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Research Article

Successive Iteration and Positive Solutions for Nonlinear m-Point Boundary Value Problems on Time Scales

Yanbin Sang

Department of Mathematics, North University of China, Taiyuan, Shanxi 030051, China

Correspondence should be addressed to Yanbin Sang, sangyanbin@126.com

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We study the existence of positive solutions for a class of m-point boundary value problems on time scales. Our approach is based on the monotone iterative technique and the cone expansion and compression fixed point theorem of norm type. Without the assumption of the existence of lower and upper solutions, we do not only obtain the existence of positive solutions of the problem, but also establish the iterative schemes for approximating the solutions.

1. Introduction

The purpose of this paper is to consider the existence of positive solutions and establish the corresponding iterative schemes for the following m-point boundary value problems (BVP) on time scales:

$$\begin{gathered} u^{\Delta\nabla}(t) + f(t, u(t)) = 0, \quad t \in [0,1]_{\mathbf{T}}, \\ \beta u(0) - \gamma u^{\Delta}(0) = 0, \qquad u(1) = \sum_{i=1}^{m-2} \alpha_i u(\xi_i), \quad m \geq 3. \end{gathered} \tag{1.1}$$

The study of dynamic equations on time scales goes back to its founder Hilger [1], and is a new area of still fairly theoretical exploration in mathematics. Motivating the subject is the notion that dynamic equations on time scales can build bridges between continuous and discrete mathematics. Some preliminary definitions and theorems on time scales can be found in [2–5] which are good references for the calculus of time scales.

In recent years, by applying fixed point theorems, the method of lower and upper solutions and critical point theory, many authors have studied the existence of positive solutions for two-point and multipoint boundary value problems on time scales, for details, see [2, 3, 6–18] and references therein. However, to the best of our knowledge, there are few papers which are concerned with the computational methods of the multipoint boundary value problems on time scales. We would like to mention some results of Sun and Li [16], Aykut Hamel and Yoruk [12], Anderson and Wong [10], Wang et al. [18], and Jankowski [13], which motivated us to consider the BVP (1.1).

In [16], Sun and Li considered the existence of positive solutions of the following dynamic equations on time scales:

$$\begin{gathered} u^{\Delta\nabla}(t) + a(t)f(t,u(t)) = 0, \quad t \in (0,T), \\ \beta u(0) - \gamma u^{\Delta}(0) = 0, \quad \alpha u(\eta) = u(T), \end{gathered} \tag{1.2}$$

where $\beta, \gamma \geq 0$, $\beta+\gamma > 0$, $\eta \in (0, \rho(T))$, $0 < \alpha < T/\eta$. They obtained the existence of single and multiple positive solutions of (1.2) by using a fixed point theorem and the Leggett-Williams fixed point theorem, respectively.

Very recently, in [12], Aykut Hamel and Yoruk discussed the following dynamic equation on time scales:

$$\begin{gathered} u^{\Delta\nabla}(t) + f(t,u(t)) = 0, \quad t \in [0,1] \subset \mathbf{T}, \\ \beta u(0) - \gamma u^{\Delta}(0) = 0, \qquad u(1) = \sum_{i=1}^{m-2} \alpha_i u(\xi_i), \quad m \geq 3. \end{gathered} \tag{1.3}$$

They obtained some results for the existence of at least two and three positive solutions to the BVP (1.3) by using fixed point theorems in a cone and the associated Green's function.

In related paper, in [10], Anderson and Wong studied the second-order time scale semipositone boundary value problem with Sturm-Liouville boundary conditions or multipoint conditions as in

$$\begin{gathered} \left(pu^{\Delta}\right)^{\nabla}(t) + \lambda f(t,u(t)) = 0, \quad t \in (a,b]_{\mathbf{T}}, \\ \alpha u(a) - \beta\left(pu^{\Delta}\right)(a) = 0, \qquad \gamma u^{\sigma}(b) + \delta\left(pu^{\Delta}\right)(b) = 0, \quad \text{or} \\ \alpha u(a) - \beta\left(pu^{\Delta}\right)(a) = \sum_{i=1}^{n} \phi_i\left(pu^{\Delta}\right)(t_i), \qquad \gamma u^{\sigma}(b) + \delta\left(pu^{\Delta}\right)(b) = \sum_{i=1}^{n} \psi_i\left(pu^{\Delta}\right)(t_i). \end{gathered} \tag{1.4}$$

