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Research Article

Global Dynamics of Discrete Competitive Models with Large Intrinsic Growth Rates

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The global dynamics of discrete competitive model of Lotka-Volterra type with two species is considered. Earlier works have shown that the unique positive equilibrium is globally attractive under the assumption that the intrinsic growth rates of the two competitive species are less than 1+ln 2, and further the unique positive equilibrium is globally asymptotically stable under the stronger condition that the intrinsic growth rates of the two competitive species are less than or equal to 1. We prove that the system can also be globally asymptotically stable when the intrinsic growth rates of the two competitive species are greater than 1 and globally attractive when the intrinsic growth rates of the two competitive species are greater than 1 + ln 2.

1. Introduction

In this paper, we further consider the global dynamics of discrete Lotka-Volterra model

$$\begin{aligned} x_1(n+1) &= x_1(n)\exp(r_1 - a_{11}x_1(n) - a_{12}x_2(n)),\\ x_2(n+1) &= x_2(n)\exp(r_2 - a_{21}x_1(n) - a_{22}x_2(n)), \end{aligned} \tag{1.1}$$

with positive initial conditions $x_1(0), x_2(0) > 0$. Here $x_i(n)$ $(i = 1,2)$ is the density of population i at nth generation, r_i $(i = 1,2)$ is the intrinsic growth rate of population i. a_{ij} $(i,j = 1,2)$ represents the intensity of intraspecific competition or interspecific action of the two species. It is assumed that r_i and a_{ij} $(i,j = 1,2)$ are positive constants throughout this paper.

The discrete Lotka-Volterra models have wide applications in applied mathematics. They were first established in biomathematical background and then have proved to be a rich source in analysis for the dynamical systems in different research fields such as physics, chemistry, and economy [1].

Model (1.1) was first introduced in May [2] then was investigated by many authors [3–14]. The difference system (1.1) is autonomous, some of the works mentioned above are the nonautonomous case of (1.1). Many results about the global dynamics of (1.1) such as permanence, global attractivity, global asymptotical stability have been obtained. For example, it is shown in [10] that (1.1) can be globally asymptotically stable when $r_i \le 1$ $(i = 1, 2)$. And from [3] we know that (1.1) can be globally attractive under the assumption that $r_i < 1 + \ln 2$ $(i = 1, 2)$.

It is well known that for the single-species Logistic model

$$x(n+1) = x(n)\exp[r - ax(n)], \tag{1.2}$$

the positive equilibrium $x^* = r/a$ is globally asymptotically stable if and only if $r \le 2$ and there exists periodic cycles when $r > 2$. When $r \ge 3.13$, (1.2) exhibits chaotic behavior (e.g., see [15]). That is, the global dynamics of (1.2) is very complex when the intrinsic growth rate r is large. It is clear that (1.1) is a coupling of two equations described by (1.2). And it is proved in [16] that (1.1) also exhibits chaotic behavior when $r_i = r \ge 3.13$ $(i = 1, 2)$. Therefore, questions can be proposed naturally: how to investigate the global dynamics of (1.1) when $1 + \ln 2 < r_i < 3.13$ $(i = 1, 2)$? Can model (1.1) be also globally asymptotically stable when $r_i > 1$ $(i = 1, 2)$? Can model (1.1) be globally attractive when $r_i > 1 + \ln 2$ $(i = 1, 2)$?

Our aim of this paper is to obtain some global dynamics of (1.1) when the intrinsic growth rate r_i $(i = 1, 2)$ is large $(r_i \ge 1, i = 1, 2)$ and give answers to the above questions. First we obtain permanent result of (1.1), then global attractivity of (1.1) is obtained through geometrical properties of (1.1). Last, we obtain the global asymptotical stability of (1.1) by applying a theorem in [10]. After these theoretical results for (1.1) obtained, we give numerical examples to confirm these theoretical results and show that our theoretical results imply that (1.1) can be globally attractive when $r_i > 1 + \ln 2$ $(i = 1, 2)$ and (1.1) can also be globally asymptotically stable when $r_i > 1$ $(i = 1, 2)$.

The paper is organized as follows. We give some preliminaries in Section 2. In Section 3, we discuss permanence, global attractivity, and global asymptotical stability of (1.1) theoretically. Numerical examples are given in Section 4 to show the feasibility of the assumptions of the main results and on the other hand, to show that our main results can be applied to larger intrinsic growth rates than earlier works. Brief conclusion is given in Section 5.

