Research Article

Dynamical Properties of a Delay Prey-Predator Model with Disease in the Prey Species Only

Xiangyun Shi,[1] Xueyong Zhou,[1,2] and Xinyu Song[1]

[1] *College of Mathematics and Information Science, Xinyang Normal University, Xinyang, Henan 464000, China*
[2] *School of Mathematical Sciences, Nanjing Normal University, Nanjing, Jiangsu 210046, China*

Correspondence should be addressed to Xueyong Zhou, xueyongzhou@126.com

A three-dimensional ecoepidemiological model with delay is considered. We first investigate the existence and stability of the equilibria. We then study the effect of the time delay on the stability of the positive equilibrium. The existence of a Hopf bifurcation at the positive equilibrium is obtained through the study of an exponential polynomial equation with delay-dependent coefficients. Numerical simulation with a hypothetical set of data has been carried out to support the analytical findings.

1. Introduction

Recently, epidemiological models have become important tools in analyzing the spread and control of infectious diseases after the seminal model of Kermack-Mckendrick on SIRS (susceptible-infected-removed-susceptible) systems [1], and numerous mathematical models were developed to study a disease transmission, to evaluate the spread of epidemics, to understand the mechanisms of epidemics in order to prevent them or minimise the transmission of diseases and other measures. After the seminal models of Vito Volterra and Alfred James Lotka in the mid 1920s for predator-prey interactions [2, 3], predator-prey models have received a great deal of interest in different fields of science, from evolutionary and behavioural ecology to conservation and population biology and even economics [4–8].

It is well known that species do not exist alone in the natural world. It is of more biological significance to study the persistence-extinction threshold of each population in systems of two or more interacting species subjected to disease. Few investigators [9–12] have paid attention to study prey-predator model with infection.

In order to study the influence of disease on an environment where two or more interacting species are present. In this paper, we will put emphasis on such an ecoepidemiological system consisting of three species, namely, the sound prey (which is susceptible), the infected prey (which becomes infective by some viruses), and the predator population.

We have two populations:

(1) the prey, whose total population density is denoted by $N(t)$,

(2) the predator, whose population density is denoted by $y(t)$.

We make the following assumptions.

(A_1) In the absence of infection and predation, the prey population density grows logistically with carrying capacity K $(K > 0)$ and an intrinsic birth rate constant r $(r > 0)$ [13, 14],

$$\frac{dS(t)}{dt} = rS(t)\left(1 - \frac{S(t)}{K}\right). \tag{1.1}$$

(A_2) In the presence of disease, the total prey population $N(t)$ are divided into two distinct classes, namely, susceptible populations, $S(t)$, and infected populations, $I(t)$ [13, 14]. Therefore, at any time t, the total density of prey population is

$$N(t) = S(t) + I(t). \tag{1.2}$$

(A_3) We assume that only susceptible prey $S(t)$ are capable of reproducing with logistic law (1.1); that is, the infected prey $I(t)$ are removed by death (say its death rate is a positive constant μ), or by predation before having the possibility of reproducing. However, the infective population $I(t)$ still contributes with $S(t)$ to population growth toward the carrying capacity [13, 14].

(A_4) We assume that the force of infection at time t is given by $\beta e^{-\mu_1 \tau} S(t) I(t - \tau)$, where β is the average number of contacts per infective per day, μ_1 is disease-induced death rate and $\tau > 0$ is a fixed time during which the infectious agents develop in the vector and it is only after that time that the infected vector can infect a susceptible prey [15]. Hence, the SI model of the infected prey is

$$\frac{dS(t)}{dt} = rS(t)\left(1 - \frac{S(t)}{K}\right) - \beta e^{-\mu_1 \tau} S(t) I(t - \tau),$$

$$\frac{dI(t)}{dt} = \beta e^{-\mu_1 \tau} S(t) I(t - \tau) - \mu I(t). \tag{1.3}$$

(A_5) Numerous field studies show that infected prey are more vulnerable to predation compared with their noninfected counterpart [16–18]. Lafferty and Morris [17] quantified that the predation rates on infected prey may be 31 times higher compared to that on susceptible prey. Thus, we consider the case when the predator mainly eats the infected prey. We assume that the predator eats only

the infected prey with Leslie-Gower ratio-dependent schemes [19–23]. That is to say, the predator consumes the prey according to the ratio-dependent functional response [24–26] and the predator grows logistically with intrinsic growth rate δ and carrying capacity proportional to the prey populations size $I(t)$.

