Research Article

The Dynamics of the Pulse Birth in an SIR Epidemic Model with Standard Incidence

Juping Zhang,[1,2] Zhen Jin,[1] Yakui Xue,[1] and Youwen Li[1]

[1] *Department of Mathematics, School of Science, North University of China, Taiyuan, Shanxi 030051, China*
[2] *School of Mechatronic Engineering, North University of China, Taiyuan, Shanxi 030051, China*

Correspondence should be addressed to Zhen Jin, jinzhn@263.net

An SIR epidemic model with pulse birth and standard incidence is presented. The dynamics of the epidemic model is analyzed. The basic reproductive number R^* is defined. It is proved that the infection-free periodic solution is global asymptotically stable if $R^* < 1$. The infection-free periodic solution is unstable and the disease is uniform persistent if $R^* > 1$. Our theoretical results are confirmed by numerical simulations.

1. Introduction

Every year billions of population suffer or die of various infectious disease. Mathematical models have become important tools in analyzing the spread and control of infectious diseases. Differential equation models have been used to study the dynamics of many diseases in wild animal population. Birth is one of the very important dynamic factors. Many models have invariably assumed that the host animals are born throughout the year, whereas it is often the case that births are seasonal or occur in regular pulse, such as the blue whale, polar bear, Orinoco crocodile, Yangtse alligator, and Giant panda. The dynamic factors of the population usually impact the spread of epidemic. Therefore, it is more reasonable to describe the natural phenomenon by means of the impulsive differential equation [1, 2].

Roberts and Kao established an SI epidemic model with pulse birth, and they found the periodic solutions and determined the criteria for their stability [3]. In view of animal life histories which exhibit enormous diversity, some authors studied the model with stage structure and pulse birth for the dynamics in some species [4–6]. Vaccination is an effective way to control the transmission of a disease. Mathematical modeling can contribute to the design and assessment of the vaccination strategies. Many infectious diseases always take on strongly infectivity during a period of the year; therefore, seasonal preventing is an effective and practicable way to control infectious disease [7]. Nokes and Swinton studied the control

of childhood viral infections by pulse vaccination [8]. Jin studied the global stability of the disease-free periodic solution for SIR and SIRS models with pulse vaccination [9]. Stone et al. presented a theoretical examination of the pulse vaccination policy in the SIR epidemic model [10]. They found a disease-free periodic solution and studied the local stability of this solution. Fuhrman et al. studied asymptotic behavior of an SI epidemic model with pulse removal [11]. d'Onofrio studied the use of pulse vaccination strategy to eradicate infectious disease for SIR and SEIR epidemic models [12–15]. Shi and Chen studied stage-structured impulsive SI model for pest management [16]. And the incidence of a disease is the number of new cases per unit time and plays an important role in the study of mathematical epidemiology. Many works have focused on the epidemic models with bilinear incidence whereas Anderson and May and De Jong et al. pointed out that the epidemic models with standard incidence provide a more natural description for humankind and gregarious animals [17–19].

The purpose of this paper is to study the dynamical behavior of an SIR model with pluse birth and standard incidence. We suppose that a mass vaccination program is introduced, under which newborn animals are vaccinated at a constant rate p $(0 < p < 1)$ and vaccination confers lifelong immunity. Immunity is not conferred at birth, and thus all newborns are susceptible.This paper is organized as follows. In the next section, we present an SIR model with pulse birth and standard incidence and obtain the existence of the infection-free periodic solution. In Section 3, the basic reproductive number R^* is defined. Local stability and the global asymptotically stable of the infection-free periodic solution are obtained when $R^* < 1$. Section 4 concentrates on the uniform persistence of the infectious disease when $R^* > 1$. Numerical simulation is given in Section 5.

2. The SIR Model with Pulse Birth

In our study, we analyze the dynamics of the SIR model of a population of susceptible (S), infective (I), and recovered (R) with immunity individuals. Immunity is not conferred at birth, and thus all newborns are susceptible. Vaccination gives lifelong immunity to pS susceptible who are, as a consequence, transferred to the recovered class (R) of the population. Using the impulsive differential equation, we have

$$S' = -\beta \frac{S}{N} I - dS - pS,$$

$$I' = \beta \frac{S}{N} I - (d + \theta + \alpha)I,$$

$$R' = \theta I - dR + pS, \quad t \neq n\tau, \tag{2.1}$$

$$S(n\tau^+) = S(n\tau) + b_1 S(n\tau) + b_1 \gamma_1 I(n\tau) + b_1 \gamma_2 R(n\tau),$$

$$I(n\tau^+) = I(n\tau) + b_1(1 - \gamma_1)I(n\tau),$$

$$R(n\tau^+) = R(n\tau) + b_1(1 - \gamma_2)R(n\tau), \quad t = n\tau.$$

The total population size is denoted by N, with $N = S + I + R$. Here the parameters β, d, p, θ, α, b_1, γ_1, γ_2 are all positive constants. β is adequate contact rate, d is the per capita death rate, θ is the removed rate, and p is vaccination, a fraction of the entire susceptible population. b_1 is the

proportion of the offspring of population. To some diseases, all the offspring of susceptible parents are still susceptible individuals, but it is different to the recovered. Because individual differences cause different immune response, a fraction γ_2 $(0 < \gamma_2 < 1)$ of their offspring are the susceptible; the rest are immunity (e.g., when Giant panda gives the breast to her baby, the immunity of Giant panda baby is obtained. But if Giant panda baby did not eat breast, their immunity to disease is very poor. They are vulnerable to suffering from respiratory and digestive disease. Therefore, they become the susceptible.). Similarly, a fraction γ_1 $(0 < \gamma_1 < 1)$ of the infectious offspring are susceptible, and the rest are infectious. Due to the effect of the diseases to the infectious, the ratio of the susceptible in their offspring is relatively low. So we assume the fraction $\gamma_1 < \gamma_2$. α represents the death rate due to disease. From biological view, we assume $\beta \geq \alpha$.