On the other hand, the method of lower and upper solutions has been effectively used for proving the existence results for dynamic equations on time scales. In [18], Wang et al. considered a method of generalized quasilinearization, with even-order k ($k \geq 2$) convergence, for the BVP

$$
\begin{gathered}
-\left(p(t)x^{\Delta}\right)^{\nabla} + q(t)x^{\sigma} = f(t,x^{\sigma}) + g(t,x^{\sigma}), \quad t \in [a,b]_{\mathbf{T}}, \\
\tau_1 x(\rho(a)) - \tau_2 x^{\Delta}(\rho(a)) = 0, \quad x(\sigma(b)) - \tau_3 x(\eta) = 0.
\end{gathered}
\tag{1.5}
$$

The main contribution in [18] relaxed the monotone conditions on $f^{(i)}(t,x), g^{(i)}(t,x)$ $(1 < i < k)$ including a more general concept of upper and lower solution in mathematical biology, so that the high-order convergence of the iterations was ensured for a larger class of nonlinear functions on time scales.

In [13], Jankowski investigated second-order differential equations with deviating arguments on time scales of the form

$$
\begin{gathered}
-x^{\Delta\Delta}(t) = f(t,x(t),x(\alpha(t))) \equiv (Fx)(t), \quad t \in J, \\
x(0) = k_1 \in \mathbf{R}, \qquad x(T) = k_2 \in \mathbf{R}.
\end{gathered}
\tag{1.6}
$$

They formulated sufficient conditions, under which such problems had a minimal and a maximal solution in a corresponding region bounded by upper-lower solutions.

We would also like to mention the result of Yao [19]. In [19], Yao considered the positive solutions to the following two classes of nonlinear second-order three-point boundary value problems:

$$
\begin{gathered}
u''(t) + f\left(t,u(t),u'(t)\right) = 0, \quad 0 \leq t \leq 1, \\
u''(t) + f(t,u(t)) = 0, \quad 0 \leq t \leq 1, \\
u(0) = 0, \quad \alpha u(\eta) = u(1),
\end{gathered}
\tag{1.7}
$$

where both η and α are given constants satisfying $0 < \eta < 1$, $0 < \alpha < 1/\eta$. By improving the classical monotone iterative technique of Amann [20], two successive iterative schemes were established for the BVP (1.7). It was worth stating that the first terms of the iterative schemes were constant functions or simple functions. We note that Ma et al. [21] and Sun et al. [22, 23] have also applied the similar methods to p-laplacian boundary value problems with $\mathbf{T} = \mathbf{R}$.

In this paper, we will investigate the iterative and existence of positive solutions for the BVP (1.1), by considering the "heights" of the nonlinear term f on some bounded sets and applying monotone iterative techniques on a Banach space, we do not only obtain the existence of positive solutions for the BVP (1.1), but also give the iterative schemes for approximating the solutions. We should point out that the monotone condition imposed on the nonlinear term f will play crucial role in obtaining the iterative schemes for approximating the solutions. In essence, we combine the method of lower and upper solutions with the cone expansion and compression fixed point theorem of norm type. The idea of this paper comes from Yao [19, 24, 25].

Let $\mathbf{T}$ be a time scale which has the subspace topology inherited from the standard topology on $\mathbf{R}$. For each interval I of $\mathbf{R}$, we define $I_{\mathbf{T}} = I \cap \mathbf{T}$.

For the remainder of this article, we denote the set of continuous functions from $[0,1]_{\mathbf{T}}$ to $\mathbf{R}$ by $C([0,1]_{\mathbf{T}},\mathbf{R})$. Let $C([0,1]_{\mathbf{T}},\mathbf{R})$ be endowed with the ordering $x \le y$ if $x(t) \le y(t)$ for all $t \in [0,1]_{\mathbf{T}}$, and $\|u\| = \max_{t\in[0,1]_{\mathbf{T}}}|u(t)|$ is defined as usual by maximum norm. The $C([0,1]_{\mathbf{T}},\mathbf{R})$ is a Banach space.

Throughout this paper, we will assume that the following assumptions are satisfied:

(H_1) $\beta,\gamma \ge 0,\ 0 < \beta+\gamma \le 1,\ \xi_i \in (0,\rho(1))_{\mathbf{T}}$ for $i = 1,2,\dots,m-2$ with $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < \rho(1)$;

(H_2) $\sum_{i=1}^{m-2}\alpha_i \in (0,1)$ with $\alpha_i \in (0,+\infty)$ for $i = 1,2,\dots,m-2$ and $d = \beta(1-\sum_{i=1}^{m-2}\alpha_i\xi_i) + \gamma(1-\sum_{i=1}^{m-2}\alpha_i) > 0$;

(H_3) $f : [0,1]_{\mathbf{T}} \times [0,+\infty) \to [0,+\infty)$ is continuous.

