\right)\Delta s\right) \\
&\geq \frac{\xi_{m-2}\int_0^1 \varphi_q(\int_0^s a(\tau) f(\tau, u(\tau))\Delta\tau)\Delta s}{1-\sum_{i=1}^{m-2} a_i\xi_i} \\
&\quad - \frac{\int_0^{\xi_{m-2}} \varphi_q(\int_0^s a(\tau) f(\tau, u(\tau))\Delta\tau)\Delta s}{1-\sum_{i=1}^{m-2} a_i\xi_i} \\
&\geq \frac{\xi_{m-2}\int_{\xi_{m-2}}^1 \varphi_q(\int_{\xi_{m-2}}^s a(\tau) f(\tau, u(\tau))\Delta\tau)\Delta s}{1-\sum_{i=1}^{m-2} a_i\xi_i} \\
&\geq \varphi_q(\varphi_p(M\gamma)\varphi_p(R))\frac{\xi_{m-2}\int_{\xi_{m-2}}^1 \varphi_q(\int_{\xi_{m-2}}^s a(\tau)\Delta\tau)\Delta s}{1-\sum_{i=1}^{m-2} a_i\xi_i} \\
&\geq \frac{M\gamma R}{1-\sum_{i=1}^{m-2} a_i\xi_i}\cdot\xi_{m-2}\int_{\xi_{m-2}}^1 \varphi_q\left(\int_{\xi_{m-2}}^s a(\tau)\Delta\tau\right)\Delta s = \|u\|.
\end{aligned} \tag{3.4}$$

Hence, $\|Au\| \geq \|u\|$ for $u \in P \cap \partial\Omega_2$. Thus by the Theorem 3.1, A has a fixed point u in $P \cap (\overline{\Omega}_2 \setminus \Omega_1)$. Therefore, the MBVP (1.5) has at least one positive solution. □

4. Existence of at least Two Positive Solutions

In this section, we apply the Avery-Henderson fixed point theorem [9] to prove the existence of at least two positive solutions to the nonlinear MBVP (1.5) .

Theorem 4.1 (see Avery and Henderson [9]). *Let P be a cone in a real Banach space E. Set*

$$P(\Phi, \rho_3) = \{u \in P \mid \Phi(u) < \rho_3\}. \tag{4.1}$$

If ν and Φ are increasing, nonnegative continuous functionals on P, let θ be a nonnegative continuous functional on P with $\theta(0) = 0$ such that, for some positive constants ρ_3 and $M > 0$, $\Phi(u) \le \theta(u) \le \nu(u)$ and $\|u\| \le M\Phi(u)$, for all $u \in \overline{P(\Phi, \rho_3)}$. Suppose that there exist positive numbers $\rho_1 < \rho_2 < \rho_3$ such that $\theta(\lambda u) = \lambda\theta(u)$ for all $0 \le \lambda \le 1$ and $u \in \partial P(\theta, \rho_2)$.

If $A : \overline{P(\Phi, \rho_3)} \to P$ is a completely continuous operator satisfying

(i) $\Phi(Au) > \rho_3$ for all $u \in \partial P(\Phi, \rho_3)$;

(ii) $\theta(Au) < \rho_2$ for all $u \in \partial P(\theta, \rho_2)$;

(iii) $P(\nu, \rho_1) \ne \phi$ and $\nu(Au) > \rho_1$ for all $u \in \partial P(\nu, \rho_1)$, then A has at least two fixed points u_1 and u_2 such that $\rho_1 < \nu(u_1)$ with $\theta(u_1) < \rho_2$ and $\rho_3 < (u_2)$ with $\Phi(u_2) < \rho_3$.

Let $l \in (0,1)_{\mathbf{T}}$ and $0 < \xi_{m-2} < l < 1$. Define the increasing, nonnegative and continuous functionals Φ, θ, and ν on P, by $\Phi(u) = u(\xi_{m-2})$, $\theta(u) = u(\xi_{m-2})$, and $\nu(u) = u(l)$.

