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Research Article

The Average Errors for the Grünwald Interpolation in the Wiener Space

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We determine the weakly asymptotically orders for the average errors of the Grünwald interpolation sequences based on the Tchebycheff nodes in the Wiener space. By these results we know that for the L_p-norm $(2 \le q \le 4)$ approximation, the p-average $(1 \le p \le 4)$ error of some Grünwald interpolation sequences is weakly equivalent to the p-average errors of the best polynomial approximation sequence.

1. Introduction and Main Results

Let F be a real separable Banach space equipped with a probability measure μ on the Borel sets of F. Let H be another normed space such that F is continuously embedded in H. By $\|\cdot\|$ we denote the norm in H. Any $A : F \to H$ such that $f \mapsto \|f - A(f)\|$ is a measurable mapping is called an approximation operator (or just approximation). The p-average error of A is defined as

$$e_p(A,H) = \left(\int_F \|f - A(f)\|^p \mu(df)\right)^{1/p}. \tag{1.1}$$

Since in practice the underlying function is usually given via its (exact or noisy) values at finitely many points, the approximation operator $A(f)$ is often considered depending on some function values about f only. Many papers such as [1–4] studied the complexity of computing an ε-approximation in average case setting. Noticed that the polynomial interpolation operators are important approximation tool in the continuous functions space, and they are depending on some function values about f only, we want to know the average error for some polynomial interpolation operators in the Wiener measure. Now we turn to describe the contents in detail.

Let X be the space of continuous function f defined on $[0,1]$ such that $f(0) = 0$. The space X is equipped with the sup norm. The Wiener measure ω is uniquely defined by the following property:

$$\omega\left(f \in X : \left(f(t_1), \ldots, f(t_n)\right) \in B\right) = \prod_{j=1}^{n} \frac{1}{\sqrt{2\pi\left(t_j - t_{j-1}\right)}} \int_B e^{\sum_{j=1}^{n}(-(u_j-u_{j-1})^2/2(t_j-t_{j-1}))} du_1 \cdots du_n, \tag{1.2}$$

for every $n \geq 1$, $B \in \mathcal{B}(\mathcal{R}^n)$, where $\mathcal{B}(\mathcal{R}^n)$ denote the set class of all Borel measurable subsets of $\mathcal{R}^n$, and $0 = t_0 < t_1 < \cdots < t_n \leq 1$ with $u_0 = 0$. Its mean is zero, and its correlation operator is given by $L_{x_1}(C_\omega L_{x_2}) = \min\{x_1, x_2\}$ for $L_{x_i}(f) = f(x_i)$, that is,

$$\int_X f(x_1) f(x_2) \omega(df) = \min\{x_1, x_2\}, \qquad \forall x_1, x_2 \in [0,1]. \tag{1.3}$$

In this paper, we specify $F = \{f \in C[-1,1] : g(t) = f(2t-1) \in X\}$, and for every measurable subset $A \subset F$, we define

$$\mu(A) = \omega\left(\left\{g(t) = f(2t-1), f \in A\right\}\right). \tag{1.4}$$

For $1 \leq p < \infty$, denote by L_p the linear normed space of L_p-integrable function f on $[-1,1]$ with the following finite norm:

$$\|f\|_p = \left(\int_{-1}^{1} |f(x)|^p dx\right)^{1/p}. \tag{1.5}$$

Let

$$t_k = t_{kn} = \cos\frac{2k-1}{2n}\pi, \quad k = 1, \ldots, n \tag{1.6}$$

be the zeros of $T_n(x) = \cos n\theta$ $(x = \cos\theta)$, the nth degree Tchebycheff polynomial of the first kind. The well-known Grünwald interpolation polynomial of f based on $\{t_k\}_{k=1}^{n}$ is given by (see [5])

$$G_n(f, x) = \sum_{k=1}^{n} f(t_k) l_k^2(x), \tag{1.7}$$

where

$$l_k(x) = \frac{(-1)^{k+1}\sqrt{1-t_k^2}\,T_n(x)}{n(x-t_k)}, \qquad k = 1, \ldots, n. \tag{1.8}$$

Theorem 1.1. *Let $G_n(f,x)$ be defined as above. Then*

$$e_p(G_n, L_p) \asymp \begin{cases} \dfrac{1}{\sqrt{n}}, & 1 \le p \le 4, \\ \dfrac{1}{n^{2/p}}, & p \ge 4, \end{cases} \tag{1.9}$$

where in the following $A(n) \asymp B(n)$ means that there exists C independent of n such that $A(n)/C \le B(n) \le CA(n)$, and the constant C may be different in the same expression.