2. Preliminaries

A pair of sequences of real positive numbers $\{x_1(n), x_2(n)\}_{n=1}^{\infty}$ that satisfies (1.1) is a positive solution of (1.1). It is clear that the solutions of system (1.1) with initial values $x_1(0) > 0, x_2(0) > 0$ are positive ones, which is accordant with the biological background of (1.1). That is, we only need to investigate the dynamics of system (1.1) in the plane domain

$$G = \{(x, y) \mid x \ge 0,\ y \ge 0\}. \tag{2.1}$$

If a solution of (1.1) is a pair of real constants $\{x_1, x_2\}$, then it is an equilibrium of (1.1).

Lemma 2.1. *Assume that*

$$D \triangleq a_{11}a_{22} - a_{12}a_{21} \neq 0, \tag{2.2}$$

then system (1.1) *has four equilibria.*

Proof. Solving the following scalar equation system:

$$x_1 = x_1 \exp(r_1 - a_{11}x_1 - a_{12}x_2), \qquad x_2 = x_2 \exp(r_2 - a_{21}x_1 - a_{22}x_2). \tag{2.3}$$

We obtain that the four equilibria of system (1.1) are

$$(0,0), \left(\frac{r_1}{a_{11}}, 0\right), \left(0, \frac{r_2}{a_{22}}\right), (x_1^*, x_2^*), \tag{2.4}$$

respectively. Here and the following, we denote

$$\begin{aligned} x_1^* &\triangleq \frac{D_1}{D}, \qquad x_2^* \triangleq \frac{D_2}{D}, \\ D_1 &\triangleq r_1a_{22} - r_2a_{12}, \qquad D_2 \triangleq r_2a_{11} - r_1a_{21}. \end{aligned} \tag{2.5}$$

□

The equilibria $(0,0), (r_1/a_{11}, 0)$ and $(0, r_2/a_{22})$ are the so-called "boundary equilibrium." If we further assume that

$$D_1 > 0, \qquad D_2 > 0, \tag{2.6}$$

which implies that $D > 0$, then (x_1^*, x_2^*) is the unique positive equilibrium of (1.1).

Lemma 2.2. *Denote* $f(x,y) = x\exp(r_1 - a_{11}x - a_{12}y)$, $r_1 \geq 1$, *then the maximum* $\widehat{M}_1$ *of* $f(x,y)$ *in the domain*

$$G_1 \triangleq \{(x,y) \mid x \geq 0, y \geq 0,\ r_1 - a_{11}x - a_{12}y \leq 0,\ r_2 - a_{21}x - a_{22}y \leq 0\} \tag{2.7}$$

is

(1) *if* $a_{22} \leq D_1$ *or* $a_{22}/D \geq r_1/a_{21}$, *then* $\widehat{M}_1 = \max\{(r_2/a_{21})\exp(r_1 - a_{11}(r_2/a_{21})), D_1/D\}$,

(2) *if* $D_1/D < a_{22}/D < r_1/a_{21}$, *then* $\widehat{M}_1 = (a_{22}/D)\exp((D_1/a_{22}) - 1)$.

Denote $g(x,y) = y\exp(r_2 - a_{21}x - a_{22}y)$, $r_2 \geq 1$, *then the maximum* $\widehat{M}_2$ *of* $g(x,y)$ *in domain* G_1 *is*

(1) *if* $a_{11} \leq D_2$ *or* $a_{11}/D \geq r_2/a_{12}$, *then* $\widehat{M}_2 = \max\{(r_1/a_{12})\exp(r_2 - a_{22}(r_1/a_{12})), D_2/D\}$,

(2) *if* $D_2/D < a_{11}/D < r_2/a_{12}$, *then* $\widehat{M}_2 = (a_{11}/D)\exp((D_2/a_{11}) - 1)$.

Proof. For any fixed x (or y), let $y \to +\infty$ (or $x \to +\infty$) we get $f(x,y) \to 0$. Note that

$$\lim_{x\to+\infty, y\to+\infty} f(x,y) = 0, \tag{2.8}$$

therefore, the maximum of $f(x,y)$ in domain G_1 exists. Direct computation gives $\widehat{M}_1$, we omit the details. Similarly, $\widehat{M}_2$ exists and its value can be obtained directly. □

Lemma 2.3. (1) *If F: $R^+ = [0,+\infty) \to R$ is monotonously increasing, then for each positive sequence $\{y(n)\}_{n=1}^{\infty}$,*

$$\limsup_{n\to\infty} F(y(n)) = F\left(\limsup_{n\to\infty} y(n)\right), \qquad \liminf_{n\to\infty} F(y(n)) = F\left(\liminf_{n\to\infty} y(n)\right). \tag{2.9}$$

If F: $R^+ = [0,+\infty) \to R$is monotonously decreasing, then for each positive sequence $\{y(n)\}_{n=1}^{\infty}$,

$$\limsup_{n\to\infty} F(y(n)) = F\left(\liminf_{n\to\infty} y(n)\right), \qquad \liminf_{n\to\infty} F(y(n)) = F\left(\limsup_{n\to\infty} y(n)\right). \tag{2.10}$$