From Lemma 2.4, for each $u \in P$, $\Phi(u) = \theta(u) \le \nu(u)$.

In addition, for each $u \in P$, Lemma 2.3 implies $\Phi(u) = u(\xi_{m-2}) \ge \gamma\|u\|$.

Thus,

$$\|u\| < \frac{1}{\gamma}\Phi(u), \quad \forall u \in P. \tag{4.2}$$

We also see that $\theta(0) = 0$ and $\theta(\lambda u) = \lambda\theta(u)$ for all $0 \le \lambda \le 1$ and $u \in \partial P(\theta, q)$.

Theorem 4.2. *Assume (H_1)–(H_3) hold, suppose there exist positive numbers $\rho_1 < \rho_2 < \rho_3$, such that the function f satisfies the following conditions:*

(B_1) $f(t,u) > \varphi_p(m\gamma)\varphi_p(\rho_1)$, *for* $t \in [\xi_{m-2}, l]$ *and* $u \in [\gamma\rho_1, \rho_1]$;

(B_2) $f(t,u) < \varphi_p(m)\varphi_p(\rho_2)$, *for* $t \in [\xi_{m-2}, 1]$ *and* $u \in [0, \rho_2]$;

(B_3) $f(t,u) > \varphi_p(M\gamma)\varphi_p(\rho_3)$, *for* $t \in [\xi_{m-2}, l]$ *and* $u \in [\rho_3, (1/\gamma)\rho_3]$.

Then the MBVP (1.5) has at least two positive solutions u_1 and u_2 such that $u_1(t) > \rho_1$ with $u_1(l) < \rho_2$ and $u_2(l) > \rho_2$ with $u_2(l) < \rho_3$.

Proof. We now verify that all of the conditions of Theorem 4.1 are satisfied.

Define the cone P as (2.5), define a completely continuous integral operator $A : P \to P$ by

$$\begin{aligned} Au(t) = &-\int_0^t \varphi_q\left(\int_0^s a(\tau) f(\tau, u(\tau))\Delta\tau\right)\Delta s \\ &- t \cdot \frac{\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} \varphi_q(\int_0^s a(\tau) f(\tau, u(\tau))\Delta\tau)\Delta s}{1 - \sum_{i=1}^{m-2} a_i\xi_i} \\ &+ t \cdot \frac{\int_0^1 \varphi_q(\int_0^s a(\tau) f(\tau, u(\tau))\Delta\tau)\Delta s}{1 - \sum_{i=1}^{m-2} a_i\xi_i}. \end{aligned} \tag{4.3}$$

M and m as in (3.1). To verify that condition (i) of Theorem 4.1 holds, we choose $u \in \partial P(\Phi, \rho_3)$, then $\Phi(u) = \rho_3$. This implies $\rho_3 \le \|u\| \le (1/\gamma)\Phi(u)$. Note that $\|u\| \le (1/\gamma)\Phi(u) =$

$(1/\gamma)\rho_3$. We have $\rho_3 \leq u(t) \leq (1/\gamma)\rho_3$, for $t \in [\xi_{m-2}, 1]_T$. As a consequence of (B_3), $f(t,u) > \varphi_p(M\gamma)\varphi_p(\rho_3)$, for $t \in [\xi_{m-2}, l]_T$. Since $Au \in P$, we have from Lemma 2.2,