By Hölder inequality, combining Theorem 1.1 with paper [2] we know that for $1 \le p,\ q \le 4$,

$$e_p(G_n, L_q) \asymp \frac{1}{\sqrt{n}}. \tag{1.10}$$

Remark 1.2. Denote by $\mathcal{P}_n$ the set of algebraic polynomials of degree $\le n$. For $f \in F$, let $T_n f$ denote the best L_q-approximation polynomial of f from $\mathcal{P}_n$. Then the p-average error of the best L_q-approximation of continuous functions by polynomials from $\mathcal{P}_n$ over the Wiener space is given by

$$e_p(T_n, L_q) = \left(\int_F \|f - T_n f\|_q^p \mu(df) \right)^{1/p}. \tag{1.11}$$

By Theorem 1.1 and paper [6] we can obtain that for $2 \le q \le 4$ and $1 \le p \le 4$, we have

$$e_p(G_n, L_q) \asymp e_p(T_n, L_q) \asymp \frac{1}{\sqrt{n}}. \tag{1.12}$$

Remark 1.3. Let us recall some fundamental notions about the information-based complexity in the average case setting. Let F be a set with a probability measure μ, and let G be a normed linear space with norm $\|\cdot\|$. Let S be a measurable mapping from F into G which is called a solution operator. Let N be a measurable mapping from F into $\mathcal{R}^n$, and let ϕ be a measurable mapping from $\mathcal{R}^n$ into G which are called an information operator and an algorithm, respectively. For $1 \le p < +\infty$, the p-average error of the approximation $\phi \circ N$ with respect to the measure μ is defined by

$$e_p(S, N, \phi, \|\cdot\|) := \left(\int_F \|Sf - \phi(N(f))\|^p \mu(df) \right)^{1/p}, \tag{1.13}$$

and the p-average radius of information N with respect to μ is defined by

$$r_p(S, N, \|\cdot\|) := \inf_{\phi} e_p(S, N, \phi, \|\cdot\|), \tag{1.14}$$

where ϕ ranges over the set of all algorithms. Futhermore, let Λ denote a class of permissible information functional and denote $\mathcal{N}_n^\Lambda$ the set of nonadaptive information operators N from Λ of cardinality n, that is,

$$N(f) = (L_1(f), L_2(f), \ldots, L_n(f)), \qquad L_i \in \Lambda, \quad i = 1, \ldots, n. \tag{1.15}$$

Let

$$r_p(n, S, \Lambda, \|\cdot\|) = \inf_{N \in \mathcal{N}_n^\Lambda} r_p(S, N, \|\cdot\|), \tag{1.16}$$

denote the nth minimal p-average radius of nonadaptive information in the class Λ.

For example, if F and μ are defined as above, S is the identity mapping I, and Λ is consist of function evaluations at fixed point; then by [2] we know that for L_q-norm approximation, if $1 \le p$, $q < \infty$, we have

$$r_p(n, I, \Lambda, L_q) \asymp \frac{1}{\sqrt{n}}. \tag{1.17}$$

It is easy to understand that $G_n(f, x)$ can be viewed as a composition of a nonadaptive information operator from $\mathcal{N}_n^\Lambda$ and a linear algorithm, and for $1 \le p$, $q \le 4$,

$$e_p(G_n, L_q) \asymp r_p(n, I, \Lambda, L_q). \tag{1.18}$$

In comparison with the result of Theorem 1.1, we consider the following Grünwald interpolation. Let