(2) *For any positive sequences $\{y(n)\}_{n=1}^{\infty}$, $\{z(n)\}_{n=1}^{\infty}$ one has*

$$\begin{aligned}\limsup_{n\to\infty} \left[y(n)z(n)\right] &\le \limsup_{n\to\infty} y(n) \limsup_{n\to\infty} z(n),\\ \liminf_{n\to\infty} \left[y(n)z(n)\right] &\ge \liminf_{n\to\infty} y(n) \liminf_{n\to\infty} z(n).\end{aligned} \tag{2.11}$$

Proof. One can refer to [17] for the proof of this lemma. □

Next we give some definitions that will be used in this paper.

Definition 2.4. System (1.1) is *permanent* if there exist positive constants m and M such that

$$m \le \liminf_{n\to\infty} x_i(n) \le \limsup_{n\to\infty} x_i(n) \le M, \quad i = 1, 2. \tag{2.12}$$

Definition 2.5. System (1.1) is *strongly persistent* if each positive solution of (1.1) satisfies

$$\liminf_{n\to\infty} x_i(n) > 0, \quad i = 1, 2. \tag{2.13}$$

Definition 2.6. The solution $\{x_1(n), x_2(n)\}$ of system (1.1) with initial values $x_1(0) > 0$, $x_2(0) > 0$ is said to be *stable* if for any $\varepsilon > 0$, there is a $\delta > 0$ such that if $\max\{|x_1(0) - \overline{x}_1(0)|, |x_2(0) - \overline{x}_2(0)|\} < \delta$, we have $|x_1(n) - \overline{x}_1(n)| + |x_2(n) - \overline{x}_2(n)| < \varepsilon$ for all positive integers n, where $\{\overline{x}_1(n), \overline{x}_2(n)\}$ is the solution of (1.1) with initial values $\overline{x}_1(0) > 0, \overline{x}_2(0) > 0$.

Definition 2.7. Suppose that $\{x_1^*, x_2^*\}$ is the positive equilibrium solution of (1.1). If for each positive solution $\{\overline{x}_1(n), \overline{x}_2(n)\}$ of system (1.1), we have $\max\{|x_1^* - \overline{x}_1(n)|, |x_2^* - \overline{x}_2(n)|\} \to 0$ as $n \to \infty$, we say (1.1) is *globally attractive* or the equilibrium $\{x_1^*, x_2^*\}$ of (1.1) is *globally attractive*.

Definition 2.8. The positive equilibrium solution $\{x_1^*, x_2^*\}$ of (1.1) or system (1.1) is said to be *globally asymptotically stable* if this equilibrium is stable and globally attractive.

The following lemma can be found in [10].

Lemma 2.9. *Consider the following difference system:*

$$x_i(n+1) = x_i(n)\exp\left(r_i(n) - \sum_{j=1}^{n} a_{ij}(n)x_j(n)\right), \quad i = 1,2,\ldots,l. \tag{2.14}$$

Assume that

(i) *there exist positive constant ν and positive constants c_i such that*

$$c_i a_{ii}(n) > \sum_{j=1, j\neq i}^{l} c_j \left|a_{ji}(n)\right| + \nu, \quad i = 1,2,\ldots,l, \tag{2.15}$$

for all large n;

(ii) *system* (2.14) *is strongly persistent;*

(iii) *for any positive solution $\{x_1(n), x_2(n), \ldots, x_l(k)\}$ of system* (2.14)*,*

$$a_{ii}(n)x_i(n) \leq 1, \quad i = 1,2,\ldots,l, \tag{2.16}$$

for all large n.

Then system (2.14) *is globally asymptotically stable.*

3. Main Results

In this section, we will obtain the permanence, global attractivity, and global asymptotical stability of system (1.1) when $r_i \geq 1$ $(i = 1,2)$.