From (2.1), we obtain

$$N' = -dN - \alpha I, \quad t \neq n\tau,$$
$$N(n\tau^+) = (1 + b_1)N(n\tau), \quad t = n\tau.$$
(2.2)

Let $s = S/N$, $i = I/N$, $r = R/N$, then systems (2.1) and (2.2) can be written as follows:

$$s' = -ps + (\alpha - \beta)si,$$
$$i' = (\beta s - \theta - \alpha)i + \alpha i^2,$$
$$r' = ps + \theta i + \alpha ir, \quad t \neq n\tau,$$
$$s(n\tau^+) = s(n\tau) + \frac{b_1 \gamma_1}{1 + b_1} i(n\tau) + \frac{b_1 \gamma_2}{1 + b_1} r(n\tau),$$
(2.3)
$$i(n\tau^+) = \frac{1 + b_1(1 - \gamma_1)}{(1 + b_1)} i(n\tau),$$
$$r(n\tau^+) = \frac{1 + b_1(1 - \gamma_2)}{(1 + b_1)} r(n\tau), \quad t = n\tau.$$

The total population size is normalized to one. By virtue of the equation $s(t) + i(t) + r(t) = 1$, we ignore the third and the sixth equations of system (2.3) to study the two-dimensional system:

$$s' = -ps + (\alpha - \beta)si,$$
$$i' = (\beta s - \theta - \alpha)i + \alpha i^2, \quad t \neq n\tau,$$
$$s(n\tau^+) = \frac{b_1 \gamma_2}{1 + b_1} + \left(1 - \frac{b_1 \gamma_2}{1 + b_1}\right)s(n\tau) + \left(\frac{b_1 \gamma_1}{1 + b_1} - \frac{b_1 \gamma_2}{1 + b_1}\right)i(n\tau),$$
(2.4)
$$i(n\tau^+) = \frac{1 + b_1(1 - \gamma_1)}{1 + b_1} i(n\tau), \quad t = n\tau.$$

From biological view, we easily see that the domain

$$\Omega = \{(s, i, r) : s \geq 0, \ i \geq 0, \ r \geq 0, \ s + i + r = 1\} \tag{2.5}$$

is the positive invariant set of system (2.3).

We first demonstrate the existence of infection-free periodic solution of system (2.4), in which infectious individuals are entirely absent from the population permanently, that is, $i(t) = 0, \ t \geq 0$. Under this condition, the growth of susceptible individuals and the population must satisfy

$$s' = -ps, \quad t \neq n\tau,$$

$$s(n\tau^+) = \frac{b_1 \gamma_2}{1 + b_1} + \left(1 - \frac{b_1 \gamma_2}{1 + b_1}\right) s(n\tau), \quad t = n\tau. \tag{2.6}$$

Integrating the first equation in system (2.6) between pulses, it is easy to obtain the solution with initial value $s(0^+) = s_0$,

$$s(t) = s(n\tau^+) e^{-p(t - n\tau)}, \quad n\tau < t \leq (n + 1)\tau. \tag{2.7}$$

Equation (2.7) holds between pulses. At each successive pulse, it yields

$$s((n + 1)\tau^+) = \frac{b_1 \gamma_2}{1 + b_1} + \left(1 - \frac{b_1 \gamma_2}{1 + b_1}\right) e^{-p\tau} s(n\tau^+) = F(s(n\tau^+)). \tag{2.8}$$

Equation (2.8) has a unique fixed point $s^* = b_1 \gamma_2 e^{p\tau} / ((1 + b_1) e^{p\tau} - (1 + b_1 - b_1 \gamma_2))$. The fixed point s^* is locally stable because $dF(s(n\tau^+))/ds|_{s(n\tau^+) = s^*} = (1 - b_1 \gamma_2 / (1 + b_1)) e^{-p\tau} < 1$, By substituting $s(n\tau^+) = s^*$ to (2.7), we obtain the complete expression for the infection-free periodic solution over the nth time-interval $n\tau < t \leq (n + 1)\tau$,

$$\widetilde{s}(t) = \frac{b_1 \gamma_2 e^{p\tau}}{(1 + b_1) e^{p\tau} - (1 + b_1 - b_1 \gamma_2)} e^{-p(t - n\tau)}, \quad \widetilde{i}(t) = 0. \tag{2.9}$$

Therefore the system (2.4) has the infection-free periodic solution $(\widetilde{s}(t), \widetilde{i}(t))$.

3. The Stability of the Infection-Free Periodic Solution

In this section, we will prove the local and global asymptotically stable of the infection-free periodic solution $(\widetilde{s}(t), \widetilde{i}(t))$.

The local stability of the infection-free periodic solution $(\widetilde{s}(t), \widetilde{i}(t))$ may be determined by considering the linearized SIR equation of system (2.4) about the known periodic solution

$(\widetilde{s}(t), \widetilde{i}(t))$ by setting $s(t) = \widetilde{s}(t) + x(t)$, $i(t) = \widetilde{i}(t) + y(t)$, where $x(t)$ and $y(t)$ are small perturbation. The variables $x(t)$ and $y(t)$ are described by the relation

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \Phi(t) \begin{pmatrix} x(0) \\ y(0) \end{pmatrix}, \tag{3.1}$$

where the fundamental solution matrix $\Phi(t) = \varphi_{ij}(t)$ $(i, j = 1, 2)$ satisfies

$$\frac{d\Phi(t)}{dt} = \begin{pmatrix} -p & (\alpha - \beta)\widetilde{s}(t) \\ 0 & \beta\widetilde{s}(t) - \theta - \alpha \end{pmatrix} \Phi(t), \tag{3.2}$$

with $\Phi(0) = E$, where E is the identity matrix. The resetting of the equations of (2.4) becomes

$$\begin{pmatrix} x(n\tau^+) \\ y(n\tau^+) \end{pmatrix} = \begin{pmatrix} 1 - \dfrac{b_1\gamma_2}{1 + b_1} & \dfrac{b_1(\gamma_1 - \gamma_2)}{1 + b_1} \\ 0 & \dfrac{1 + b_1 - b_1\gamma_1}{1 + b_1} \end{pmatrix} \begin{pmatrix} x(n\tau) \\ y(n\tau) \end{pmatrix}. \tag{3.3}$$

Hence, according to the Floquet theory, if all eigenvalues of

$$M(\tau) = \begin{pmatrix} 1 - \dfrac{b_1\gamma_2}{1 + b_1} & \dfrac{b_1(\gamma_1 - \gamma_2)}{1 + b_1} \\ 0 & \dfrac{1 + b_1 - b_1\gamma_1}{1 + b_1} \end{pmatrix} \Phi(\tau) \tag{3.4}$$

are less than one, then the infection-free periodic solution $(\widetilde{s}(t), 0)$ is locally stable. By calculating, we have

$$\Phi(t) = \begin{pmatrix} 1 & \Phi_{12} \\ 0 & \Phi_{22} \end{pmatrix}, \tag{3.5}$$

where $\Phi_{22}(t) = \exp(\beta \int \widetilde{s}(\sigma)d\sigma - (\theta + \alpha)t)$.