2. Preliminaries and Several Lemmas

To prove the main results in this paper, we will employ several lemmas. These lemmas are based on the linear boundary value problem

$$\begin{gathered} u^{\Delta\nabla}(t) + h(t) = 0, \quad t \in [0,1]_{\mathbf{T}}, \\ \beta u(0) - \gamma u^{\Delta}(0) = 0, \qquad u(1) = \sum_{i=1}^{m-2}\alpha_i u(\xi_i), \quad m \ge 3. \end{gathered} \tag{2.1}$$

Lemma 2.1 (see [12]). *It holds $d = \beta(1-\sum_{i=1}^{m-2}\alpha_i\xi_i) + \gamma(1-\sum_{i=1}^{m-2}\alpha_i) \ne 0$; then the Green's function for the BVP*

$$\begin{gathered} -u^{\Delta\nabla}(t) = 0, \quad t \in [0,1]_{\mathbf{T}}, \\ \beta u(0) - \gamma u^{\Delta}(0) = 0, \qquad u(1) = \sum_{i=1}^{m-2}\alpha_i u(\xi_i), \quad m \ge 3, \end{gathered} \tag{2.2}$$

is given by

$$G(t,s)=\frac{1}{d}\begin{cases}(\beta s+\gamma)\left[(1-t)-\displaystyle\sum_{j=1}^{m-2}a_j(\xi_j-t)\right], \\ \quad \text{if } 0\le t\le 1,\ 0\le s\le \xi_1,\ s\le t; \\ (\beta s+\gamma)(1-t)-\displaystyle\sum_{j=i}^{m-2}\alpha_j(\xi_j-t)(\beta s+\gamma)+\sum_{j=1}^{i-1}\alpha_j(\beta\xi_j+\gamma)(t-s), \\ \quad \text{if } \xi_{r-1}\le t\le \xi_r,\ 2\le r\le m-1,\ \xi_{i-1}\le s\le \xi_i,\ 2\le i\le r,\ s\le t; \\ (\beta t+\gamma)\left[(1-s)-\displaystyle\sum_{j=i}^{m-2}\alpha_j(\xi_j-s)\right], \\ \quad \text{if } \xi_{r-1}\le t\le \xi_r,\ 2\le r\le m-2,\ \xi_{i-1}\le s\le \xi_i,\ r\le i\le m-2,\ t\le s; \\ (\beta t+\gamma)(1-s), \\ \quad \text{if } 0\le t\le 1,\ \xi_{m-2}\le s\le 1,\ t\le s.\end{cases} \tag{2.3}$$

Here for the sake of convenience, one writes $\sum_{i=m_1}^{m_2} h(i)=0$ *for* $m_2<m_1$.

Lemma 2.2 (see [12]). *Assume that conditions* (H_1)–(H_3) *are satisfied. Then*

(i) $G(t,s)\ge 0$ *for* $t,s\in[0,1]_{\mathbf{T}}$;

(ii) *there exist a number* $\Psi\in(0,1)$ *and a continuous function* $\theta:[0,1]_{\mathbf{T}}\to\mathbf{R}^+$ *such that*

$$\begin{gathered}G(t,s)\le\theta(s)\quad \text{for } t,s\in[0,1]_{\mathbf{T}}, \\ G(t,s)\ge\Psi\theta(s)\quad \text{for } t\in[\xi_1,1]_{\mathbf{T}},\ s\in[0,1]_{\mathbf{T}},\end{gathered} \tag{2.4}$$

where

$$\begin{aligned}\theta(s)&=\max\left\{1,\frac{\sum_{i=1}^{m-2}\alpha_i}{\xi_1}\right\}\frac{(\beta s+\gamma)(1-s)}{d}, \\ \Psi&=\frac{1}{\max\left\{1,\sum_{i=1}^{m-2}\alpha_i/\xi_1\right\}} \\ &\quad\times\min_{2\le s\le m-2}\left\{(\beta\xi_1+\gamma)(1-\xi_{m-2}),\sum_{j=1}^{m-2}\alpha_j(1-\xi_j),\sum_{j=1}^{s-1}\alpha_j(\beta\xi_j+\gamma)+\sum_{j=s}^{m-2}\alpha_j(1-\xi_j)\right\}.\end{aligned} \tag{2.5}$$