From the above assumptions we have the following model:

$$\frac{dS(t)}{dt} = rS(t)\left(1 - \frac{S(t)}{K}\right) - \beta e^{-\mu_1\tau}S(t)I(t-\tau),$$

$$\frac{dI(t)}{dt} = \beta e^{-\mu_1\tau}S(t)I(t-\tau) - \frac{cy(t)I(t)}{my(t)+I(t)} - \mu I(t), \quad (1.4)$$

$$\frac{dy(t)}{dt} = \delta y(t)\left(1 - \frac{hy(t)}{I(t)}\right).$$

All the parameters of system (1.4) are positive. The initial conditions for system (1.4) defined in the Banach space

$$C^+ = \left\{\varphi \in C\left([-\tau,0], \mathbb{R}^3_+\right) \mid \varphi_1(\theta) = S(\theta), \; \varphi_2(\theta) = I(\theta), \; \varphi_3(\theta) = y(\theta)\right\}, \quad (1.5)$$

where $\mathbb{R}^3_+ = \{(S,I,y) \in \mathbb{R}^3 \mid S \geq 0, \; I \geq 0, \; y \geq 0\}$. We assume that $\varphi_i(0) > 0$ $(i = 1,2,3)$ by the biological meaning.

It is well known by the fundamental theorem of functional differential equations that system (1.4) has a unique solution $(S(t), I(t), y(t))$ satisfying initial conditions (1.5).

In this paper, we will discuss the dynamical properties (e.g., the stability of equilibrium, the existence of Hopf bifurcation around the positive equilibrium) of the delay ecoepidemic model (1.4). This paper is organized as follows. Section 2 gives positivity and boundedness of solutions of system (1.4). Section 3 gives the analysis of existence and uniqueness of positive equilibrium as well as its stability, by studying the characteristic equation associated with system (1.4) which takes the form of a third-degree exponential polynomial with delay-dependent coefficients. In Section 4, some numerical simulations will be given to illustrate our theoretical results. Section 5 discusses the results.

2. Positivity and Boundedness of Solutions

It is important to show positivity and boundedness for the system (1.4) because the variables S, I and y represent populations. Positivity implies that the populations survive and boundedness may be interpreted as a natural restriction to growth as a consequence of limited resources.

Let us put system (1.4) in a vector form by setting $X = \mathrm{col}(S, I, y) \in \mathbb{R}^3$,

$$G(X) = \begin{pmatrix} G_1(X) \\ G_2(X) \\ G_3(X) \end{pmatrix} = \begin{pmatrix} rS\left(1 - \dfrac{S}{K}\right) - \beta e^{-\mu_1\tau}SI(t-\tau) \\ \beta e^{-\mu_1\tau}SI(t-\tau) - \dfrac{cyI}{my+I} - \mu I \\ \delta y\left(1 - \dfrac{hy}{I}\right) \end{pmatrix}, \quad (2.1)$$

where $G : \mathcal{C}^+ \to \mathbb{R}^3$ and $G \in \mathcal{C}^\infty(\mathbb{R}^3)$. Then system (1.4) becomes

$$\dot{X} = G(X), \tag{2.2}$$

with $X(\theta) = (\phi_1(\theta), \phi_2(\theta), \phi_3(\theta)) \in \mathcal{C}^+$ and $\phi_i(0) > 0$ $(i = 1, 2, 3)$. It is easy to check in (2.1) that whenever choosing $X(\theta) \in \mathcal{C}^+$ such that $x_i = 0$, then $G_i(X)|_{x_i(t)=0, X \in \mathbb{R}^3_+} \geq 0$, $(i = 1, 2, 3)$, where $x_1 = S(t)$, $x_2 = I(t)$ and $x_3 = y(t)$.

Due to lemma in [27] any solutions of (2.2) with $X(\theta) \in \mathcal{C}^+$, that is, $X(t) = X(t, X(\theta))$, is such that $X(\theta) \in \mathbb{R}^3$ for all $t > 0$.

Because the analysis of the boundedness of solutions of system (1.4) is similar to the one in Section 2 in [28], we omit it. Let $M \geq 0$, and we can get that all solutions of system (1.4) with initial condition enter the region $B = \{(S(t), I(t), y(t)) \mid 0 \leq S(t), I(t), y(t) \leq M\}$.