$$x_k = x_{kn} = \cos\frac{k\pi}{n+1}, \qquad k = 1, \ldots, n \tag{1.19}$$

be the zeros of $u_n(x) = \sin(n+1)\theta/\sin\theta$, $(x = \cos\theta)$, the nth Tchebycheff polynomial of the second kind. The Grünwald interpolation polynomial of f based on $\{x_k\}_{k=1}^n$ is given by

$$\overline{G}_n(f, x) = \sum_{k=1}^{n} f(x_k)\bar{l}_k^2(x), \tag{1.20}$$

where

$$\bar{l}_k(x) = \frac{u_n(x)}{u_n'(x_k)(x - x_k)} = \frac{(-1)^{k+1}(1 - x_k^2)u_n(x)}{(n+1)(x - x_k)}, \qquad k = 1, \ldots, n. \tag{1.21}$$

Theorem 1.4. *Let $\overline{G}_n(f, x)$ be defined as above. Then*

$$e_2\left(\overline{G}_n, L_2\right) \asymp 1. \tag{1.22}$$

2. The Proof of Theorem 1.1

We consider the upper estimate first. From [7, page 107, (28)] we obtain

$$e_p^p(G_n, \|\cdot\|_p) = v_p^p \cdot \int_{-1}^{1} \left(\int_F \left| f(x) - G_n(f,x) \right|^2 \mu(df) \right)^{p/2} dx, \tag{2.1}$$

where v_p^p is the pth absolute moment of the standard normal distribution. It is easy to verify

$$f(x) - G_n(f,x) = \left(1 - \sum_{k=1}^{n} l_k^2(x) \right) f(x) + \sum_{k=1}^{n} (f(x) - f(t_k)) l_k^2(x). \tag{2.2}$$

From (2.2) and Hölder inequality we can obtain

$$\begin{aligned} \int_F |f(x) - G_n(f,x)|^2 \mu(df) &\le 2\left(1 - \sum_{k=1}^{n} l_k^2(x) \right)^2 \int_F f^2(x)\mu(df) \\ &\quad + 2\int_F \left(\sum_{k=1}^{n} (f(x) - f(t_k)) l_k^2(x) \right)^2 \mu(df) \\ &= 2I_1(x) + 2I_2(x). \end{aligned} \tag{2.3}$$

By (1.3) we obtain

$$\int_F f^2(x)\mu(df) = \int_X g^2\left(\frac{1+x}{2}\right)\omega(dg) = \frac{1+x}{2}. \tag{2.4}$$

Let $x = \cos\theta$, then it is easy to verify

$$\sum_{k=1}^{n} l_k^2(x) - 1 = \frac{T_n(x)}{n^2}\left(xT_n'(x) - nT_n(x)\right) = \frac{\cos n\theta \sin(n-1)\theta}{n \sin\theta}. \tag{2.5}$$

By (2.4), (2.5), and a simple computation we can obtain

$$\begin{aligned} \int_{-1}^{1} |I_1(x)|^{p/2} dx &\le \frac{1}{n^p} \int_0^{\pi} \frac{|\cos n\theta \sin(n-1)\theta|^p}{|\sin\theta|^{p-1}} d\theta \\ &\le \begin{cases} \dfrac{1}{n^p}, & 1 \le p \le 2, \\ \dfrac{\ln n}{n^p}, & p = 2, \\ \dfrac{1}{n^2}, & p > 2. \end{cases} \end{aligned} \tag{2.6}$$

By (1.3), it is easy to verify that for $k \geq j$,

$$\int_F (f(x) - f(t_k))(f(x) - f(t_j))\mu(df) = \begin{cases} \dfrac{t_k - x}{2}, & x < t_k, \\ 0, & t_k \leq x \leq t_j, \\ \dfrac{x - t_j}{2}, & x > t_j. \end{cases} \tag{2.7}$$

Let $t_0 = 1$, $t_{n+1} = -1$. From (2.7) and a simple computation we know that for $x \in [t_{m+1}, t_m]$, $m = 0, \ldots, n$,

$$\begin{aligned} I_2(x) &= \frac{1}{2}\sum_{k=1}^{n} |x - t_k| l_k^4(x) + \sum_{k=s+1}^{n-1} (x - t_k) l_k^2(x) \sum_{j=k+1}^{n} l_j^2(x) + \sum_{k=1}^{m} (t_k - x) l_k^2(x) \sum_{j=1}^{k-1} l_j^2(x) \\ &= J_1(x) + J_2(x) + J_3(x). \end{aligned} \tag{2.8}$$