$$
\begin{aligned}
\Phi(Au) &= (Au)(\xi_{m-2}) \\
&= -\int_0^{\xi_{m-2}} \varphi_q\left(\int_0^s a(\tau) f(\tau, u(\tau))\Delta\tau\right)\Delta s \\
&\quad - \xi_{m-2}\cdot \frac{\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} \varphi_q(\int_0^s a(\tau) f(\tau,u(\tau))\Delta\tau)\Delta s}{1-\sum_{i=1}^{m-2} a_i\xi_i} \\
&\quad + \xi_{m-2}\cdot \frac{\int_0^1 \varphi_q(\int_0^s a(\tau) f(\tau,u(\tau))\Delta\tau)\Delta s}{1-\sum_{i=1}^{m-2} a_i\xi_i} \\
&= \frac{\xi_{m-2}\int_0^1 \varphi_q(\int_0^s a(\tau) f(\tau,u(\tau))\Delta\tau)\Delta s - \int_0^{\xi_{m-2}} \varphi_q(\int_0^s a(\tau) f(\tau,u(\tau))\Delta\tau)\Delta s}{1-\sum_{i=1}^{m-2} a_i\xi_i} \\
&\quad + \frac{1}{1-\sum\limits_{i=1}^{m-2} a_i\xi_i} \sum_{i=1}^{m-2} a_i \left(\xi_i \int_0^{\xi_{m-2}} \varphi_q\left(\int_0^s a(\tau) f(\tau,u(\tau))\Delta\tau\right)\Delta s\right. \\
&\qquad\qquad \left. - \xi_{m-2}\int_0^{\xi_i} \varphi_q\left(\int_0^s a(\tau) f(\tau,u(\tau))\Delta\tau\right)\Delta s\right) \\
&\geq \frac{\xi_{m-2}\int_0^1 \varphi_q(\int_0^s a(\tau) f(\tau,u(\tau))\Delta\tau)\Delta s - \int_0^{\xi_{m-2}} \varphi_q(\int_0^s a(\tau) f(\tau,u(\tau))\Delta\tau)\Delta s}{1-\sum_{i=1}^{m-2} a_i\xi_i} \\
&\geq \frac{\xi_{m-2}\int_{\xi_{m-2}}^1 \varphi_q(\int_0^s a(\tau) f(\tau,u(\tau))\Delta\tau)\Delta s}{1-\sum_{i=1}^{m-2} a_i\xi_i} \\
&\geq \varphi_q(\varphi_p(M\gamma)\varphi_p(\rho_3))\frac{\xi_{m-2}\int_{\xi_{m-2}}^1 \varphi_q(\int_0^s a(\tau)\Delta\tau)\Delta s}{1-\sum_{i=1}^{m-2} a_i\xi_i} \\
&\geq \frac{M\gamma\rho_3\xi_{m-2}}{1-\sum_{i=1}^{m-2} a_i\xi_i}\int_{\xi_{m-2}}^1 \varphi_q\left(\int_0^s a(\tau)\Delta\tau\right)\Delta s \geq \rho_3.
\end{aligned}
\tag{4.4}
$$

Then condition (i) of Theorem 4.1 holds.

Let $u \in \partial P(\theta, \rho_2)$. Then $\theta(u) = \rho_2$. This implies $0 \leq u(t) \leq \|u\| \leq (1/\gamma)\rho_2$, for $t \in [\xi_{m-2}, 1]$. From (B_2), we have

$$
\begin{aligned}
\theta(Au) &= (Au)(\xi_{m-2}) \\
&\leq \xi_{m-2}\cdot \frac{\int_0^1 \varphi_q(\int_0^s a(\tau) f(\tau,u(\tau))\Delta\tau)\Delta s}{1-\sum_{i=1}^{m-2} a_i\xi_i} \\
&\leq \xi_{m-2}\cdot \frac{\int_0^1 \varphi_q(\int_0^s a(\tau) f(\tau,u(\tau))\Delta\tau)\Delta s}{1-\sum_{i=1}^{m-2} a_i\xi_i} \\
&\leq m\rho_2\cdot \frac{\int_0^1 \varphi_q(\int_0^s a(\tau)\Delta\tau)\Delta s}{1-\sum_{i=1}^{m-2} a_i\xi_i} = \rho_2 = \|u\|.
\end{aligned}
\tag{4.5}
$$

Hence condition (ii) of Theorem 4.1 holds.