Lemma 3.1. *For every positive solution $\{x_1(n), x_2(n)\}$ of system* (1.1) *with initial values $x_1(0) > 0$, $x_2(0) > 0$, one has*

$$\limsup_{n\to\infty} x_1(n) \leq s_1, \qquad \limsup_{n\to\infty} x_2(n) \leq s_2, \tag{3.1}$$

where

$$s_1 = \frac{1}{a_{11}}\exp(r_1 - 1), \qquad s_2 = \frac{1}{a_{22}}\exp(r_2 - 1). \tag{3.2}$$

Proof. Note that

$$\begin{aligned} x_1(n+1) &= x_1(n)\exp(r_1 - a_{11}x_1(n) - a_{12}x_2(n)) \\ &= x_1(n)\exp(r_1 - a_{11}x_1(n))\exp(-a_{12}x_2(n)), \quad \exp(-a_{12}x_2(n)) \leq 1 \end{aligned} \tag{3.3}$$

for all n, therefore

$$x_1(n+1) \le x_1(n)\exp(r_1 - a_{11}x_1(n)) \le \frac{1}{a_{11}}\exp(r_1 - 1). \tag{3.4}$$

Here we used

$$\max_{x\ge 0} x\exp(r - ax) = \frac{1}{a}\exp(r-1) \tag{3.5}$$

for $a > 0$. Then

$$\limsup_{n\to\infty} x_1(n) \le \frac{1}{a_{11}}\exp(r_1 - 1). \tag{3.6}$$

The proof of

$$\limsup_{n\to\infty} x_2(n) \le \frac{1}{a_{22}}\exp(r_2 - 1) \tag{3.7}$$

is similar. □

Lemma 3.2. *Assume that* $\{x_1(n), x_2(n)\}$ *is the solution of* (1.1) *with initial values* $x_1(0) > 0, x_2(0) > 0$ *and*

$$\frac{1}{a_{11}}\exp(r_1 - 1) < \frac{r_2}{a_{21}}, \qquad \frac{1}{a_{22}}\exp(r_2 - 1) < \frac{r_1}{a_{12}}, \tag{3.8}$$

then

$$\liminf_{n\to\infty} x_1(n) \ge t_1 > 0, \qquad \liminf_{n\to\infty} x_2(n) \ge t_2 > 0, \tag{3.9}$$

where

$$\begin{aligned} t_1 &= \frac{r_1}{a_{11}}\left(1 - \frac{a_{12}}{r_1}s_2\right)\exp\left(r_1 - a_{12}s_2 - \frac{a_{11}}{r_1}s_1\right), \\ t_2 &= \frac{r_2}{a_{22}}\left(1 - \frac{a_{21}}{r_2}s_1\right)\exp\left(r_2 - a_{21}s_1 - \frac{a_{22}}{r_2}s_2\right), \end{aligned} \tag{3.10}$$

and s_1, s_2 *are the same as in Lemma 3.1.*

Proof. The proof of this lemma is similar to that of [3, Proposition 2]. □

Note that $t_1 > 0, t_2 > 0$, therefore, system (1.1) is permanent from Lemmas 3.1 and 3.2 under the assumption (3.8).

Theorem 3.3. *Assume that* (3.8) *is satisfied then system* (1.1) *with initial values* $x_1(0) > 0, x_2(0) > 0$ *is permanent.*

Theorem 3.4. *Assume that* (2.6), *and* (3.8) *hold. The coefficients of* (1.1) *satisfy* $r_i \geq 1$ $(i = 1, 2)$ *and*

(1) $a_{22} \leq D_1$ *or* $a_{22}/D \geq r_1/a_{21}$,

(2) $a_{11} \leq D_2$ *or* $a_{11}/D \geq r_2/a_{12}$.

Further, assume that

$$\widehat{M}_1 \leq \frac{D_1}{D} \neq \frac{1}{a_{11}}, \qquad \widehat{M}_2 \leq \frac{D_2}{D} \neq \frac{1}{a_{22}}, \tag{3.11}$$

where $\widehat{M}_1$ *and* $\widehat{M}_2$ *are defined in Lemma 2.2. Then the unique positive equilibrium* (x_1^*, x_2^*) *of* (1.1) *is globally attractive.*

Proof. If we denote

$$\begin{aligned} l_1 &= \liminf_{n\to\infty} x_1(n), \qquad l_2 = \liminf_{n\to\infty} x_2(n), \\ L_1 &= \limsup_{n\to\infty} x_1(n), \qquad L_2 = \limsup_{n\to\infty} x_2(n) \end{aligned} \tag{3.12}$$

for any positive solution $\{x_1(n), x_2(n)\}$ of system (1.1) with initial conditions $x_1(0) > 0, x_2(0) > 0$, we have

$$0 < l_1 \leq L_1 < +\infty, \qquad 0 < l_2 \leq L_2 < +\infty \tag{3.13}$$

from Theorem 3.3 and Definition 2.4. Moreover,

$$l_1 \geq l_1 \exp(r_1 - a_{11}L_1 - a_{12}L_2), \tag{3.14}$$

$$l_2 \geq l_2 \exp(r_2 - a_{21}L_1 - a_{22}L_2), \tag{3.15}$$

$$L_1 \leq L_1 \exp(r_1 - a_{11}l_1 - a_{12}l_2), \tag{3.16}$$

$$L_2 \leq L_2 \exp(r_2 - a_{21}l_1 - a_{22}l_2) \tag{3.17}$$

from (2) of Lemma 2.3.