3. Stability Analysis and Hopf Bifurcation

In this section, we focus on investigating the stability of the equilibria and Hopf bifurcation around the positive equilibrium of the system (1.4). System (1.4) has the boundary equilibrium $E_1(\mu/\beta e^{-\mu_1\tau}, (r/\beta e^{-\mu_1\tau})(1 - \mu/\beta e^{-\mu_1\tau}K), 0) \triangleq (S_1, I_1, y_1)$ and the positive equilibrium $E_2(S_2, I_2, y_2)$, where $S_2 = (\mu h + c + \mu m)/\beta e^{-\mu_1\tau}(m+h)$, $I_2 = r(\beta e^{-\mu_1\tau}mK + \beta e^{-\mu_1\tau}hK - \mu h - c - \mu m)/(\beta e^{-\mu_1\tau})^2 K(m+h)$, $y_2 = r(\beta e^{-\mu_1\tau}mK + \beta e^{-\mu_1\tau}hK - \mu h - c - \mu m)/(\beta e^{-\mu_1\tau})^2 hK(m+h)$. Clearly, if $1 - \mu/\beta e^{-\mu_1\tau}K > 0$, then E_1 exists and remains positive, and E_2 exists and remains positive if $\beta > (1/Ke^{-\mu_1\tau})(\mu + c/(m+h)) \triangleq \beta_0$.

Let $E^*(S^*, I^*, y^*)$ be any arbitrary equilibrium. Then the characteristic equation about E^* is given by

$$\begin{vmatrix} r - \dfrac{2rS^*}{K} - \beta e^{-\mu_1\tau}I^* - \lambda & -\beta e^{-\mu_1\tau}S^* e^{-\lambda\tau} & 0 \\[3mm] \beta e^{-\mu_1\tau}I^* & \beta e^{-\mu_1\tau}S^* e^{-\lambda\tau} - \dfrac{cmy^{*2}}{(my^* + I^*)^2} - \mu - \lambda & -\dfrac{cI^{*2}}{(my^* + I^*)^2} \\[3mm] 0 & \dfrac{\delta h y^{*2}}{I^{*2}} & \delta - \dfrac{2\delta h y^*}{I^*} - \lambda \end{vmatrix} = 0. \tag{3.1}$$

For equilibrium E_1, (3.1) reduces to

$$\begin{vmatrix} -\dfrac{rS_1}{K} - \beta e^{-\mu_1\tau}I_1 - \lambda & -\beta e^{-\mu_1\tau}S_1 e^{-\lambda\tau} & 0 \\[3mm] \beta e^{-\mu_1\tau}I_1 & \beta e^{-\mu_1\tau}S_1 e^{-\lambda\tau} - \mu - \lambda & -c \\[3mm] 0 & 0 & \delta - \lambda \end{vmatrix} = 0. \tag{3.2}$$

It is easy to see that the equilibrium E_1 is unstable.
For equilibrium E_2, (3.1) reduces to

$$P(\lambda, \tau) + Q(\lambda, \tau)e^{-\lambda\tau} = 0 \tag{3.3}$$

with

$$P(\lambda, \tau) = \lambda^3 + A_1(\tau)\lambda^2 + A_2(\tau)\lambda + A_3(\tau),$$

$$Q(\lambda, \tau) = B_1(\tau)\lambda^2 + B_2(\tau)\lambda + B_3(\tau),$$

(3.4)

where

$$A_1(\tau) = \delta + \mu + \frac{cmy_2^2}{(my_2 + I_2)^2} + \frac{rS_2}{K},$$

$$A_2(\tau) = \delta\left(\mu + \frac{cmy_2^2}{(my_2 + I_2)^2}\right) + \frac{rS_2}{K}\left(\delta + \mu + \frac{cmy_2^2}{(my_2 + I_2)^2}\right) + \frac{\delta}{h}\frac{cI_2^2}{(my_2 + I_2)^2},$$

$$A_3(\tau) = \frac{rS_2}{K}\left[\delta\left(\mu + \frac{cmy_2^2}{(my_2 + I_2)^2}\right) + \frac{\delta}{h}\frac{cI_2^2}{(my_2 + I_2)^2}\right],$$

$$B_1(\tau) = -\beta e^{-\mu_1\tau}S_2,$$

$$B_2(\tau) = -\delta\beta e^{-\mu_1\tau}S_2 - \frac{r\beta e^{-\mu_1\tau}S_2^2}{K} + \left(\beta e^{-\mu_1\tau}\right)^2 S_2 I_2,$$

$$B_3(\tau) = -\frac{r\delta\beta e^{-\mu_1\tau}S_2^2}{K} + \delta\left(\beta e^{-\mu_1\tau}\right)^2 S_2 I_2.$$