If we first define $u(t) = \rho_1/2$, for $t \in [0,1]_T$, then $\nu(u) = \rho_1/2 < \rho_1$. So $P(\nu, \rho_1) \neq \phi$.

Now, let $u \in \partial P(\nu, \rho_1)$, then $\nu(u) = u(l) = \rho_1$. This mean that $\rho_1/\gamma \le u(t) \le \|u\| \le \rho_1$. From (B_1) and Lemma 2.4, we get

$$\begin{aligned}
\nu(Au) &= (Au)(l) \ge (Au)(\xi_{m-2}) \\
&= -\int_0^{\xi_{m-2}} \varphi_q\left(\int_0^s a(\tau) f(\tau, u(\tau))\Delta\tau\right)\Delta s \\
&\quad - \xi_{m-2} \cdot \frac{\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} \varphi_q(\int_0^s a(\tau) f(\tau, u(\tau))\Delta\tau)\Delta s}{1 - \sum_{i=1}^{m-2} a_i \xi_i} \\
&\quad + \xi_{m-2} \cdot \frac{\int_0^1 \varphi_q(\int_0^s a(\tau) f(\tau, u(\tau))\Delta\tau)\Delta s}{1 - \sum_{i=1}^{m-2} a_i \xi_i} \\
&= \frac{\xi_{m-2}\int_0^1 \varphi_q(\int_0^s a(\tau) f(\tau, u(\tau))\Delta\tau)\Delta s - \int_0^{\xi_{m-2}} \varphi_q(\int_0^s a(\tau) f(\tau, u(\tau))\Delta\tau)\Delta s}{1 - \sum_{i=1}^{m-2} a_i \xi_i} \\
&\quad + \frac{1}{1 - \sum_{i=1}^{m-2} a_i \xi_i} \sum_{i=1}^{m-2} a_i \left(\xi_i \int_0^{\xi_{m-2}} \varphi_q\left(\int_0^s a(\tau) f(\tau, u(\tau))\Delta\tau\right)\Delta s\right. \\
&\qquad\qquad \left. - \xi_{m-2}\int_0^{\xi_i} \varphi_q\left(\int_0^s a(\tau) f(\tau, u(\tau))\Delta\tau\right)\Delta s\right) \\
&\ge \frac{\xi_{m-2}\int_0^1 \varphi_q(\int_0^s a(\tau) f(\tau, u(\tau))\Delta\tau)\Delta s - \int_0^{\xi_{m-2}} \varphi_q(\int_0^s a(\tau) f(\tau, u(\tau))\Delta\tau)\Delta s}{1 - \sum_{i=1}^{m-2} a_i \xi_i} \\
&\ge \frac{\xi_{m-2}\int_{\xi_{m-2}}^1 \varphi_q(\int_{\xi_{m-2}}^s a(\tau) f(\tau, u(\tau))\Delta\tau)\Delta s}{1 - \sum_{i=1}^{m-2} a_i \xi_i} \\
&\ge \varphi_q(\varphi_p(m\gamma)\varphi_p(\rho_1)) \frac{\xi_{m-2}\int_{\xi_{m-2}}^1 \varphi_q(\int_{\xi_{m-2}}^s a(\tau)\Delta\tau)\Delta s}{1 - \sum_{i=1}^{m-2} a_i \xi_i} \\
&= \frac{m\gamma\rho 1}{1 - \sum_{i=1}^{m-2} a_i \xi_i} \cdot \xi_{m-2} \int_{\xi_{m-2}}^1 \varphi_q\left(\int_{\xi_{m-2}}^s a(\tau)\Delta\tau\right)\Delta s \ge \rho_1.
\end{aligned} \tag{4.6}$$

Then condition (iii) of Theorem 4.1 holds.