Let $\mathbf{B} = C([0,1]_{\mathbf{T}}, \mathbf{R})$. It is easy to see that the BVP (1.1) has a solution $u = u(t)$ if and only if u is a fixed point of the operator equation:

$$Tu(t) = \int_0^1 G(t,s) f(s,u(s)) \nabla s. \tag{2.6}$$

Denote

$$K = \left\{ u \in \mathbf{B} : u \text{ is nonnegative, concave, and } \min_{t\in[\xi_1,1]_{\mathbf{T}}} u(t) \geq \Psi \|u\| \right\}, \tag{2.7}$$

where Ψ is the same as in Lemma 2.2. By [12, Lemma 3.1], we can obtain that $T(K) \subset K$ and $T : K \to K$ is completely continuous.

3. Successive Iteration and One Positive Solution for (1.1)

For notational convenience, we denote

$$A = \left[\max_{t\in[0,1]_{\mathbf{T}}} \int_0^1 G(t,s) \nabla s \right]^{-1}, \qquad B = \left[\max_{t\in[0,1]_{\mathbf{T}}} \int_{\xi_1}^1 G(t,s) \nabla s \right]^{-1}. \tag{3.1}$$

Constants A, B are not easy to compute explicitly. For convenience, we can replace A by A', B by B', where

$$A' = \left[\int_0^1 \theta(s) \nabla s \right]^{-1}, \qquad B' = \left[\Psi \int_{\xi_1}^1 \theta(s) \nabla s \right]^{-1}. \tag{3.2}$$

Obviously, $0 < A' < A < B < B'$.

Theorem 3.1. *Assume (H_1)–(H_3) hold, and there exist two positive numbers a, b with $b < a$ such that*

(C_1) $\max\{f(t,a) : t \in [0,1]_{\mathbf{T}}\} \leq aA$, $\min\{f(t,\Psi b) : t \in [\xi_1,1]_{\mathbf{T}}\} \geq bB$;

(C_2) $f(t,u_1) \leq f(t,u_2)$ *for any* $t \in [0,1]_{\mathbf{T}}$, $0 \leq u_1 \leq u_2 \leq a$.

Then the BVP (1.1) *has at least one positive solution u^* such that $b \leq \|u^*\| \leq a$ and $\lim_{n\to\infty} T^n \tilde{u} = u^*$, that is, $T^n\tilde{u}$ converges uniformly to u^* in $[0,1]_{\mathbf{T}}$, where $\tilde{u}(t) \equiv a$, $t \in [0,1]_{\mathbf{T}}$.*

Remark 3.2. The iterative scheme in Theorem 3.1 is $u_1 = T\tilde{u}$, $u_{n+1} = Tu_n$, $n = 1, 2, \ldots$. It starts off with constant function $\tilde{u}(t) \equiv a$, $t \in [0,1]_{\mathbf{T}}$.

Proof of Theorem 3.1. Denote $K[b,a] = \{u \in K : b \leq \|u\| \leq a\}$. If $u \in K[b,a]$, then

$$\max_{t\in[0,1]_{\mathbf{T}}} u(t) \leq a, \qquad \min_{t\in[\xi_1,1]_{\mathbf{T}}} u(t) \geq \Psi\|u\| \geq b\Psi. \tag{3.3}$$

By Assumptions (C_1) and (C_2), we have

$$\begin{aligned} &f(t,u(t)) \le f(t,a) \le aA, \quad t \in [0,1]_{\mathbf{T}}; \\ &f(t,u(t)) \ge f(t,b\Psi) \ge bB, \quad t \in [\xi_1,1]_{\mathbf{T}}. \end{aligned} \tag{3.4}$$

It follows that

$$\begin{aligned} \|Tu\| &= \max_{t\in[0,1]_{\mathbf{T}}} \left| \int_0^1 G(t,s) f(s,u(s)) \nabla s \right| \\ &\le aA \max_{t\in[0,1]_{\mathbf{T}}} \int_0^1 G(t,s) \nabla s = a; \\ \|Tu\| &\ge \max_{t\in[0,1]_{\mathbf{T}}} \int_{\xi_i}^1 G(t,s) f(s,u(s)) \nabla s \\ &\ge bB \max_{t\in[0,1]_{\mathbf{T}}} \int_{\xi_i}^1 G(t,s) \nabla s = b. \end{aligned} \tag{3.5}$$

Thus, we assert that $T : K[b,a] \to K[b,a]$.