The eigenvalues of M denoted by μ_1, μ_2 are $\mu_1 = (1 - b_1\gamma_2/(1 + b_1))e^{-p\tau} < 1$, and $\mu_2 = ((1 + b_1 - b_1\gamma_1)/(1 + b_1)) \exp\{\beta\int_0^\tau \widetilde{s}(\sigma)d\sigma - (\theta + \alpha)\tau\}$, if and only if $\mu_2 < 1$. Define threshold of model (2.4) as follows:

$$R^* = \frac{\beta\int_0^\tau \widetilde{s}(\sigma)d\sigma}{\ln((1 + b_1)/(1 + b_1 - b_1\gamma_1)) + (\theta + \alpha)\tau}, \tag{3.6}$$

where $\tilde{s}(t)$ is the infection-free periodic solution. That is, the infection-free periodic solution $(\tilde{s}(t), \tilde{i}(t))$ is locally asymptotically stable if $R^* < 1$. So we obtained following theorem.

Theorem 3.1. *If $R^* < 1$, then the infection-free periodic solution $(\tilde{s}(t), \tilde{i}(t))$ of system (2.4) is locally asymptotically stable.*

   Now we give the global asymptotically stable of the infection-free periodic solution. In order to prove the global stability of the infection-free periodic solution $(\tilde{s}(t), \tilde{i}(t))$, we need to use to comparison theory and impulsive differential inequality [1, 2].

Theorem 3.2. *If $R^* < 1$, then the infection-free periodic solution $(\tilde{s}(t), \tilde{i}(t))$ of system (2.4) is global asymptotically stable.*

Proof. Because of $\alpha \leq \beta$, and $\gamma_1 \leq \gamma_2$, we have

$$
\begin{aligned}
s' &\leq -ps, \quad t \neq n\tau, \\
s(n\tau^+) &\leq \frac{b_1 \gamma_2}{1 + b_1} + \left(1 - \frac{b_1 \gamma_2}{1 + b_1}\right) s(n\tau), \quad t = n\tau.
\end{aligned}
\tag{3.7}
$$

By impulsive differential inequality, we see that

$$
\begin{aligned}
s(t) &\leq s(0^+) \prod_{o < n\tau < t} \left(1 - \frac{b_1 \gamma_2}{1 + b_1}\right) \exp\left\{\int_0^t (-p) \, d\sigma\right\} \\
&\quad + \sum_{o < n\tau < t} \left\{\prod_{n\tau < j\tau < t} \left(1 - \frac{b_1 \gamma_2}{1 + b_1}\right) \exp\left[\int_{n\tau}^t (-p) \, d\sigma\right] \frac{b_1 \gamma_2}{1 + b_1}\right\} \\
&= \tilde{s}(t) + s(0^+) \left(\frac{1 - b_1 \gamma_2}{1 + b_1}\right)^{[t/T]} e^{-pt} \\
&\quad - \frac{(b_1 \gamma_2 / (1 + b_1))(1 - b_1 \gamma_2 / (1 + b_1))^{[t/T]} e^{-pt}}{(1 - b_1 \gamma_2 / (1 + b_1)) e^{-pT}}.
\end{aligned}
\tag{3.8}
$$

Since

$$
\lim_{t \to \infty} \left\{s(0^+) \left(1 - \frac{b_1 \gamma_2}{1 + b_1}\right)^{[t/T]} e^{-pt} - \frac{(b_1 \gamma_2 / (1 + b_1))(1 - b_1 \gamma_2 / (1 + b_1))^{[t/T]} e^{-pt}}{(1 - b_1 \gamma_2 / (1 + b_1)) e^{-pT}}\right\} = 0, \tag{3.9}
$$

for any given $\epsilon_1 > 0$, there exists $T_1 > 0$, such that $s(t) < \tilde{s}(t) + \epsilon_1$, for all $t > T_1$. $\square$

Introduce the new variable $u = s + r$, then

$$u' = (-\beta s + \alpha u + \theta)(1 - u), \quad t \neq n\tau,$$

$$u(n\tau^+) = \frac{b_1 \gamma_1}{1 + b_1} + \left(1 - \frac{b_1 \gamma_1}{1 + b_1}\right) u(n\tau), \quad t = n\tau. \tag{3.10}$$

Consider the following comparison system with pulse:

$$v' = -(\beta \tilde{s}(t) + \beta \epsilon_1 + \alpha - \theta)v - \alpha v^2 + \theta - \beta(\tilde{s}(t) + \epsilon_1), \quad t \neq n\tau,$$

$$v(n\tau^+) = \frac{b_1 \gamma_1}{1 + b_1} + \left(1 - \frac{b_1 \gamma_1}{1 + b_1}\right) v(n\tau), \quad t = n\tau. \tag{3.11}$$

The first equation of (3.11) is Riccati equation. It is easy to see that $v(t) = 1$ is a solution of system (3.11). Let $y = v - 1$, then

$$y' = -(\beta \tilde{s}(t) - \alpha - \theta + \beta \epsilon_1)y - \alpha y^2, \quad t \neq n\tau,$$

$$y(n\tau^+) = \left(1 - \frac{b_1 \gamma_1}{1 + b_1}\right) y(n\tau), \quad t = n\tau. \tag{3.12}$$

Let $z = 1/y$, then

$$z' = -(\beta \tilde{s}(t) - \alpha - \theta + \beta \epsilon_1)z - \alpha, \quad t \neq n\tau,$$

$$z(n\tau^+) = \left(\frac{1 + b_1}{1 + b_1 - b_1 \gamma_1}\right) z(n\tau), \quad t = n\tau. \tag{3.13}$$

Let $q(t) = \beta \tilde{s}(t) - \alpha - \theta + \beta \epsilon_1$, solving system (3.13) between pulses $(T_1 + n\tau, T_1 + (n+1)\tau]$, we have

$$z(t) = e^{-\int_{T_1+n\tau}^{t} q(\sigma)d\sigma} \left[\alpha \int_{T_1+n\tau}^{t} e^{\int_{T_1+n\tau}^{u} q(\sigma)d\sigma} du + \frac{1 + b_1}{1 + b_1 - b_1 \gamma_1} z(T_1 + n\tau)\right], \tag{3.14}$$

when $t = T_1 + (n + 1)\tau$, (3.14) can be written as follows:

$$z(T_1 + (n + 1)\tau) = e^{-\int_{T_1+n\tau}^{T_1+(n+1)\tau} q(\sigma)d\sigma} \left[\alpha \int_{T_1+n\tau}^{T_1+(n+1)\tau} e^{\int_{T_1+n\tau}^{u} q(\sigma)d\sigma} du + \frac{1 + b_1}{1 + b_1 - b_1 \gamma_1} z(T_1 + n\tau)\right]. \tag{3.15}$$