Note (3.13), the inequalities (3.14)–(3.17) can be written as follows:

$$r_1 - a_{11}l_1 - a_{12}l_2 \geq 0, \tag{3.18}$$

$$r_2 - a_{21}l_1 - a_{22}l_2 \geq 0, \tag{3.19}$$

$$r_1 - a_{11}L_1 - a_{12}L_2 \leq 0, \tag{3.20}$$

$$r_2 - a_{21}L_1 - a_{22}L_2 \leq 0. \tag{3.21}$$

From (3.18)–(3.21), it is clear that (l_1, l_2) lies in the domain

$$G_2 = \{(x,y) \mid x \geq 0,\ y \geq 0,\ r_1 - a_{11}x - a_{12}y \geq 0,\ r_2 - a_{21}x - a_{22}y \geq 0\}, \tag{3.22}$$

while (L_1, L_2) lies in the domain G_1 (see (2.7)). Therefore, from (3.11) and Lemma 2.2, the maximum of $f(x,y) = x\exp(r_1 - a_{11}x - a_{12}y)$ in domain G_1 is D_1/D, the maximum of $g(x,y) = y\exp(r_2 - a_{21}x - a_{22}y)$ in domain G_1 is D_2/D. Then

$$L_1 \leq \frac{D_1}{D}, \qquad L_2 \leq \frac{D_2}{D}. \tag{3.23}$$

But in domain G_1, only the point $(x_1^*, x_2^*) = (D_1/D, D_2/D)$ satisfies these two inequalities, then

$$L_1 = \frac{D_1}{D} = x_1^*, \qquad L_2 = \frac{D_2}{D} = x_2^*. \tag{3.24}$$

At this point, we claim that

$$l_1 = \frac{D_1}{D} = x_1^*, \qquad l_2 = \frac{D_2}{D} = x_2^*. \tag{3.25}$$

Note (3.11), we must consider the following four cases to prove claim (3.25):

Case (i): $D_1/D < 1/a_{11},\ D_2/D < 1/a_{22}$,

Case (ii): $D_1/D < 1/a_{11},\ D_2/D > 1/a_{22}$,

Case (iii): $D_1/D > 1/a_{11},\ D_2/D < 1/a_{22}$,

Case (iv): $D_1/D > 1/a_{11},\ D_2/D > 1/a_{22}$.

It is easy to verify that the function $h(x) = x\exp(r - ax), a > 0$ is monotonously increasing when $0 < x < 1/a$ and monotonously decreasing when $x > 1/a$. With this fact and Lemma 2.3, the proof of the claim is given as below.

Case (i)

We rearrange the two equations of (1.1) as

$$\begin{aligned} x_1(n+1) &= x_1(n)\exp(r_1 - a_{11}x_1(n))\exp(-a_{12}x_2(n)), \\ x_2(n+1) &= x_2(n)\exp(r_2 - a_{22}x_2(n))\exp(-a_{21}x_1(n)). \end{aligned} \tag{3.26}$$

Note that

$$L_1 = x_1^* = \frac{D_1}{D} < \frac{1}{a_{11}}, \qquad L_2 = x_2^* = \frac{D_2}{D} < \frac{1}{a_{22}}, \tag{3.27}$$

we have $x_1(n) < 1/a_{11}, x_2(n) < 1/a_{22}$ for n sufficiently large. Then

$$\begin{aligned} l_1 &\geq l_1 \exp(r_1 - a_{11}l_1 - a_{12}L_2), \\ l_2 &\geq l_2 \exp(r_2 - a_{21}L_1 - a_{22}l_2). \end{aligned} \tag{3.28}$$

That is

$$r_1 - a_{11}l_1 - a_{12}L_2 \leq 0, \tag{3.29}$$

$$r_2 - a_{21}L_1 - a_{22}l_2 \leq 0. \tag{3.30}$$

The inequalities (3.24), (3.29) together with (3.30) imply that

$$l_1 \geq \frac{D_1}{D} = x_1^*, \qquad l_2 \geq \frac{D_2}{D} = x_2^*. \tag{3.31}$$

From (3.13) and (3.24), we get

$$l_1 = \frac{D_1}{D} = x_1^*, \qquad l_2 = \frac{D_2}{D} = x_2^*. \tag{3.32}$$

Case (ii)

Similarly, we have

$$l_1 \geq l_1 \exp(r_1 - a_{11}l_1 - a_{12}L_2), \tag{3.33}$$

$$l_2 \geq L_2 \exp(r_2 - a_{21}L_1 - a_{22}L_2). \tag{3.34}$$

From (3.13), (3.24), and (3.33), we get $l_1 = D_1/D = x_1^*$. And from (3.13), (3.24), and (3.34), $l_2 = L_2 = D_2/D = x_2^*$ follows.