(3.5)

For $\tau = 0$, the transcendental equation (3.3) reduces to (3.4)

$$\lambda^3 + (A_1(0) + B_1(0))\lambda^2 + (A_2(0) + B_2(0))\lambda + A_3(0) + B_3(0) = 0.$$

(3.6)

We can easily get

$$A_1(0) + B_1(0) = \frac{rS_2}{K} + \delta - \delta^* = \frac{r\bar{\beta}_0}{\beta} + \delta - \delta^* > 0,$$

$$A_2(0) + B_2(0) = \frac{rS_2}{K}(\delta - \delta^*) + \beta^2 S_2 I_2 = r\left(\mu + \frac{c}{m+h}\right)\left(1 + \frac{\delta - \delta^*}{K\beta} - \frac{\bar{\beta}_0}{\beta}\right),$$

(3.7)

$$A_3(0) + B_3(0) = \delta\beta^2 S_2 I_2 = r\delta\left(\mu + \frac{c}{m+h}\right)\left(1 - \frac{\bar{\beta}_0}{\beta}\right) > 0$$

if $\delta > \delta^*$ and $\beta > \beta_0$ (obviously, $\beta_0 > \bar{\beta}_0$), where $\delta^* = ch/(m+h)^2$ and $\bar{\beta}_0 = (1/K)(\mu + c/(m+h))$. By Routh-Hurwitz criterion, we can get the following theorem.

Theorem 3.1. *When $\tau = 0$, the positive equilibrium E_2 is locally asymptotically stable provided that the conditions*

(H_1) $r\beta_0 > \delta^*\beta$,

(H_2) $\delta > \delta^*$,

(H_3) $\beta > \beta_0$.

In the following, we investigate the existence of purely imaginary roots $\lambda = i\omega$ $(\omega > 0)$ to (3.3). Equation (3.3) takes the form of a third-degree exponential polynomial in λ, which all the coefficients of P and Q depending on τ. Beretta and Kuang [29] established a geometrical criterion which gives the existence of purely imaginary of a characteristic equation with delay-dependent coefficients.

In order to apply the criterion due to Beretta and Kuang [29], we need to verify the following properties for all $\tau \in [0, \tau_{\max})$, where $\tau_{\max}$ is the maximum value which E_2 exists. We can easily obtain that $\tau_{\max} = (1/\mu_1) \ln(K\beta(m + h)/\mu(m + h) + c)$.

(a) $P(0, \tau) + Q(0, \tau) \neq 0$;

(b) $P(i\omega, \tau) + Q(i\omega, \tau) \neq 0$;

(c) $\limsup\{|P(\lambda, \tau)/Q(\lambda, \tau)| : |\lambda| \to \infty, \ \operatorname{Re}\lambda \geq 0\} < 1$;

(d) $F(\omega, \tau) = |P(i\omega, \tau)|^2 - |Q(i\omega, \tau)|^2$ has a finite number of zeros;

(e) Each positive root $\omega(\tau)$ of $F(\omega, \tau) = 0$ is continuous and differentiable in τ whenever it exists.

Here, $P(\lambda, \tau)$ and $Q(\lambda, \tau)$ are defined as in (3.4).
Let $\tau \in [0, \tau_{\max})$. It is easy to see that

$$P(0, \tau) + Q(0, \tau) = A_3(\tau) + B_3(\tau) > 0. \tag{3.8}$$

This implies that (a) is satisfied, and (b) is obviously true because

$$P(i\omega, \tau) + Q(i\omega, \tau) = -i\omega^3 - A_1(\tau)\omega^2 + iA_2(\tau)\omega + A_3(\tau) - B_1(\tau)\omega^2 + iB_2(\tau)\omega + B_3(\tau)$$

$$= \left[-(A_1(\tau) + B_1(\tau))\omega^2 + A_3(\tau) + B_3(\tau)\right] + i\left[-\omega^3 + (A_2(\tau) + B_2(\tau))\omega\right]$$

$$\neq 0. \tag{3.9}$$

From (3.4) we know that

$$\lim_{|\lambda| \to +\infty} \left| \frac{Q(\lambda, \tau)}{P(\lambda, \tau)} \right| = 0. \tag{3.10}$$

Therefore (c) follows.