Let $\widetilde{u}(t) \equiv a,\ t \in [0,1]_{\mathbf{T}}$, then $\widetilde{u} \in K[b,a]$. Let $u_1 = T\widetilde{u}$, then $u_1 \in K[b,a]$. Denote $u_{n+1} = Tu_n, n = 1,2,\ldots$. Since $T(K[b,a]) \subset K[b,a]$, we have $u_n \in T(K[b,a]) \subset K[b,a], n = 1,2,\ldots$. Since T is completely continuous, we assert that $\{u_n\}_{n=1}^{\infty}$ has a convergent subsequence $\{u_{n_k}\}_{k=1}^{\infty}$ and there exists $u^* \in K[b,a]$, such that $u_{n_k} \to u^*$.

Now, since $u_1 \in K[b,a]$, we have

$$u_1(t) \le \|u_1\| \le a = \widetilde{u}(t), \quad t \in [0,1]_{\mathbf{T}}. \tag{3.6}$$

By Assumption (C_2),

$$\begin{aligned} u_2(t) &= Tu_1(t) \\ &= \int_0^1 G(t,s) f(s,u_1(s)) \nabla s \\ &\le \int_0^1 G(t,s) f(s,\widetilde{u}(s)) \nabla s \\ &= T\widetilde{u}(t) = u_1(t). \end{aligned} \tag{3.7}$$

By the induction, then

$$u_{n+1}(t) \le u_n(t), \quad t \in [0,1]_{\mathbf{T}},\ n = 1,2,\ldots. \tag{3.8}$$

Hence, $T^n\widetilde{u} = u_n \to u^*$. Applying the continuity of T and $u_{n+1} = Tu_n$, we get $Tu^* = u^*$. Since $\|u^*\| \ge b > 0$ and u^* is a nonnegative concave function, we conclude that u^* is a positive solution of the BVP (1.1). □

Corollary 3.3. *Assume that* (H_1)–(H_3) *hold, and the following conditions are satisfied:*

(C_1') $\overline{\lim}_{l\to 0}\min_{t\in[\xi_1,1]_{\mathbf{T}}} f(t,l)/l > \Psi^{-1}B, \underline{\lim}_{l\to+\infty}\max_{t\in[0,1]_{\mathbf{T}}} f(t,l)/l < A$ *(particularly,* $\lim_{l\to 0}\min_{t\in[\xi_1,1]_{\mathbf{T}}} f(t,l)/l = +\infty, \lim_{l\to+\infty}\max_{t\in[0,1]_{\mathbf{T}}} f(t,l)/l = 0$*);*

(C_2') $f(t,u_1) \le f(t,u_2)$ *for any* $t \in [0,1]_{\mathbf{T}},\ u_1 \le u_2, u_1,\ u_2 \in [0,+\infty)$.

Then the BVP (1.1) *has at least one positive solution* $u^* \in K$ *and there exists a positive number* a *such that* $\lim_{n\to\infty} T^n\widetilde{u} = u^*$, *that is,*

$$\lim_{n\to\infty}\sup_{t\in[0,1]_{\mathbf{T}}} |T^n\widetilde{u}(t) - u^*(t)| = 0, \tag{3.9}$$

where $\widetilde{u}(t) \equiv a,\ t \in [0,1]_{\mathbf{T}}$.

Theorem 3.4. *Assume* (H_1)–(H_3) *hold, and the following conditions are satisfied:*

(D_1) *there exists* $a > 0$ *such that* $f(t,\cdot) : [0,a] \to (0,+\infty)$ *is nondecreasing for any* $t \in [0,1]_{\mathbf{T}}$ *and* $\max\{f(t,a) : t \in [0,1]_{\mathbf{T}}\} \le aA$*;*

(D_2) $f(t,0) > 0$, *for any* $t \in [0,1]_{\mathbf{T}}$.

Then the BVP (1.1) *has one positive solution* u^* *such that* $0 < \|u^*\| \le a$ *and* $\lim_{n\to\infty} T^n 0 = u^*$, *that is,* $T^n 0$ *converges uniformly to* u^* *in* $[0,1]_{\mathbf{T}}$. *Furthermore, if there exists* $0 < \omega < 1$ *such that*

$$\left|f(t,l_2) - f(t,l_1)\right| \le \omega A|l_2 - l_1|, \quad t \in [0,1]_{\mathbf{T}},\ 0 \le l_1,\ l_2 \le a. \tag{3.10}$$

Then $\|T^{n+1}0 - u^*\| \le \omega^n/(1-\omega)\|T0\|$.