On the other hand, solving system (3.13) between pulses $(T_1 + (n-1)\tau, T_1 + n\tau]$, we obtain

$$z(t) = e^{-\int_{T_1+(n-1)\tau}^{t} q(\sigma)d\sigma}\left[\alpha\int_{T_1+(n-1)\tau}^{t} e^{\int_{T_1+(n-1)\tau}^{u} q(\sigma)d\sigma} du + \frac{1+b_1}{1+b_1-b_1\gamma_1}z(T_1+(n-1)\tau)\right], \qquad (3.16)$$

then

$$z(T_1+n\tau) = e^{-\int_{T_1+n\tau}^{T_1+(n+1)\tau} q(\sigma)d\sigma}\left[\alpha\int_{T_1+(n-1)\tau}^{T_1+n\tau} e^{\int_{T_1+(n-1)\tau}^{u} q(\sigma)d\sigma} du + \frac{1+b_1}{1+b_1-b_1\gamma_1}z(T_1+(n-1)\tau)\right]. \qquad (3.17)$$

Similarly, we can get the expressions of $z(T_1 + (n-1)\tau), z(T_1 + (n-2)\tau), \ldots, z(T_1)$. Then using iterative technique step by step,

$$z(T_1+n\tau) = e^{-\int_0^{n\tau} q(\sigma)d\sigma}\left(\frac{1+b_1}{1+b_1-b_1\gamma_1}\right)^n\left[\sum_{1\leq k\leq n}\left(\frac{1+b_1}{1+b_1-b_1\gamma_1}\right)^{k-n-1} e^{\int_{T_1}^{T_1+(n-k)\tau} q(\sigma)d\sigma} du \right.$$
$$\left. + \int_{T_1+(n-k)\tau}^{T_1+(n-k+1)\tau} e^{\int_{T_1+(n-k)\tau}^{u} q(\sigma)d\sigma} du + z(T_1)\right], \qquad (3.18)$$

where

$$e^{-\int_0^{n\tau} q(\sigma)d\sigma}\left(\frac{1+b_1}{1+b_1-b_1\gamma_1}\right)^n = \exp\left\{n\left[-\beta\int_0^{\tau}\tilde{s}(\sigma)d\sigma + (\theta+\alpha)\tau - \beta\epsilon_1\tau + \ln\frac{1+b_1}{1+b_1-b_1\gamma_1}\right]\right\}. \qquad (3.19)$$

The condition $R^* < 1$ implies that $\lim_{n\to\infty}z(n\tau) = \infty$, then $\lim_{t\to\infty}x(t) = 1$. The comparison principle and the condition $u(t) < 1$ imply that $\lim_{t\to\infty}u(t) = 1$, so we have $\lim_{t\to\infty}i(t) = 0$.

Because we have proved that $\lim_{t\to\infty}i(t) = 0$ when $R^* < 1$, for any given $\epsilon_2 > 0$, there exists $T_2 > 0$, such that $-\epsilon_2 < i(t) < \epsilon_2$, for all $t > T_2$.

When $t > T_2$, from system (2.4), we have

$$s' \geq -ps + (\alpha - \beta)\epsilon_2 s, \quad t \neq n\tau,$$
$$s(n\tau^+) \geq \frac{b_1\gamma_1}{1+b_1} + \left(1 - \frac{b_1\gamma_2}{1+b_1}\right)s(n\tau), \quad t = n\tau. \qquad (3.20)$$

Therefore

$$
\begin{aligned}
s(t) \geq s(T_2^+) &\prod_{T_2 < n\tau < t} \left(1 - \frac{b_1\gamma_2}{1+b_1}\right) \exp\left\{\int_{T_2}^t (-p + (\alpha - \beta)\epsilon_2)d\sigma\right\} \\
&+ \sum_{T_2 < n\tau < t} \left\{ \prod_{T_2 < j\tau < t} \left(1 - \frac{b_1\gamma_2}{1+b_1}\right) \exp\left\{\int_{n\tau}^t (-p + (\alpha - \beta)\epsilon_2)d\sigma\right\} \frac{b_1\gamma_1}{1+b_1}\right\} \\
= s(T_2^+) &\left(1 - \frac{b_1\gamma_2}{1+b_1}\right)^{[t/T]-[T_2/T]} \exp(-p + (\alpha - \beta)\epsilon_2)(t - T_2) \\
&+ \frac{b_1\gamma_1/(1+b_1)}{1 - (1 - b_1\gamma_2/(1+b_1))\exp(-p + (\alpha - \beta)\epsilon_2)T} \exp(-p + (\alpha - \beta)\epsilon_2)\left(t - \left[\frac{t}{T}\right]T\right) \\
&- \frac{b_1\gamma_1/(1+b_1)}{1 - (1 - b_1\gamma_2/(1+b_1))\exp(-p + (\alpha - \beta)\epsilon_2)T} \\
&\times \left(1 - \frac{b_1\gamma_2}{1+b_1}\right)^{([t/T]-[T_2/T]-1)} \exp(-p + (\alpha - \beta)\epsilon_2)T.
\end{aligned}
\tag{3.21}
$$

For any given $\epsilon_2 > 0$, we have

$$
\begin{aligned}
\lim_{t \to \infty} &\left\{ s(T_2^+)\left(1 - \frac{b_1\gamma_2}{1+b_1}\right)^{[t/T]-[T_2/T]} \exp(-p + (\alpha - \beta)\epsilon_2)(t - T_2) \right. \\
&- \frac{b_1\gamma_1/(1+b_1)}{1 - (1 - b_1\gamma_2/(1+b_1))\exp(-p + (\alpha - \beta)\epsilon_2)T} \\
&\left. \times \left(1 - \frac{b_1\gamma_2}{1+b_1}\right)^{([t/T]-[T_2/T]-1)} \exp(-p + (\alpha - \beta)\epsilon_2)(T) \right\} = 0, \\
\lim_{t \to \infty} &\frac{b_1\gamma_1/(1+b_1)}{1 - (1 - b_1\gamma_2/(1+b_1))\exp(-p + (\alpha - \beta)\epsilon_2)T} \exp(-p + (\alpha - \beta)\epsilon_2)\left(t - \left[\frac{t}{T}\right]T\right) = \widetilde{s}(t).
\end{aligned}
\tag{3.22}
$$

Therefore, for any given $\epsilon_3 > 0$, there exists $T_3 > 0$, when $t > T_3$, then we have

$$
s(t) \geq \widetilde{s}(t) - \epsilon_3.
\tag{3.23}
$$

For any given $\epsilon > 0$. Let $T = \max\{T_1, T_2, T_3\}$, then $t > T$, then we have

$$
\widetilde{s}(t) - \epsilon \geq s(t) \geq \widetilde{s}(t) + \epsilon,
\tag{3.24}
$$

that is $\lim_{t \to \infty} s(t) = \widetilde{s}(t)$.

Therefore the infection-free periodic solution $(\widetilde{s}(t), 0)$ is global asymptotically stable.

4. The Uniform Persistence of the Infectious Disease

In this section, we will discuss the uniform persistence of the infectious disease, that is, $\lim_{t\to\infty} \inf i(t) \geq \rho > 0$ if $R^* > 1$.