From [8] we know $\sum_{k=1}^{n} l_k^2(x) \leq 2$, hence

$$\sum_{k=1}^{n} l_k^p(x) \leq C, \qquad \forall p \geq 2. \tag{2.9}$$

From (1.8) it follows that

$$|(x - x_k) l_k(x)| \leq \frac{1}{n}, \qquad k = 1, \ldots, n. \tag{2.10}$$

From (2.7) and (2.10) it follows that

$$|J_1(x)| \leq \frac{1}{2n} \sum_{k=1}^{n} \int_{-1}^{1} \left| l_k^3(x) \right| dx \leq \frac{C}{n}. \tag{2.11}$$

From (2.10) it follows that

$$J_2(x) \leq \frac{1}{n} \sum_{j=m+1}^{n-1} \left| l_j(x) \right| \sum_{k=j+1}^{n} l_k^2(x). \tag{2.12}$$

Let $x = \cos\theta$, we have

$$\begin{aligned}
\sum_{j=m+1}^{n-1} \left|l_j(x)\right| \sum_{k=j+1}^{n} l_k^2(x) &= \sum_{j=m+1}^{n-1} \left| \frac{\sin\left((2j-1)\pi/2n\right)\cos n\theta}{n(\cos\theta - \cos\left((2j-1)\pi/2n\right))} \right| \\
&\quad \times \sum_{k=j+1}^{n} \frac{\sin^2((2k-1)\pi/2n)\cos^2 n\theta}{n^2(\cos\theta - \cos((2k-1)\pi/2n))^2} \\
&\le \frac{1}{n^3} \sum_{j=m+1}^{n-1} \left| \frac{\sin\left((2j-1)\pi/2n\right)}{(\cos((2m-1)\pi/2n) - \cos\left((2j-1)\pi/2n\right))} \right| \\
&\quad \times \sum_{k=j+1}^{n} \frac{\sin^2((2k-1)\pi/2n)}{(\cos((2m-1)\pi/2n) - \cos((2k-1)\pi/2n))^2} \\
&= \frac{1}{4n^3} \sum_{j=m+1}^{n-1} \left| \frac{\sin\left((2j-1)\pi/2n\right)}{(\sin\left((j-m)\pi/2n\right)\sin\left((j+m-1)\pi/2n\right))} \right| \\
&\quad \times \sum_{k=j+1}^{n} \frac{\sin^2((2k-1)\pi/2n)}{(\sin((k-m)\pi/2n)\sin((k+m-1)\pi/2n))^2}.
\end{aligned} \tag{2.13}$$

By $\sin x + \sin y = 2\sin((x+y)/2)\cos(x-y)/2$ we know that for $0 < x,\ y < \pi$, thus

$$0 < \sin x \le 2\sin\frac{x+y}{2}. \tag{2.14}$$

It is easy to know

$$\sum_{k=j+1}^{n} \frac{1}{(k-m)^2} < \sum_{k=j+1}^{n} \frac{1}{(k-m)(k-m-1)} = \frac{1}{j-m} - \frac{1}{n-m} \le \frac{1}{j-m}. \tag{2.15}$$

By $2x/\pi \le \sin x$, $\forall x \in (0, \pi/2]$, (2.16), (2.17), and (2.18) we can obtain

$$\sum_{j=m}^{n-1} \left|l_j(x)\right| \sum_{k=j+1}^{n} l_k^2(x) = |l_m(x)| \sum_{k=m+1}^{n} l_k^2(x) + \sum_{j=m+1}^{n-1} \left|l_j(x)\right| \sum_{k=j+1}^{n} l_k^2(x) \le C. \tag{2.16}$$

From (2.12) and (2.16) we can obtain

$$|J_2(x)| \le \frac{C}{n}. \tag{2.17}$$

Similarly

$$|J_3(x)| \le \frac{C}{n}. \tag{2.18}$$

From (2.3), (2.8), (2.11), (2.17), and (2.18) we can obtain

$$\int_{-1}^{1} |I_2(x)|^{p/2} dx \le \frac{C}{n^{p/2}}. \tag{2.19}$$

By (2.1), (2.3), (2.6), and (2.19) we can obtain the upper estimate.