Proof. Denote $K[0,a] = \{u \in K : \|u\| \le a\}$. Similarly to the proof of Theorem 3.1, we can know that $T : K[0,a] \to K[0,a]$. Let $\widetilde{u}_1 = T0$, then $\widetilde{u}_1 \in K[0,a]$. Denote $\widetilde{u}_{n+1}(t) = T\widetilde{u}_n, n = 1,2,\ldots$. Copying the corresponding proof of Theorem 3.1, we can prove that

$$\widetilde{u}_{n+1}(t) \ge \widetilde{u}_n(t), \quad t \in [0,1]_{\mathbf{T}},\ n = 1,2,\ldots. \tag{3.11}$$

Since T is completely continuous, we can get that there exists $u^* \in K[0,a]$ such that $\widetilde{u}_n \to u^*$. Applying the continuity of T and $\widetilde{u}_{n+1}(t) = T\widetilde{u}_n$, we can obtain that $Tu^* = u^*$. We note that $f(t,0) > 0$, for all $t \in [0,1]_{\mathbf{T}}$, it implies that the zero function is not the solution of the problem (1.1). Therefore, u^* is a positive solution of (1.1).

Now, since

$$\left|f(t,l_2) - f(t,l_1)\right| \le \omega A|l_2 - l_1|, \quad t \in [0,1]_{\mathbf{T}},\ 0 \le l_1,\ l_2 \le a. \tag{3.12}$$

If $u_1, u_2 \in K[0,a]$ and $u_2(t) \geq u_1(t),\ t \in [0,1]_{\mathbf{T}}$, then

$$\begin{aligned}
\|Tu_2 - Tu_1\| &= \max_{t\in[0,1]_{\mathbf{T}}} \left| \int_0^1 G(t,s)\left[f(s,u_2(s)) - f(s,u_1(s))\right]\nabla s \right| \\
&\leq \omega A \max_{t\in[0,1]_{\mathbf{T}}} \int_0^1 G(t,s)|u_2(s) - u_1(s)|\nabla s \\
&\leq \omega A \|u_2 - u_1\| A^{-1} \\
&= \omega \|u_2 - u_1\|.
\end{aligned} \tag{3.13}$$

Hence, we can deduce that

$$\begin{gathered}
\|\widetilde{u}_{n+2} - \widetilde{u}_{n+1}\| = \|T\widetilde{u}_{n+1} - T\widetilde{u}_n\| \leq \omega^n \|T0 - 0\| = \omega^n \|T0\|, \\
\|\widetilde{u}_{n+k+2} - \widetilde{u}_{n+1}\| \leq \left(\omega^{n+k} + \omega^{n+k-1} + \cdots + \omega^n\right)\|T0\| < \frac{\omega^n}{1-\omega}\|T0\|.
\end{gathered} \tag{3.14}$$

It implies that

$$\left\| T^{n+1}0 - u^* \right\| \leq \frac{\omega^n}{1-\omega}\|T0\|. \tag{3.15}$$

The proof is complete. □

4. Existence of n positive solutions

Theorem 4.1. *Assume* (H_1)–(H_3) *hold, and there exist* $2n$ *positive numbers* $a_1, \dots, a_n, b_1, \dots, b_n$ *with* $b_1 < a_1 < b_2 < a_2 < \cdots < b_n < a_n$ *such that*

- (E_1) $\max\{f(t,a_i) : t \in [0,1]_{\mathbf{T}}\} \leq a_i A$, $\min\{f(t,\Psi b_i) : t \in [\xi_1, 1]_{\mathbf{T}}\} \geq b_i B,\ i = 1,2,\dots,n;$
- (E_2) $f(t,u_1) \leq f(t,u_2)$ *for any* $t \in [0,1]_{\mathbf{T}},\ 0 \leq u_1 \leq u_2 \leq a_n$.

Then the BVP (1.1) *has* n *positive solutions* $u_i^*, i = 1,2,\dots,n$ *such that* $b_i \leq \|u_i^*\| \leq a_i$ *and* $\lim_{n\to\infty} T^n \widetilde{u}_i = u_i^*$, *that is,*

$$\lim_{n\to\infty} \sup_{t\in[0,1]_{\mathbf{T}}} \left| T^n \widetilde{u}_i(t) - u_i^*(t) \right| = 0, \tag{4.1}$$

where $\widetilde{u}_i(t) \equiv a_i,\ t \in [0,1]_{\mathbf{T}},\ i = 1,2,\dots,n.$