To discuss the uniform persistence, we need the following lemma.

Lemma 4.1. *For the following impulsive equation,*

$$x' = -gx - h, \quad t \neq n\tau,$$

$$x(n\tau^+) = \frac{b_1\gamma_1}{1+b_1} + \left(1 - \frac{b_1\gamma_2}{1+b_1}\right)x(n\tau), \quad t = n\tau, \tag{4.1}$$

has a unique positive τ-periodic solution $\tilde{x}(t)$ for which $\tilde{x}(0) > 0$, $t \in R_+$, and $\tilde{x}(t)$ is global asymptotically stable in the sense that $\lim_{t\to\infty}|x(t) - \tilde{x}(t)| = 0$, where $x(t)$ is any solution of system (2.2) with positive initial value $x(0) > 0$ and g, h are positive constants.

Proof. Solving (4.1), we have

$$x(t) = W(t,0)x(0) - h\int_0^t W(t,\sigma)d\sigma + \frac{b_1\gamma_1}{1+b_1}\sum_{0<n\tau<t} W(t,n\tau^+), \tag{4.2}$$

where

$$W(t,t_0) = \prod_{t_0\leq n\tau<t}\left(1 - \frac{b_1\gamma_2}{1+b_1}\right)e^{-g(t-t_0)}. \tag{4.3}$$

Since $W(\tau,0) = (1-b_1\gamma_2/(1+b_1))e^{-g\tau} < 1$, (4.1) has a unique positive τ-periodic solution $\tilde{x}(t)$ with the initial value $\tilde{x}(0) = (-h\int_0^\tau W(\tau^+,\sigma)d\sigma + (b_1\gamma_1/(1+b_1))W(\tau,\tau))/(1 - W(\tau,0))$. Next, we only need to prove that $\lim_{t\to\infty}|x(t) - \tilde{x}(t)| = 0$.

Since

$$|x(t) - \tilde{x}(t)| = W(t,0)|x(0) - \tilde{x}(0)|, \tag{4.4}$$

the result is obtained if $W(t,0) \to 0$ as $t \to \infty$. Suppose $t \in (n\tau, (n+1)\tau]$, then

$$W(t,0) = \prod_{0\leq j\tau<t}\left(1 - \frac{b_1\gamma_2}{1+b_1}\right)e^{-gt} = \left(1 - \frac{b_1\gamma_2}{1+b_1}\right)^{[t/\tau]}e^{-gt}. \tag{4.5}$$

Thus $\lim_{t\to\infty}W(t,0) = 0$. The proof is complete. □

Lemma 4.2. *If $R^* > 1$, then the disease uniformly weakly persists in the population, in the sense that there exists some $c > 0$ such that $\lim_{t\to\infty} \sup i(t) > c$ for all solutions of (2.4).*

Proof. Suppose that for every $\epsilon > 0$, there is some solution with $\lim_{t\to\infty} \sup i(t) < \epsilon$. From the first equation of (2.4), we have

$$s' = -ps + (\alpha - \beta)si \geq -ps + (\alpha - \beta)\epsilon, \quad t \neq n\tau. \tag{4.6}$$

Consider the following equation:

$$w' = -pw + (\alpha - \beta)\epsilon, \quad t \neq n\tau,$$

$$w(n\tau^+) = \frac{b_1\gamma_1}{1 + b_1} + \left(1 - \frac{b_1\gamma_2}{1 + b_1}\right)w(n\tau), \quad t = n\tau. \tag{4.7}$$

By Lemma 4.1, we see that (4.7) has a unique positive τ-periodic solution $\tilde{w}(t)$, and $\tilde{w}(t)$ is global asymptotically stable. Solving (4.7), we have

$$w(t) = W(t,0)w(0) + (\alpha - \beta)\epsilon \int_0^t W(t,\sigma)d\sigma + \frac{b_1\gamma_1}{1 + b_1}\sum_{0<n\tau<t} W(t,n\tau^+), \tag{4.8}$$

$$\tilde{w}(t) = (\alpha - \beta)\epsilon\left(\frac{W(t,0)\int_0^\tau W(\tau^+,\sigma)d\sigma}{1 - W(\tau,0)} + \int_0^t W(t,\sigma)d\sigma\right)$$

$$+ \left(1 - \frac{b_1\gamma_2}{1 + b_1}\right)\frac{W(t,0)W(\tau,\tau)}{1 - W(\tau,0)} + \frac{b_1\gamma_1}{1 + b_1}\sum_{0<n\tau<t} W(t,n\tau^+), \tag{4.9}$$

and $\tilde{w}(t)$ is global asymptotically stable. By (4.9), let $\alpha = \beta$, we obtain the periodic solution of (2.6) that

$$\tilde{s}(t) = \left(1 - \frac{b_1\gamma_2}{1 + b_1}\right)\frac{W(t,0)W(\tau,\tau)}{1 - W(\tau,0)} + \frac{b_1\gamma_1}{1 + b_1}\sum_{0<n\tau<t} W(t,n\tau^+). \tag{4.10}$$

and we have

$$\tilde{s}(t) - \tilde{w}(t) = (\beta - \alpha)\epsilon\left(\frac{W(t,0)\int_0^\tau W(\tau^+,\sigma)d\sigma}{1 - W(\tau,0)} + \int_0^t W(t,\sigma)d\sigma\right). \tag{4.11}$$

Let

$$\Delta = (\beta - \alpha)\max_{0\leq t\leq\tau}\left\{\frac{W(t,0)\int_0^\tau W(\tau^+,\sigma)d\sigma}{1 - W(\tau,0)} + \int_0^t W(t,\sigma)d\sigma\right\}, \tag{4.12}$$

by (4.11), we can see that

$$\tilde{w}(t) \geq \tilde{s}(t) - \Delta\epsilon. \tag{4.13}$$

By comparison theory, we obtain that

$$i' \geq -(\theta + \alpha)i + \beta i w(t) + \alpha i^2. \tag{4.14}$$

Since $\tilde{w}(t)$ is global asymptotically stable, for above ϵ, there exists $T_4 > 0$, such that $w(t) \geq \tilde{w}(t) - \epsilon, t > T_4$. From (4.13) and (4.14), we have that

$$i' \geq [\beta \tilde{s}(t) - (\theta + \alpha) - (1 + \Delta)\beta \epsilon]i. \tag{4.15}$$

Consider the following equation:

$$i' \geq [\beta \tilde{s}(t) - (\theta + \alpha) - (1 + \Delta)\beta \epsilon]i, \quad t \neq n\tau,$$

$$i(n\tau^+) = \frac{1 + b_1(1 - \gamma_1)}{1 + b_1}i(n\tau), \quad t = n\tau. \tag{4.16}$$