The proof of Case (iii) is similar to that of Case (ii).

Case (iv)

We have

$$\begin{aligned} l_1 &\geq L_1 \exp(r_1 - a_{11}L_1 - a_{12}L_2), \\ l_2 &\geq L_2 \exp(r_2 - a_{21}L_1 - a_{22}L_2). \end{aligned} \tag{3.35}$$

Therefore,

$$l_1 = \frac{D_1}{D} = x_1^*, \qquad l_2 = \frac{D_2}{D} = x_2^* \tag{3.36}$$

are consequent from (3.13), (3.24), and (3.35).

The proof of claim (3.25) is completed. Note (3.24) and (3.25),

$$\lim_{n\to\infty} x_1(n) = x_1^*, \qquad \lim_{n\to\infty} x_2(n) = x_2^* \tag{3.37}$$

for any positive solution $\{x_1(n), x_2(n)\}$ of system (1.1). That is, (1.1) is globally attractive according to Definition 2.7. □

Theorem 3.5. *Assume that the assumptions of Theorem 3.4 are satisfied, moreover,*

$$\frac{D_1}{D} < \frac{1}{a_{11}}, \qquad \frac{D_2}{D} < \frac{1}{a_{22}}, \tag{3.38}$$

then the unique positive equilibrium of system (1.1) *is globally asymptotically stable.*

Proof. From Theorem 3.3, system (1.1) is strongly persistent. That is, condition (ii) of Lemma 2.9 is satisfied.

$D_i > 0, i, j = 1, 2$ implies that $r_1a_{22} - r_2a_{12} > 0, r_2a_{11} - r_1a_{21} > 0$. Set $c_1 = r_2, c_2 = r_1$, it is clear that $c_1a_{11} > c_2a_{21}, c_2a_{22} > c_1a_{12}$. Thus, condition (i) of Lemma 2.9 is satisfied.

Let $\{x_1(n), x_2(n)\}$ be any positive solution of system (1.1). We show below that

$$a_{11}x_1(n+1) \le 1, \qquad a_{22}x_2(n+1) \le 1 \tag{3.39}$$

for all large n. By Theorem 3.4, we know that $(x_1^*, x_2^*) = (D_1/D, D_2/D)$ is globally attractive. That is

$$\lim_{n\to\infty} x_1(n) = \frac{D_1}{D}, \qquad \lim_{n\to\infty} x_2(n) = \frac{D_2}{D}. \tag{3.40}$$

From (3.38) we first select $\varepsilon > 0$, such that

$$\frac{D_1}{D} + \varepsilon < \frac{1}{a_{11}}, \qquad \frac{D_2}{D} + \varepsilon < \frac{1}{a_{22}}. \tag{3.41}$$

Further from (3.40), we know that there exists N_1 and N_2, such that

$$\begin{aligned} x_1(n+1) &< \frac{D_1}{D} + \varepsilon, \quad \text{for } n \ge N_1, \\ x_2(n+1) &< \frac{D_2}{D} + \varepsilon, \quad \text{for } n \ge N_2, \end{aligned} \tag{3.42}$$

respectively. Then denote $N = \max\{N_1, N_2\}$, we get

$$x_1(n+1) < \frac{D_1}{D} + \varepsilon < \frac{1}{a_{11}}, \qquad x_2(n+1) < \frac{D_2}{D} + \varepsilon < \frac{1}{a_{22}} \tag{3.43}$$

for $n \ge N$ from (3.41). That is, (3.39) is true for all sufficiently large n. Therefore, condition (iii) of Lemma 2.9 is satisfied. The proof is completed by applying Lemma 2.9. □

Theorem 3.6. *Assume that* (2.6)*, and* (3.8) *hold, the coefficients of* (1.1) *satisfy* $r_i \geq 1$ $(i = 1, 2)$ *and*

$$\frac{D_1}{D} < \frac{a_{22}}{D} < \frac{r_1}{a_{21}}, \qquad \frac{D_2}{D} < \frac{a_{11}}{D} < \frac{r_2}{a_{12}}, \tag{3.44}$$

$$\frac{a_{22}}{D} \exp\left(\frac{D_1}{a_{22}} - 1\right) < \frac{1}{a_{11}}, \qquad \frac{a_{11}}{D} \exp\left(\frac{D_2}{a_{11}} - 1\right) < \frac{1}{a_{22}}, \tag{3.45}$$

then the positive equilibrium of system (1.1) *is globally asymptotically stable.*