Corollary 4.2. *Assume that* (H_1)–(H_3) *and* (C_1')–(C_2') *hold, and the following condition is satisfied*

(E') *there exist* $2(n-1)$ *positive numbers* $a_1 < b_2 < a_2 < \cdots < b_{n-1} < a_{n-1} < b_n$ *such that*

$$\begin{gathered}
\max\left\{f(t,a_i) : t \in [0,1]_{\mathbf{T}}\right\} < a_i A, \quad i = 1,\dots,n-1, \\
\min\left\{f(t,\Psi b_i) : t \in [\xi_i, 1]_{\mathbf{T}}\right\} > b_i B, \quad i = 2,\dots,n.
\end{gathered} \tag{4.2}$$

Then the BVP (1.1) *has n positive solutions $u_i^*, i = 1, 2, \ldots, n$, and there exists a positive number a_n with $a_n > b_n$ such that $\lim_{n\to\infty} T^n \widetilde{u}_i = u_i^*$, where $\widetilde{u}_i(t) \equiv a_i,\ t \in [0,1]_{\mathbf{T}},\ i = 1, 2, \ldots, n$.*

5. Examples

Example 5.1. Let $\mathbf{T} = [0, 1/3] \bigcup [1/2, 1]$. Considering the following BVP:

$$\begin{gathered} u^{\Delta\nabla}(t) + f(t,u) = 0, \quad t \in [0,1]_{\mathbf{T}}, \\ u(0) = 0, \qquad u(1) = \frac{1}{8}u\left(\frac{1}{3}\right) + \frac{1}{6}u\left(\frac{1}{2}\right), \end{gathered} \tag{5.1}$$

where $f(t,u) = (200/109)u^2 + 1$, it is easy to check that $f(t,0) = 1 > 0$, for any $t \in [0,1]_{\mathbf{T}}$.

Further calculations give us

$$\begin{aligned} d &= \frac{7}{8}, \\ A' &= \left[\int_0^1 \theta(s)\nabla s\right]^{-1} \\ &= \left[\frac{8}{7}\int_0^1 s(1-s)\nabla s\right]^{-1} \\ &= \left\{\frac{8}{7}\left[\int_0^{1/3} s(1-s)ds + \int_{\rho(1/2)}^{1/2} s(1-s)\nabla s + \int_{1/2}^1 s(1-s)ds\right]\right\}^{-1} \\ &= \frac{567}{109}. \end{aligned} \tag{5.2}$$

Choose $a = 1$, it is easy to check that $f(t,\cdot) : [0,1]_{\mathbf{T}} \to [0,+\infty)$ is nondecreasing for any $t \in [0,1]_{\mathbf{T}}$ and

$$\max_{t\in[0,1]_{\mathbf{T}}} f(t,1) = \frac{200}{109} + 1 \le 1 \cdot \frac{567}{109}. \tag{5.3}$$

Let $\widetilde{u}_0(t) \equiv 0$, for $n = 0, 1, 2, \ldots$, we have

$$\begin{aligned} \widetilde{u}_{n+1}(t) = &-\int_0^t (t-s)\left(\frac{200}{109}\widetilde{u}_n(s) + 1\right)\nabla s + \frac{8}{7}t\int_0^1 (1-s)\left(\frac{200}{109}\widetilde{u}_n(s) + 1\right)\nabla s \\ &- \frac{8}{7}t\left[\frac{1}{8}\int_0^{1/3}\left(\frac{1}{3} - s\right)\left(\frac{200}{109}\widetilde{u}_n(s) + 1\right)\nabla s + \frac{1}{6}\int_0^{1/2}\left(\frac{1}{2} - s\right)\left(\frac{200}{109}\widetilde{u}_n(s) + 1\right)\nabla s\right]. \end{aligned} \tag{5.4}$$

By Theorem 3.4, the BVP (5.1) has one positive solution u^* such that $0 < \|u^*\| \leq 1$ and $T^n 0 \to u^*$. On the other hand, for any $0 \leq u_1,\ u_2 \leq 1$, we have

$$\begin{aligned}\left|f(u_1)-f(u_2)\right| &= \frac{200}{109}\left|u_1^2-u_2^2\right| \\ &\leq \frac{400}{109}|u_1-u_2| = \frac{567}{109}\cdot\frac{400}{567}|u_1-u_2| \\ &= \frac{400}{567}A'|u_1-u_2|.\end{aligned} \tag{5.5}$$

Then,

$$\left\|T^{n+1}0-u^*\right\| \leq \frac{(400/567)^n}{1-400/567}\|T0\| = \frac{567}{167}\left(\frac{400}{567}\right)^n\|T0\|. \tag{5.6}$$

The first and second terms of this scheme are as follows:

$$\begin{gathered}\widetilde{u}_0(t) = 0, \\ \widetilde{u}_1(t) = \begin{cases}-\dfrac{t^2}{2}+\dfrac{199t}{378}, & t\in\left[0,\dfrac{1}{3}\right], \\ -\dfrac{t^2}{2}+\dfrac{199t}{378}+\dfrac{1}{72}, & t\in\left[\dfrac{1}{2},1\right].\end{cases}\end{gathered} \tag{5.7}$$

Now, we compute the third term of this scheme.