By impulsive differential inequality, for $t \in (T_4 + n\tau, T_4 + (n+1)\tau]$, we see that

$$
\begin{aligned}
i(t) &\geq i(T_4) \prod_{T_4 < j\tau < t} \frac{1 + b_1(1 - \gamma_1)}{1 + b_1} \exp\left\{\int_{T_4}^{t} [\beta \tilde{s}(\sigma) - (\theta + \alpha) - (1 + \Delta)\beta \epsilon] d\sigma\right\} \\
&= i(T_4)\left(\frac{1 + b_1(1 - \gamma_1)}{1 + b_1}\right)^n \exp\left\{\int_{T_4}^{T_4 + n\tau} [\beta \tilde{s}(\sigma) - (\theta + \alpha) - (1 + \Delta)\beta \epsilon] d\sigma\right. \\
&\qquad\qquad\qquad\qquad\qquad\left. + \int_{T_4 + n\tau}^{t} [\beta \tilde{s}(\sigma) - (\theta + \alpha) - (1 + \Delta)\beta \epsilon] d\sigma\right\} \\
&\geq C \exp\left\{n\left[(R^* - 1)\left(\ln\frac{1 + b_1}{1 + b_1(1 - \gamma_1)} + (\theta + \alpha)\tau\right) - (1 + \Delta)\beta \epsilon \tau\right]\right\},
\end{aligned}
\tag{4.17}
$$

where $C = i(T_4) \exp\{-[(\theta + \alpha) + (1 + \Delta)\beta \epsilon]\tau\}$. Taking

$$0 < \epsilon < \frac{(R^* - 1)[\ln((1 + b_1)/(1 + b_1(1 - \gamma_1))) + (\theta + \alpha)\tau]}{(1 + \Delta)\beta \tau}, \tag{4.18}$$

thus $i(t) \to \infty$ as $t \to \infty$, a contradiction to the fact that $i(t)$ is bounded. The proof is complete. $\square$

Theorem 4.3. *If $R^* > 1$, then the disease is uniformly persistent, that is, there exists a positive constant ρ such that for every positive solution of (2.4), $\lim_{t \to \infty} \inf i(t) \geq \rho > 0$.*

Proof. Let

$$0 < \eta \leq \frac{1}{2}\left(1 - \frac{1}{R^*}\right)\frac{\tilde{s}(t)}{M(\theta + \alpha)}, \tag{4.19}$$

where

$$M = \max_{0 \le t \le \tau} \left\{ \frac{W(t,0)\int_0^\tau W(\tau^+,\sigma)d\sigma}{1 - W(\tau,0)} + \int_0^t W(t,\sigma)d\sigma \right\},$$

$$W(t,t_0) = \prod_{t_0 \le n\tau < t} \frac{1 + b_1(1 - \gamma_1)}{1 + b_1} e^{-p(t-t_0)}. \tag{4.20}$$

It can be obtained from Lemma 4.1 that for any positive solution of (2.4) there exists at least one $t_0 > 0$ such that $i(t_0) > \eta > 0$. Then, we are left to consider two case. The first case is $i(t) \ge \eta$ for all large $t \ge t_0$. The second case is $i(t)$ oscillates about η for large t. The conclusion of Theorem 4.3 is obvious in the first case since we can choose $\rho = \eta$. For the second case, let $t_1 > t_0$, and let $t_2 > t_1$ satisfy

$$i(t_1) = i(t_2) = \eta, \quad i(t) < \eta \quad \text{for } t_1 < t < t_2. \tag{4.21}$$

Next, we introduce the new variable $V = s + i$, and it follows from the first two equations of (2.4) that

$$V' = -pV - (\theta + \alpha)i + (\alpha V + p)i,$$

$$i' = -(\theta + \alpha)i + \beta(V - i)i + \alpha i^2, \quad t \ne n\tau,$$

$$V(n\tau^+) = \frac{b_1\gamma_2}{1 + b_1} + \left(1 - \frac{b_1\gamma_2}{1 + b_1}\right)V(n\tau), \tag{4.22}$$

$$i(n\tau^+) = \frac{1 + b_1(1 - \gamma_1)}{1 + b_1}i(n\tau), \quad t = n\tau.$$

If $i(t) \le \eta$, then $V' \ge -pV - (\theta + \alpha)\eta$, $t \ne n\tau$.

Consider the following equation:

$$x' = -px - (\theta + \alpha)\eta, \quad t \ne n\tau,$$

$$x(n\tau^+) = \frac{b_1\gamma_2}{1 + b_1} + \left(1 - \frac{b_1\gamma_2}{1 + b_1}\right)x(n\tau), \quad t = n\tau. \tag{4.23}$$

By Lemma 4.1, we see that (4.23) has a unique positive τ-periodic solution $\tilde{x}(t)$, and $\tilde{x}(t)$ is global asymptotically stable. Solving (4.23), we have

$$\tilde{x}(t) = -(\theta + \alpha)\eta\left(\frac{W(t,0)\int_0^\tau W(\tau^+,\sigma)d\sigma}{1 - W(\tau,0)} + \int_0^t W(t,\sigma)d\sigma\right)$$

$$+ \left(1 - \frac{b_1\gamma_2}{1 + b_1}\right)\frac{W(t,0)W(\tau^+,\tau^+)}{(1 - W(\tau,0))} + \frac{b_1\gamma_2}{1 + b_1}\sum_{0 < n\tau < t} W(t,n\tau^+). \tag{4.24}$$

From (4.10) and (4.24), it is easy to see that

$$\tilde{x}(t) - \tilde{s}(t) \geq -(\theta + \alpha)\eta \left(\frac{W(t,0)\int_0^\tau W(\tau^+,\sigma)d\sigma}{1 - W(\tau,0)} + \int_0^t W(t,\sigma)d\sigma \right). \tag{4.25}$$

By $0 < \eta \leq (1/2)(1-1/R^*)(\tilde{s}(t)/M(\theta+\alpha))$, $M = \max_{0 \leq t \leq \tau}\{W(t,0)\int_0^\tau W(\tau^+,\sigma)d\sigma/(1-W(\tau,0))+ \int_0^t W(t,\sigma)d\sigma\}$, and $(W(t,0)\int_0^\tau W(\tau^+,\sigma)d\sigma/(1 - W(\tau,0)) + \int_0^t W(t,\sigma)d\sigma)(1/M) \leq 1$, we obtain

$$-(\theta + \alpha)\eta \left(\frac{W(t,0)\int_0^\tau W(\tau^+,\sigma)d\sigma}{1 - W(\tau,0)} + \int_0^t W(t,\sigma)d\sigma \right) \geq -\frac{1}{2}\left(1 - \frac{1}{R^*}\right)\tilde{s}(t), \tag{4.26}$$

namely,

$$\tilde{x}(t) \geq \frac{1}{2}\left(1 + \frac{1}{R^*}\right)\tilde{s}(t). \tag{4.27}$$