Proof. From the proof of Theorem 3.4, we know that (L_1, L_2) lies in domain G_1. Therefore, we obtain

$$\limsup_{n\to\infty} x_1(n) \leq \frac{a_{22}}{D} \exp\left(\frac{D_1}{a_{22}} - 1\right) < \frac{1}{a_{11}}, \qquad \limsup_{n\to\infty} x_2(n) \leq \frac{a_{11}}{D} \exp\left(\frac{D_2}{a_{11}} - 1\right) < \frac{1}{a_{22}} \tag{3.46}$$

from Lemma 2.2. That is, condition (iii) of Lemma 2.9 is satisfied. Conditions (i) and (ii) of Lemma 2.9 are also satisfied. Then the positive equilibrium of system (1.1) is globally asymptotically stable by applying Lemma 2.9. □

4. Numerical Examples

In this section, we give two numerical examples to show the feasibility of the assumptions of the results. The first example also shows that system (1.1) can be globally attractive when the intrinsic growth rates of the two species are greater than $1 + \ln 2 \doteq 1.6931$, and this result can be obtained by Theorem 3.4.

Example 4.1. Consider the following case of system (1.1):

$$r_1 = 1.95, \quad r_2 = 1.8, \quad a_{11} = 0.5, \quad a_{12} = 0.1, \quad a_{22} = 0.5, \quad a_{21} = 0.09, \tag{4.1}$$

then

$$\begin{gathered} D_1 = 0.7950, \qquad D_2 = 0.7245, \\ \frac{D_1}{D} = x_1^* = 3.2988, \qquad \frac{D_2}{D} = x_2^* = 3.0062, \\ \frac{1}{a_{11}} \exp(r_1 - 1) = 5.1714, \qquad \frac{1}{a_{22}} \exp(r_2 - 1) = 4.4511, \\ \frac{r_2}{a_{21}} = 20.0000, \qquad \frac{r_1}{a_{12}} = 19.5000, \qquad \frac{1}{a_{11}} = \frac{1}{a_{22}} = 2.0000, \\ \frac{r_2}{a_{21}} \exp\left(r_1 - a_{11}\frac{r_2}{a_{21}}\right) = 0.0064, \qquad \frac{r_1}{a_{12}} \exp\left(r_2 - a_{22}\frac{r_1}{a_{12}}\right) = 0.0069. \end{gathered} \tag{4.2}$$

We see that the conditions of Theorem 3.4 are satisfied. Therefore, the positive equilibrium of system (1.1) is globally attractive (see Figure 1). But this result cannot be

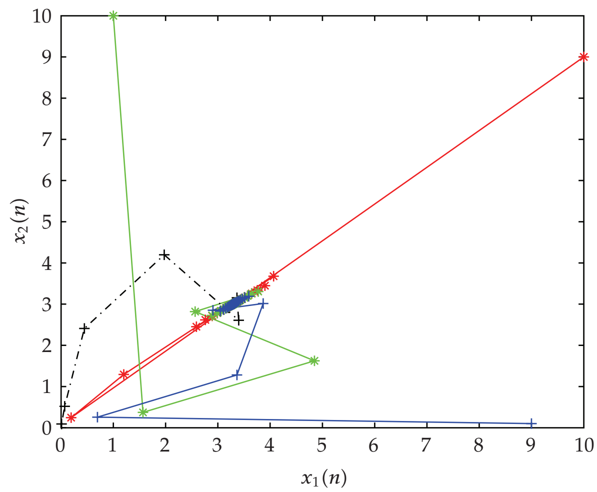

Figure 1: Solutions of system (1.1) with initial values $(x_1(0), x_2(0))$ =(10,9), (1,10), (0.01,0.09), (9,0.1), and $r_1 = 1.95, r_2 = 1.8$.

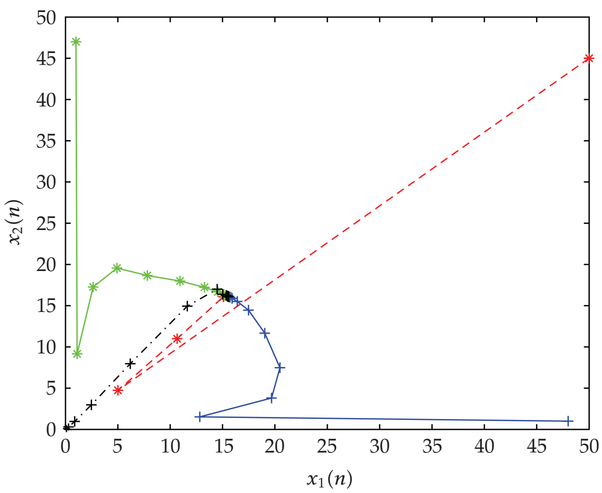

Figure 2: Solutions of system (1.1) with initial values $(x_1(0), x_2(0)) = (50,45), (1,47), (0.1,0.09), (48,0.1)$, and $r_1 = 1.1, r_2 = 1.2$.