For $t \in [0, 1/3]$,

$$\widetilde{u}_2(t) = \frac{175}{327}t^4-\frac{9950}{61803}t^3-\frac{t^2}{2}+\left(\frac{6720175}{70084602}+\frac{199}{378}\right)t. \tag{5.8}$$

For $t \in [1/2, 1]$,

$$\widetilde{u}_2(t) = \frac{175}{327}t^4-\frac{9950}{61803}t^3-\frac{503t^2}{981}+\left(\frac{6720175}{70084602}+\frac{100097}{185409}\right)t. \tag{5.9}$$

Example 5.2. Let $\mathbf{T} = \{0\}\bigcup\{1/3^n : n \in \mathbf{N_0}\}$. Considering the BVP on $\mathbf{T}$,

$$\begin{gathered}u^{\Delta\nabla}+\sqrt{u(t)} = 0, \quad t\in[0,1]_{\mathbf{T}}, \\ u^{\Delta}(0) = 0, \qquad u(1) = \frac{1}{3}u\left(\frac{1}{9}\right)+\frac{1}{9}u\left(\frac{1}{3}\right).\end{gathered} \tag{5.10}$$

By direct computation, we can get

$$d = \frac{5}{9}, \qquad A' = \frac{9}{16}, \qquad B' = \frac{81}{8}, \qquad \Psi = \frac{5}{54}. \tag{5.11}$$

Choose $a = 100,\ b = 1/1875$, it is easy to see that the nonlinear term $f(t,u) = f(u) = \sqrt{u(t)}$ possesses the following properties

(a) $f : [0,1]_{\mathbf{T}} \times [0,+\infty) \to [0,+\infty)$ is continuous;

(b) $f(t,u_1) \le f(t,u_2)$ for any $t \in [0,1]_{\mathbf{T}}$ and $0 \le u_1 \le u_2 \le 100$;

(c) $\max\{f(t,100) : t \in [0,1]_{\mathbf{T}}\} = \sqrt{100} \le aA' = 100 \times 9/16$, $\min\{f(t,\Psi b) : t \in [\xi_1,1]_{\mathbf{T}}\} = \sqrt{5/54 \times 1/1875} > bB' = 1/1875 \times 81/8$.

By Theorem 3.1, the BVP (5.10) has one positive solution u^* such that $1/1875 \le \|u^*\| \le 100$ and $\lim_{n\to\infty} T^n \widetilde{u} = u^*$, where $\widetilde{u}(t) \equiv 100,\ t \in [0,1]_{\mathbf{T}}$. Let $u_0(t) \equiv 100,\ t \in [0,1]_{\mathbf{T}}$. For $n = 0,1,2,\ldots$, we have

$$\begin{aligned} u_{n+1} &= \int_0^1 G(t,s) f(s,u_n(s)) \nabla s \\ &= -\int_0^t (t-s)\sqrt{u_n(s)}\nabla s + \frac{9}{5}\int_0^1 (1-s)\sqrt{u_n(s)}\nabla s \\ &\quad - \frac{3}{5}\int_0^{1/9}\left(\frac{1}{9}-s\right)\sqrt{u_n(s)}\nabla s - \frac{1}{5}\int_0^{1/3}\left(\frac{1}{3}-s\right)\sqrt{u_n(s)}\nabla s. \end{aligned} \tag{5.12}$$

Remark 5.3. By Theorems 3.1, 3.2, and 3.3 in [12, 16, 17], the existence of positive solutions for the BVP (5.1) can be obtained, however, we cannot give a way to find the solutions which will be useful from an application viewpoint. Therefore, Theorem 3.1 improves and extends the main results of [12, 16, 17]. On the other hand, in Example 5.2, since $f(0) = 0$, we cannot obtain the above mentioned results by use of Theorem 3.4, thus, Theorems 3.1 and 3.4 do not contain each other.

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