The comparison principle and the global asymptotically stable of $\tilde{x}(t)$ imply that there exists a positive constant $T_5 > 0$ such that

$$V(t) \geq \frac{1}{2}\left(1 + \frac{1}{R^*}\right)\tilde{s}(t), \quad \forall t > t_1 + T_5. \tag{4.28}$$

From (4.28) and the second equation of (4.22), we can see that

$$i' \geq \left[\frac{\beta}{2}\left(1 + \frac{1}{R^*}\right)\tilde{s}(t) - (\theta + \alpha)\right]i + (\alpha - \beta)i^2. \tag{4.29}$$

Consider the following equation:

$$y' = \left[\frac{\beta}{2}\left(1 + \frac{1}{R^*}\right)\tilde{s}(t) - (\theta + \alpha)\right]y + (\alpha - \beta)y^2, \quad t \neq n\tau,$$

$$y(n\tau^+) = \frac{1 + b_1(1 - \gamma_1)}{1 + b_1}y(n\tau), \quad t = n\tau. \tag{4.30}$$

Let $z = y^{-1}$, then we have

$$z' = \left[(\theta + \alpha) - \frac{\beta}{2}\left(1 + \frac{1}{R^*}\right)\tilde{s}(t)\right]z + (\beta - \alpha), \quad t \neq n\tau,$$

$$z(n\tau^+) = \frac{1 + b_1}{1 + b_1(1 - \gamma_1)}z(n\tau), \quad t = n\tau. \tag{4.31}$$
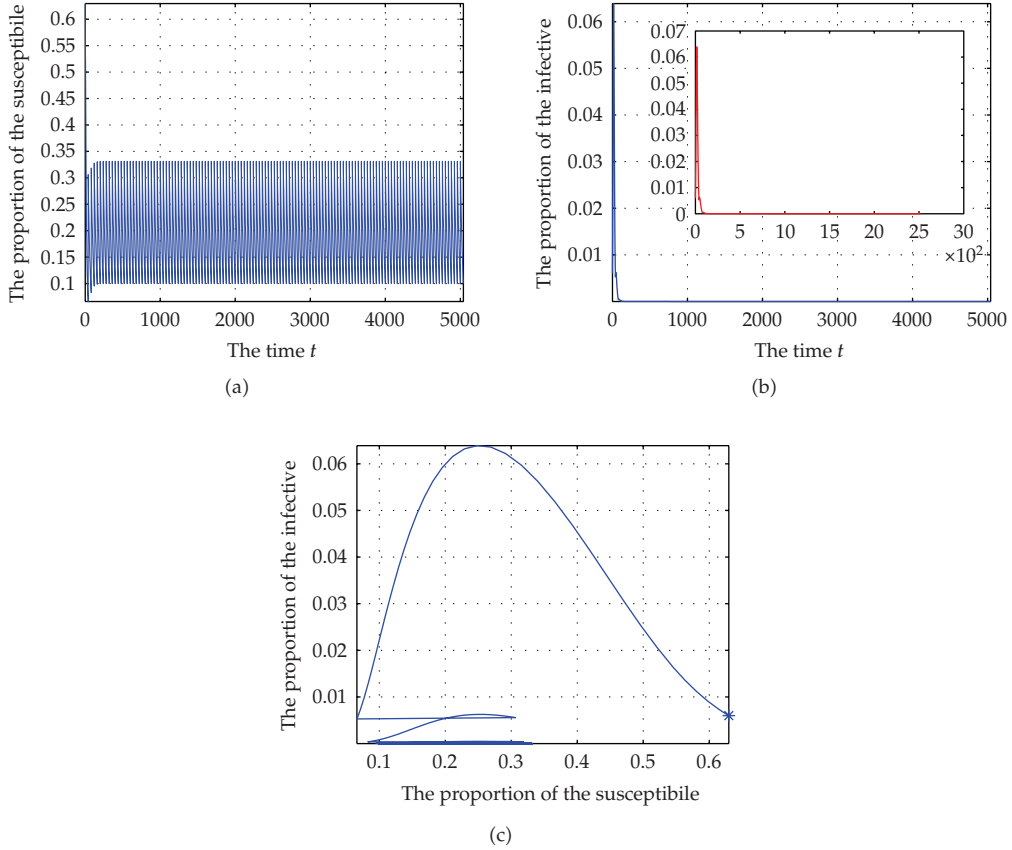
(a)

(b)

(c)

Figure 1: The time series and the orbits of the system (2.4) with $R^* < 1$. (a) and (b) show the time series for susceptible and infective, respectively. (c) shows the orbits $s - i$ plane.

By the same method of Lemma 4.1, we can get a conclusion that $\tilde{z}(t) = (\beta - \alpha)(W(t,0) \times \int_0^\tau W(\tau^+, \sigma)d\sigma/(1 - W(\tau, 0)) + \int_0^t W(t, \sigma)d\sigma) + (1 + b_1)/(1 + b_1(1 - \gamma_1))(W(t, 0)W(\tau, \tau)/(1 - W(\tau, 0)))$ is global asymptotically stable. Thus system (4.30) has a unique positive τ-periodic solution $\tilde{y}(t)$, and $\tilde{y}(t)$ is global asymptotically stable,

$$\lim_{t \to \infty} |y(t) - \tilde{y}(t)| = 0. \tag{4.32}$$

From (4.32) we see that there exists a positive constant $T_6 > 0$ such that

$$y(t) > \rho \equiv \frac{1}{2} \min_{t_1 \le t \le t_1 + \tau} \tilde{y}(t) > 0, \quad \forall t > t_1 + T_6. \tag{4.33}$$