Now we consider the lower estimate. For $1 \le p \le 4$, we can obtain the lower estimate from [2]. For $p > 4$, from (2.4) we know that

$$\int_{-1}^{1} \left(\int_F \left| \left(1 - \sum_{k=1}^{n} l_k^2(x) \right) f(x) \right|^2 \mu(df) \right)^{p/2} dx = \int_{-1}^{1} \left| \frac{1+x}{2} \right|^{p/2} \left| \left(1 - \sum_{k=1}^{n} l_k^2(x) \right) \right|^p dx. \tag{2.20}$$

Let $x = \cos\theta$, then from (2.5) we know that

$$\sum_{k=1}^{n} l_k^2(x) - 1 = \frac{\cos\theta \sin 2n\theta}{2n\sin\theta} - \frac{\cos^2 n\theta}{n}. \tag{2.21}$$

Hence we can verify that for $\theta \in [5\pi/8n, 7\pi/8n]$,

$$\left| \sum_{k=1}^{n} l_k^2(x) - 1 \right| \ge \frac{|\sin 2n\theta|}{4n\sin\theta} \ge \frac{1}{7}. \tag{2.22}$$

From (2.20) and (2.22) and a simple computation we can obtain

$$\begin{aligned} \int_{-1}^{1} \left(\int_F \left| \left(1 - \sum_{k=1}^{n} l_k^2(x) \right) f(x) \right|^2 \mu(df) \right)^{p/2} dx &\ge \frac{\cos 5\pi/8n - \cos 7\pi/8n}{14^p} \\ &\ge \frac{3}{8 \cdot 14^p n^2}. \end{aligned} \tag{2.23}$$

From (2.2), (2.3), and (2.19) it follows that

$$\int_{-1}^{1} \left(\int_F \left| \sum_{k=1}^{n} (f(x) - f(t_k)) l_k^2(x) \right|^2 \mu(df) \right)^{p/2} dx \le \frac{C}{n^{p/2}}. \tag{2.24}$$

From (2.1), (2.2), (2.23), and (2.24) we can obtain the lower estimate for $p > 4$.

3. The Proof of Theorem 1.4

Let

$$\begin{aligned} Q_n(f,x) &= \left(\frac{1+x}{2}f(1)+\frac{1-x}{2}f(-1)\right)\frac{u_n^2(x)}{(n+1)^2} \\ &\quad+\sum_{k=1}^{n} f(x_k)\left(1-x^2\right)(1-xx_k)\left(\frac{u_n(x)}{(n+1)(x-x_k)}\right)^2 \end{aligned} \tag{3.1}$$

be the quasi-Hermite-Fejer interpolation polynomial of degree $\leq 2n+1$ based on the extended Tchebycheff nodes of the second kind (see [8]); then by a simple computation we obtain

$$\begin{aligned} \overline{G}_n(f,x_k)-Q_n(f,x_k) &= 0, \quad k=1,\dots,n, \\ \overline{G}_n'(f,x_k)-Q_n'(f,x_k) &= f(x_k)\frac{3x_k}{1-x_k^2}, \quad k=1,\dots,n, \\ \overline{G}_n(f,1)-Q_n(f,1) &= \sum_{k=1}^{n} f(x_k)(1+x_k)^2-f(1), \\ \overline{G}_n(f,-1)-Q_n(f,-1) &= \sum_{k=1}^{n} f(x_k)(1-x_k)^2-f(-1). \end{aligned} \tag{3.2}$$