Since all conditions of Theorem 4.1 are satisfied, the MBVP (1.5) has at least two positive solutions u_1 and u_2 such that $u_1(t) > \rho_1$ with $u_1(l) < \rho_2$ and $u_2(l) > \rho_2$ with $u_2(l) < \rho_3$. □

5. Existence of at least Three Positive Solutions

We will use the Leggett-Williams fixed point theorem [10] to prove the existence of at least three positive solutions to the nonlinear MBVP (1.5).

Theorem 5.1 (see Leggett and Williams [10]). *Let P be a cone in the real Banach space E. Set*

$$\begin{aligned}
Pr &= \{x \in P \mid \|x\| < r\}, \\
P(\Psi, a, b) &= \{x \in P \mid a \le \Psi(x),\ \|x\| \le b\}.
\end{aligned} \tag{5.1}$$

Suppose $A : \overline{P}r \to \overline{P}r$ be a completely continuous operator and be a nonnegative continuous concave functional on P with $\Psi(u) \le \|u\|$ for all $u \in \overline{P}r$. If there exists $0 < \rho_1 < \rho_2 < (1/\gamma)\rho_2 \le \rho_3$ such that the following condition hold:

(i) $\{u \in P(\Psi, \rho_2, (1/\gamma)\rho_2) \mid \Psi(u) > \rho_2\} \neq \phi$ and $\Psi(Au) > \rho_2$ for all $u \in P(\Psi, \rho_2, (1/\gamma)\rho_2)$;

(ii) $\|Au\| < \rho_1$ for $\|u\| \le \rho_1$;

(iii) $\Psi(Au) > \rho_2$ for $u \in P(\Psi, \rho_2, (1/\gamma)\rho_2)$ with $\|Au\| > (1/\gamma)\rho_2$, then A has at least three fixed points u_1, u_2 and u_3 in $\overline{P}r$ satisfying $\|u_1\| < \rho_1, \Psi(u_2) > \rho_2, \rho_1 < \|u_3\|$ with $\Psi(u_2) < \rho_2$.

Theorem 5.2. *Assume (H$_1$)–(H$_3$) hold . Suppose that there exist constants* $0 < \rho_1 < \rho_2 < (1/\gamma)\rho_2 \le \rho_3$ *such that*

(C$_1$) $f(t,u) \le \varphi_p(m)\varphi_p(\rho_3)$, *for* $t \in [\xi_{m-2}, l]$ *and* $u \in [0, \rho_3]$;

(C$_2$) $f(t,u) > \varphi_p(M\gamma)\varphi_p(\rho_2)$, *for* $t \in [\xi_{m-2}, l]$ *and* $u \in [\rho_2, (1/\gamma)\rho_2]$;

(C$_3$) $f(t,u) < \varphi_p(m)\varphi_p(\rho_1)$, *for* $t \in [\xi_{m-2}, 1]$ *and* $u \in [0, \rho_1]$.

Then the MBVP (1.5) has at least three positive solutions u_1, u_2, and u_3 such that $u_1(\xi) < \rho_1$, $u_2(l) > \rho_2$, $u_3(\xi) > \rho_1$ with $u_3(l) < \rho_2$.

Proof. The conditions of Theorem 5.1 will be shown to be satisfied. Define the nonnegative continuous concave functional $\Psi : P \to [0, \infty)$ to be $\Psi(u) = u(\xi_{m-2})$, the cone P as in (2.5), M and m as in (3.1). We have $\Psi(u) \le \|u\|$ for all $u \in P$. If $u \in \overline{P}_{\rho_3}$, then $\|u\| \le \rho_3$, and from assumption (C$_1$) , then we have