obtained by [3, Theorem 3] when consider the autonomous case of this theorem(the model studied in [3] is nonautonomous). In fact, the condition of [3, Theorem 3] must satisfy $\exp(r_i - 1) - 1 < 1$ $(i = 1,2)$ when $r_i > 1$, that is, $r_i < 1 + \ln 2 \doteq 1.6931$ $(i = 1,2)$. In Example 4.1, $r_i > 1 + \ln 2$ $(i = 1,2)$.

The following example shows that system (1.1) can be globally asymptotically stable when the intrinsic growth rates of the two species are greater than 1, and this result can be obtained by Theorem 3.6.

Example 4.2. Consider the following case of system (1.1):

$$r_1 = 1.1, \quad r_2 = 1.2, \quad a_{11} = 0.05, \quad a_{12} = 0.02, \quad a_{22} = 0.06, \quad a_{21} = 0.015, \tag{4.3}$$

then

$$\begin{aligned}
D_1 &= 0.0420, & D_2 &= 0.0435,\\
x_1^* &= 15.5556, & x_2^* &= 16.1111,\\
\frac{1}{a_{11}}\exp(r_1-1) &= 22.1034, & \frac{1}{a_{22}}\exp(r_2-1) &= 20.3567,\\
\frac{r_2}{a_{21}} &= 80.0000, & \frac{r_1}{a_{12}} &= 55.0000,\\
\frac{r_2}{a_{21}}\exp\left(r_1-a_{11}\frac{r_2}{a_{21}}\right) &= 4.4019, & \frac{r_1}{a_{12}}\exp\left(r_2-a_{22}\frac{r_1}{a_{12}}\right) &= 6.7351,\\
\frac{a_{22}}{D}\exp\left(\frac{D_1}{a_{22}}-1\right) &= 16.4626, & \frac{a_{11}}{D}\exp\left(\frac{D_2}{a_{11}}-1\right) &= 16.2610,\\
\frac{1}{a_{11}} &= 20.0000, & \frac{1}{a_{22}} &= 16.6667.
\end{aligned} \tag{4.4}$$

It is clear that the conditions of Theorem 3.6 are satisfied. Thus by Theorem 3.6 the positive equilibrium of system (1.1) is globally asymptotically stable (see Figure 2).

Example 4.2 shows that our results improve [12, Theorem 3] by providing estimates for the smallness of r_1, r_2. The work in [10, Theorem 2] states that if $D_1 > 0, D_2 > 0, r_i \leq 1$ $(i = 1,2)$, then the positive equilibrium (x_1^*, x_2^*) is globally asymptotically stable. Thus the global asymptotical stability of system (1.1) in the case of Example 4.2 cannot be obtained by [10, Theorem 2] because of $r_i > 1$ $(i = 1,2)$.

5. Conclusion

In this paper, we further discuss the global dynamics of a discrete autonomous competitive model of Lotka-Volterra type. Sufficient conditions are obtained to guarantee the permanence, global attractivity, and global asymptotical stability of the system. These conditions are expressed by the coefficients of the model and can be easily verified. Numerical examples are also given to show the feasibility of the conditions.

Earlier works have shown that the system of this type can be globally attractive when the intrinsic growth rates of the two species are less than $1+\ln 2$ ([3], for single-species system see [18]). It is shown in [10] that the system can be globally asymptotically stable when the intrinsic growth rates of the two species are less than 1. In [16], it is shown that the system can exhibit chaotic behavior when the intrinsic growth rates of the two species are equal and greater than 3.13. But the global dynamics of the system is not clear enough when the intrinsic growth rates of the two species are greater than 1 and less than 3.13. We obtain that the system can also be globally asymptotically stable when the intrinsic growth rates of the two competitive species are greater than 1 and globally attractive when the intrinsic growth rates of the two competitive species are greater than $1 + \ln 2$.

For the global stability of the system, the following condition in Theorem 3.5:

$$\frac{D_1}{D} < \frac{1}{a_{11}}, \qquad \frac{D_2}{D} < \frac{1}{a_{22}}, \tag{5.1}$$

can be reduced to the following by direct computation:

$$r_1 < 1 + \frac{a_{12}}{a_{22}}, \qquad r_2 < 1 + \frac{a_{21}}{a_{11}}. \tag{5.2}$$

And the above inequalities imply that r_i $(i = 1, 2)$ can be greater than 1 while the system is globally asymptotically stable.

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