Denote

$$\begin{aligned} \varphi_k(x) &= \left(1-x^2\right)(1-xx_k)\left(\frac{u_n(x)}{(n+1)(x-x_k)}\right)^2, \quad k=1,\dots,n, \\ \varphi_0(x) &= \frac{1+x}{2}\frac{u_n^2(x)}{(n+1)^2}, \qquad \varphi_{n+1}(x)=\frac{1-x}{2}\frac{u_n^2(x)}{(n+1)^2}, \\ \phi_k(x) &= \frac{(1-x^2)(1-x_k^2)u_n^2(x)}{(n+1)^2(x-x_k)}, \quad k=1,\dots,n. \end{aligned} \tag{3.3}$$

By (3.2) and the unique of the Hermite interpolation polynomial $H_n(f,x)$ which satisfies interpolation conditions,

$$\begin{aligned} H_n(f,x_k) &= f(x_k), \quad k=0,\dots,n+1, \\ H_n'(f,x_k) &= f'(x_k), \quad k=1,\dots,n, \end{aligned} \tag{3.4}$$

we obtain

$$\begin{aligned}
&\overline{G}_n(f,x) - Q_n(f,x) \\
&\quad = \varphi_0(x)\left[\sum_{k=1}^{n} f(x_k)(1+x_k)^2 - f(1)\right] + \varphi_{n+1}(x)\left[\sum_{k=1}^{n} f(x_k)(1-x_k)^2 - f(-1)\right] \\
&\qquad + \sum_{k=1}^{n} f(x_k)\frac{3x_k}{1-x_k^2}\phi_k(x) \\
&\quad = \frac{u_n^2(x)}{(n+1)^2}\sum_{k=1}^{n} f(x_k)\left(1+x_k^2\right) + \frac{2xu_n^2(x)}{(n+1)^2}\sum_{k=1}^{n} f(x_k) - \frac{u_n^2(x)}{2(n+1)^2}\left[f(1)+f(-1)\right] \\
&\qquad - \frac{xu_n^2(x)}{2(n+1)^2}\left[f(1)-f(-1)\right] + \sum_{k=1}^{n} f(x_k)\frac{3x_k}{1-x_k^2}\phi_k(x) \\
&\quad = A_1(x) + A_2(x) + A_3(x) + A_4(x) + A_5(x).
\end{aligned} \tag{3.5}$$

By (3.5) and $(a+b+c+d)^2 \le 4(a^2+b^2+c^2+d^2)$ we know that

$$\begin{aligned}
e_2\left(\overline{G}_n, L_2\right) &= \int_F\int_{-1}^{1} \left|f(x) - \overline{G}_n(f,x)\right|^2 dx\,\mu(df) \\
&\le 4\int_F\int_{-1}^{1} |f(x) - Q_n(f,x)|^2 dx\,\mu(df) + 4\int_F\int_{-1}^{1} [A_1(x)+A_2(x)]^2 dx\,\mu(df) \\
&\quad + 4\int_F\int_{-1}^{1} [A_3(x)+A_4(x)]^2 dx\,\mu(df) + 4\int_F\int_{-1}^{1} [A_5(x)]^2 dx\,\mu(df) \\
&= 4I_1 + 4I_2 + 4I_3 + 4I_4.
\end{aligned} \tag{3.6}$$

From [8] we know that for every $f \in C[-1,1]$,

$$\int_{-1}^{1} |f(x) - Q_n(f,x)|^2 dx \le \int_{-1}^{1} |f(x) - Q_n(f,x)|^2 (1-x^2)^{-1/2} dx \le C\omega^2\left(f, \frac{1}{n}\right), \tag{3.7}$$

where $\omega(f,t)$ is the modulus of continuity of f on $[-1,1]$ defined for every $t \ge 0$, and C is independent of n and f. By (3.7) and [6] we can obtain

$$I_1 \le C\int_F \omega^2\left(f, \frac{1}{n}\right)\mu(df) \le \frac{C\ln n}{n}. \tag{3.8}$$

By using $A_1(x)$ and $A_2(x)$ we obtain

$$\begin{aligned} I_2 &= \int_F \int_{-1}^{1} [A_1(x)]^2 dx\, \mu(df) + \int_F \int_{-1}^{1} [A_2(x)]^2 dx\, \mu(df) \\ &= \frac{1}{(n+1)^4} \int_{-1}^{1} u_n^4(x) dx \int_F \left[\sum_{k=1}^{n} f(x_k)(1+x_k^2) \right]^2 \mu(df) \\ &\quad + \frac{4}{(n+1)^4} \int_{-1}^{1} x^2 u_n^4(x) dx \int_F \left[\sum_{k=1}^{n} f(x_k) \right]^2 \mu(df). \end{aligned} \tag{3.9}$$