$$\begin{aligned}
(Au)(t) &= -\int_0^t \varphi_q\left(\int_0^s a(\tau) f(\tau, u(\tau))\Delta\tau\right)\Delta s \\
&\quad - t \cdot \frac{\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} \varphi_q(\int_0^s a(\tau) f(\tau, u(\tau))\Delta\tau)\Delta s}{1 - \sum_{i=1}^{m-2} a_i \xi_i} \\
&\quad + t \cdot \frac{\int_0^1 \varphi_q(\int_0^s a(\tau) f(\tau, u(\tau))\Delta\tau)\Delta s}{1 - \sum_{i=1}^{m-2} a_i \xi_i} \\
&\le t \cdot \frac{\int_0^1 \varphi_q(\int_0^s a(\tau) f(\tau, u(\tau))\Delta\tau)\Delta s}{1 - \sum_{i=1}^{m-2} a_i \xi_i} \\
&\le \varphi_q(\varphi_p(m)\varphi_p(\rho_3)) \cdot \frac{\int_0^1 \varphi_q(\int_0^s a(\tau)\Delta\tau)\Delta s}{1 - \sum_{i=1}^{m-2} a_i \xi_i} \\
&\le m\rho_3 \cdot \frac{\int_0^1 \varphi_q(\int_0^s a(\tau)\Delta\tau)\Delta s}{1 - \sum_{i=1}^{m-2} a_i \xi_i} = \rho_3.
\end{aligned} \tag{5.2}$$

This implies that $\|Au\| \le \rho_3$. Thus, we have $A : \overline{P}_{\rho_3} \to \overline{P}_{\rho_3}$. Since $(1/\gamma)\rho_2 \in P(\Psi, \rho_2, (1/\gamma)\rho_2)$ and $\Psi((1/\gamma)\rho_2) = (1/\gamma)\rho_2 > \rho_2, \{u \in P(\Psi, \rho_2, (1/\gamma)\rho_2) | \Psi(u) > \rho_2\} \neq \phi$.

For $u \in P(\Psi, \rho_2, (1/\gamma)\rho_2)$ we have $\rho_2 \le u(\xi_{m-2}) \le \|u\| \le (1/\gamma)\rho_2$. Using assumption (C_2), $f(t,u) > \varphi_p(M\gamma)\varphi_p(\rho_2)$, we obtain

$$\begin{aligned}
\Psi(Au) &= (Au)(\xi_{m-2}) \\
&= -\int_0^{\xi_{m-2}} \varphi_q\left(\int_0^s a(\tau) f(\tau, u(\tau))\Delta\tau\right)\Delta s \\
&\quad - \xi_{m-2} \cdot \frac{\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} \varphi_q(\int_0^s a(\tau) f(\tau, u(\tau))\Delta\tau)\Delta s}{1 - \sum_{i=1}^{m-2} a_i\xi_i} \\
&\quad + \xi_{m-2} \cdot \frac{\int_0^1 \varphi_q(\int_0^s a(\tau) f(\tau, u(\tau))\Delta\tau)\Delta s}{1 - \sum_{i=1}^{m-2} a_i\xi_i} \\
&= \frac{\xi_{m-2}\int_0^1 \varphi_q(\int_0^s a(\tau) f(\tau, u(\tau))\Delta\tau)\Delta s - \int_0^{\xi_{m-2}} \varphi_q(\int_0^s a(\tau) f(\tau, u(\tau))\Delta\tau)\Delta s}{1 - \sum_{i=1}^{m-2} a_i\xi_i} \\
&\quad + \frac{1}{1 - \sum_{i=1}^{m-2} a_i\xi_i} \sum_{i=1}^{m-2} a_i \left(\xi_i \int_0^{\xi_{m-2}} \varphi_q\left(\int_0^s a(\tau) f(\tau, u(\tau))\Delta\tau\right)\Delta s \right. \\
&\qquad\qquad \left. - \xi_{m-2} \int_0^{\xi_i} \varphi_q\left(\int_0^s a(\tau) f(\tau, u(\tau))\Delta\tau\right)\Delta s\right) \\
&\ge \frac{\xi_{m-2}\int_0^1 \varphi_q(\int_0^s a(\tau) f(\tau, u(\tau))\Delta\tau)\Delta s - \int_0^{\xi_{m-2}} \varphi_q(\int_0^s a(\tau) f(\tau, u(\tau))\Delta\tau)\Delta s}{1 - \sum_{i=1}^{m-2} a_i\xi_i} \\
&\ge \frac{\xi_{m-2}\int_{\xi_{m-2}}^1 \varphi_q(\int_{\xi_{m-2}}^s a(\tau) f(\tau, u(\tau))\Delta\tau)\Delta s}{1 - \sum_{i=1}^{m-2} a_i\xi_i} \\
&\ge \varphi_q(\varphi_p(M\gamma)\varphi_p(\rho_2)) \frac{\xi_{m-2}\int_{\xi_{m-2}}^1 \varphi_q(\int_{\xi_{m-2}}^s a(\tau)\Delta\tau)\Delta s}{1 - \sum_{i=1}^{m-2} a_i\xi_i} \\
&\ge \frac{M\gamma\rho_2}{1 - \sum_{i=1}^{m-2} a_i\xi_i} \cdot \xi_{m-2} \int_{\xi_{m-2}}^1 \varphi_q\left(\int_{\xi_{m-2}}^s a(\tau)\Delta\tau\right)\Delta s \ge \rho_2.
\end{aligned} \tag{5.3}$$