Let $T^* = \max\{T_5, T_6\}$, and define $\rho = \min\{\rho, \eta \exp(-(\theta + \alpha)\tau)\}$. If $t_2 - t_1 < T^*$, from the second equation of (4.22), we have the inequality
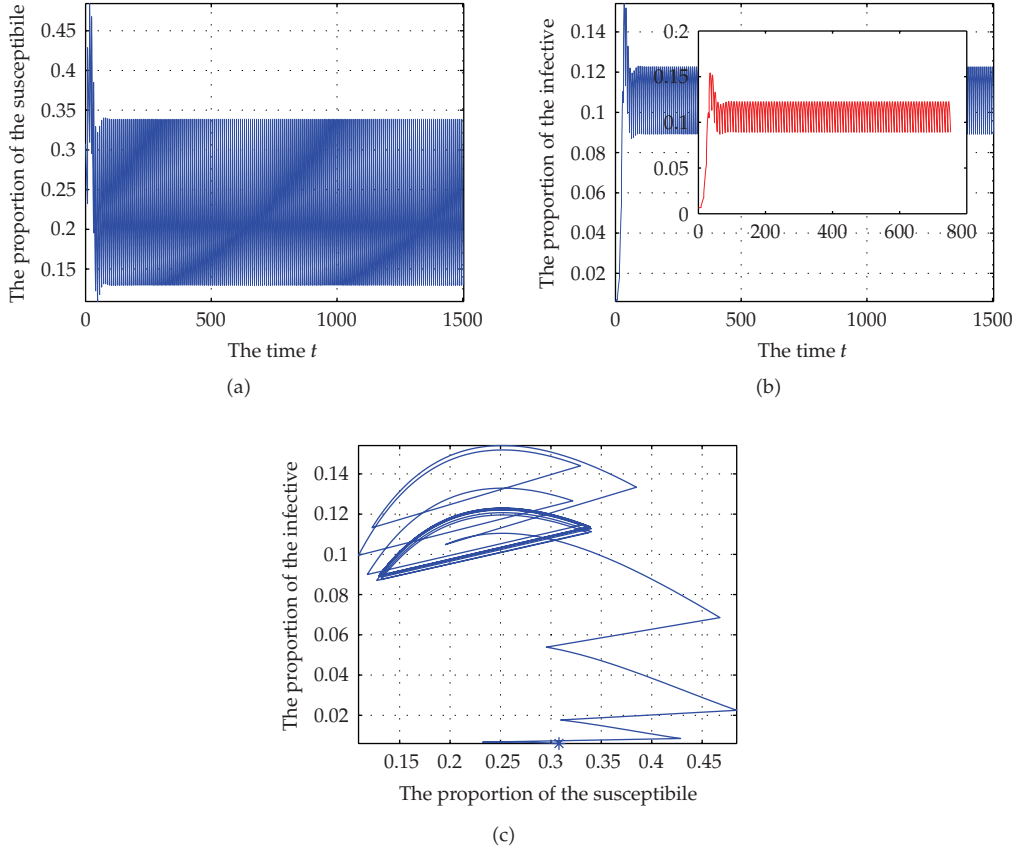
$$i'(t) \ge -(\theta + \alpha)i, \tag{4.34}$$

(a)

(b)

(c)

Figure 2: The time series and the orbits of the system (2.4) with $R^* > 1$. (a) and (b) show the time series for susceptible and infective, respectively. (c) shows the orbits $s - i$ plane.

and the comparison principle implies that $i(t) \geq \eta \exp\{-(\theta + \alpha)(t - t_1)\} \geq \eta \exp\{-(\theta + \alpha)T^*\}$, that is, $i(t) \geq \rho$ for all $t \in (t_1, t_2)$.

If $t_2 - t_1 > T^*$, we divide the interval $[t_1, t_2]$ into two subintervals $[t_1, t_1 + T^*]$ and $[t_1 + T^*, t_2]$, $i(t) \geq \rho$ is obvious in the interval $[t_1, t_1 + T^*]$. In the interval $[t_1 + T^*, t_2]$, we have the inequality (4.29) and (4.33). The comparison principle shows that $i(t) \geq y(t) \geq \rho \geq \rho$ for $t \in [t_1 + T^*, t_2]$. The analysis above is the independent of the selection of interval $[t_1, t_2]$, and the choice of ρ is the independent of the selection of interval independent of any positive solution of (2.4). The persistence is uniform to all positive solution. The proof is complete. $\square$

5. Numerical Simulation

For the birth pulses of SIR model with standard incidence, we know that the periodic infection-free solution is global asymptotically stable if the basic reproductive number $R^* < 1$. The periodic infection-free solution is unstable if the basic reproductive number $R^* > 1$, in this case, the disease will be uniform persistent. Here we do computer simulation to give a geometric impression on our results. In all simulation unit was set to unity (scaled to unity).

In Figure 1, we show the case report with the outcome of the system (2.4) when the basic reproductive number $R^* < 1$. The parameters are chosen as $p = 0.03$, $\beta = 0.8$, $\alpha = 0.002$,

$\theta = 0.2$, $b_1 = 0.4$, $\gamma_1 = 0.86$, $\gamma_2 = 0.9$, and $\tau = 40$. The three Figures 1(a), 1(b), and 1(c) in have the same initial value as $s(0) = 0.6296$, $i(0) = 0.006$. We fixed $p = 0.03$ and changed parameter τ. Figures 1(a), 1(b), and 1(c) show the solutions for $\tau = 40$ and $R^* = 0.9876$. It suggests that the disease-free periodic solution is global asymptotically stable when $R^* < 1$.

Figure 2 shows that the positive periodic solution is existence when $R^* \to 1^+$, moreover, the positive periodic solution is global asymptotically stable. The parameters are chosen as $p = 0.005$, $\beta = 0.8$, $\alpha = 0.002$, $\theta = 0.2$, $b_1 = 0.4$, $\gamma_1 = 0.32$, $\gamma_2 = 0.9$, and $\tau = 8$. Here we choose the initial value of (2.4) $s(0) = 0.3080$, $i(0) = 0.006$. In Figures 2(a), 2(b) and 2(c) with $\tau = 15$ and $R^* = 1.0236$, the other parameters are the same as Figure 1.

6. Discussion

In this paper, we have investigated the dynamic behaviors of the classical SIR model. A distinguishing feature of the SIR model considered here is that the epidemic incidence is standard form instead of bilinear form as usual. The basic reproductive number R^* is identified and is established as a sharp threshold parameter. If $R^* < 1$, the infection-free periodic solution is global asymptotically stable which implies that the disease will extinct. If $R^* > 1$, the disease will have uniform persistence and lead to epidemic disease eventually. Our theoretical results are confirmed by numerical results.

When we are modeling the transmission of some infectious diseases with pulse birth, the introduction of the standard incidence can make the model more realistic, whereas it raises hardness of the problem at the same time. For example, we attempted to achieve the global stability of infection-free periodic solution in Section 3, and we found it is impossible to prove $\lim_{t \to \infty} i(t) = 0$ by traditional techniques. In this case, we made the conclusion by making use of the new variable $v = s + r$. The SIR epidemic model with pulse birth is one of the simple and important epidemic models.

At the same time, the paper assumes the susceptible, infectious, and recovered have the same birth rate. But by the effect of the infectious diseases to the fertility of the infected, we can also assume that the susceptible and recovered have the same birth rate, which is higher than the infectious birth rate. Furthermore, we can assume that the infectious has a lower fertility than the susceptible and recovered due to the effect of the disease. So a distinguishing feature of the model considered here is that the susceptible, infectious, and recovered have different birth rates, which makes the model more realistic. For the above models we could get the similar condition for the stability of the infection-free periodic solution.

Acknowledgments

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