By (1.3) we obtain

$$\begin{gathered} \int_F \left[\sum_{k=1}^{n} f(x_k) \right]^2 \mu(df) = \int_X \left[\sum_{k=1}^{n} g\left(\frac{1+x_k}{2} \right) \right]^2 \omega(dg) = \sum_{k=1}^{n} \sum_{j=1}^{n} \min\left\{ \frac{1+x_k}{2}, \frac{1+x_j}{2} \right\} \asymp n^2. \\ \int_F \left[\sum_{k=1}^{n} f(x_k)(1+x_k^2) \right]^2 \mu(df) \asymp n^2. \end{gathered} \tag{3.10}$$

From (3.9) and (3.10) we obtain

$$I_2 \asymp \frac{1}{n^2} \int_{-1}^{1} u_n^4(x) dx \asymp 1. \tag{3.11}$$

Similar to (3.11), we have

$$|I_3| = \frac{1}{4(n+1)^4} \int_{-1}^{1} \left(1+x^2\right) u_n^4(x) dx \le \frac{C}{n^2}. \tag{3.12}$$

By (3.3) and $0 \le (1-x^2)u_n^2(x) \le 1$ we obtain

$$\begin{aligned} 0 &\le I_4 \\ &= \frac{9}{(n+1)^4} \int_F \int_{-1}^{1} (1-x^2)^2 u_n^2(x) \left[\sum_{k=1}^{n} f(x_k) x_k \frac{u_n(x)}{x-x_k} \right]^2 dx\, \mu(df) \\ &\le \frac{9}{(n+1)^4} \int_F \int_{-1}^{1} (1-x^2)^{1/2} \left[\sum_{k=1}^{n} f(x_k) x_k \frac{u_n(x)}{x-x_k} \right]^2 dx\, \mu(df) \\ &= \frac{9}{(n+1)^4} \sum_{k=1}^{n} \sum_{j=1}^{n} \int_{-1}^{1} (1-x^2)^{1/2} \frac{u_n^2(x)}{(x-x_k)(x-x_j)} dx \int_F x_k x_j f(x_k) f(x_j) \mu(df). \end{aligned} \tag{3.13}$$

By [8] we know that

$$\int_{-1}^{1} \frac{(1-x^2)u_n^2(x)}{(x-x_k)(x-x_j)} \frac{1}{\sqrt{1-x^2}} dx = \begin{cases} \dfrac{(n+1)\pi}{1-x_j^2}, & j = k, \\ 0, & j \neq k. \end{cases} \tag{3.14}$$

By (1.3), (3.13), (3.14), and $(2/\pi)x < \sin x < x,\ \ 0 < x < \pi/2$, we obtain

$$\begin{aligned} 0 &\le I_4 \\ &\le \frac{9\pi}{2(n+1)^3} \sum_{k=1}^{n} \frac{x_k^2(1+x_k)}{1-x_k^2} \\ &\le \frac{9\pi}{(n+1)^3} \sum_{k=1}^{n} \frac{1}{\sin^2 k\pi/(n+1)} \\ &\le \frac{C}{n}. \end{aligned} \tag{3.15}$$

By (3.6), (3.8), (3.11), (3.12), and (3.15) we can obtain the upper estimate of Theorem 1.4. On the other hand, by (3.5) we can verify that

$$\begin{aligned} \left| f(x) - \overline{G}_n(f,x) \right|^2 \ge \frac{[A_1(x) + A_2(x)]^2}{4} &- \left| f(x) - Q_n(f,x) \right|^2 \\ &- [A_3(x) + A_4(x)]^2 - [A_5(x)]^2. \end{aligned} \tag{3.16}$$

From (3.16), (3.8), (3.11), (3.12), and (3.15) we can obtain the lower estimate of Theorem 1.4.

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