Hence, condition (i) of Theorem 5.1 holds.

If $\|u\| \le \rho_1$, from assumption (C_3), we obtain

$$\begin{aligned}
(Au)(t) &\le \varphi_q(\varphi_p(m)\varphi_p(\rho_1)) \cdot \frac{\int_0^1 \varphi_q(\int_0^s a(\tau)\Delta\tau)\Delta s}{1 - \sum_{i=1}^{m-2} a_i\xi_i} \\
&\le m\rho_1 \cdot \frac{\int_0^1 \varphi_q(\int_0^s a(\tau)\Delta\tau)\Delta s}{1 - \sum_{i=1}^{m-2} a_i\xi_i} = \rho_1.
\end{aligned} \tag{5.4}$$

This implies that $\|Au\| \le \rho_1$.

Consequently, condition (ii) of Theorem 5.1 holds.

We suppose that $u \in P(\Psi, \rho_2, \rho_3)$, with $\|Au\| > (1/\gamma)\rho_2$. Then we get

$$\begin{aligned}\Psi(Au) &= (Au)(\xi_{m-2}) \\ &\geq \frac{\xi_{m-2}\int_{\xi_{m-2}}^{1}\varphi_q\left(\int_{\xi_{m-2}}^{s}a(\tau)f(\tau,u(\tau))\Delta\tau\right)\Delta s}{1-\sum_{i=1}^{m-2}a_i\xi_i} \\ &\geq \varphi_q\left(\varphi_p(M\gamma)\varphi_p(\rho_2)\right)\frac{\xi_{m-2}\int_{\xi_{m-2}}^{1}\varphi_q\left(\int_{\xi_{m-2}}^{s}a(\tau)\Delta\tau\right)\Delta s}{1-\sum_{i=1}^{m-2}a_i\xi_i} \\ &\geq \frac{M\gamma\rho_2}{1-\sum_{i=1}^{m-2}a_i\xi_i}\cdot\xi_{m-2}\int_{\xi_{m-2}}^{1}\varphi_q\left(\int_{\xi_{m-2}}^{s}a(\tau)\Delta\tau\right)\Delta s \geq \rho_2.\end{aligned} \tag{5.5}$$

Hence, condition (iii) of Theorem 5.1 holds.

Because all of the hypotheses of the Leggett-Williams fixed point theorem are satisfied, the nonlinear MBVP (1.5) has at least three positive solutions u_1, u_2, and u_3 such that $u_1(\xi) < \rho_1$, $u_2(l) > \rho_2$, and $u_3(\xi) > \rho_1$ with $u_3(l) < \rho_2$. □

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