Let F be defined as in (d). From

$$|P(i\omega, \tau)|^2 = \left(-\omega^3 + A_2(\tau)\omega\right)^2 + \left(-A_1(\tau)\omega^2 + A_3(\tau)\right)^2$$

$$= \omega^6 + \left(A_1^2(\tau) - 2A_2(\tau)\right)\omega^4 + \left(A_2^2(\tau) - 2A_1(\tau)A_3(\tau)\right)\omega^2 + A_3^2(\tau), \quad (3.11)$$

$$|Q(i\omega, \tau)|^2 = B_1^2(\tau)\omega^4 + \left[b_2^2(\tau) - 2B_1(\tau)B_3(\tau)\right]\omega^2 + B_3^2(\tau),$$

we have

$$F(\omega, \tau) = \omega^6 + a_1(\tau)\omega^4 + a_2(\tau)\omega^2 + a_3(\tau), \quad (3.12)$$

where

$$a_1(\tau) = A_1^2(\tau) - 2A_2(\tau) - B_1^2(\tau)$$

$$= \delta^2 + \mu^2 + \frac{c^2 m^2}{(m+h)^2} + \left(\frac{rS_2}{K}\right)^2 + \frac{2\mu cm}{(m+h)^2} - \frac{2\delta ch}{(m+h)^2} - \beta^2(e^{-\mu_1\tau})^2 S_2^2,$$

$$a_2(\tau) = A_2^2(\tau) + 2B_1(\tau)B_3(\tau) - 2A_1(\tau)A_2(\tau) - B_2^2(\tau)$$

$$= \left[\delta\left(\mu + \frac{cmy_2^2}{(my_2 + I_2)^2}\right) + \frac{rS_2}{K}\left(\delta + \mu + \frac{cmy_2^2}{(my_2 + I_2)^2}\right) + \frac{\delta}{h}\frac{cI_2^2}{(my_2 + I_2)^2}\right]$$

$$\times \left[\delta\left(\mu + \frac{cmy_2^2}{(my_2 + I_2)^2}\right) + \frac{rS_2}{K}\left(\delta + \mu + \frac{cmy_2^2}{(my_2 + I_2)^2}\right) + \frac{\delta}{h}\frac{cI_2^2}{(my_2 + I_2)^2}\right.$$

$$\left. - 2\left(\delta + \mu + \frac{cmy_2^2}{(my_2 + I_2)^2} + \frac{rS_2}{K}\right)\right]$$

$$- \left[\delta\beta e^{-\mu_1\tau}S_2\right]^2 - \left[\frac{r\beta e^{-\mu_1\tau}S_2^2}{K} - (\beta e^{-\mu_1\tau})^2 S_2 I_2\right]^2,$$

$$a_3(\tau) = A_3^2(\tau) - B_3^2(\tau)$$

$$= \left(\frac{rS_2}{K}\right)^2\left[\delta\left(\mu + \frac{cmy_2^2}{(my_2 + I_2)^2}\right) + \frac{\delta}{h}\frac{cI_2^2}{(my_2 + I_2)^2}\right]^2 - \left[\delta(\beta e^{-\mu_1\tau})^2 S_2 I_2 - \frac{r\delta\beta e^{-\mu_1\tau}S_2^2}{K}\right]^2.$$

$$(3.13)$$

It is obvious that property (d) is satisfied, and by Implicit Function Theorem, (e) is also satisfied.

Now let $\lambda = i\omega$ ($\omega > 0$) be a root of (3.3). Substituting it into (3.3) and separating the real and imaginary parts yields

$$-A_3(\tau) + A_1(\tau)\omega^2 = B_2(\tau)\omega \sin \omega\tau + \left(B_3(\tau) - B_1(\tau)\omega^2\right)\omega \cos \omega\tau,$$

$$\omega^3 - A_2(\tau)\omega = B_2(\tau)\omega \cos \omega\tau + \left(B_3(\tau) - B_1(\tau)\omega^2\right)\omega \sin \omega\tau. \tag{3.14}$$

From (3.14) it follows that

$$\sin \omega\tau = \frac{B_1(\tau)\omega^5 + [A_1(\tau)B_2(\tau) - A_2(\tau)B_1(\tau) - B_3(\tau)]\omega^3}{B_1^2(\tau)\omega^4\left(B_2^2(\tau) - 2B_1(\tau)B_3(\tau)\right)\omega^2 + B_3^2(\tau)}$$

$$+ \frac{(A_2(\tau)B_3(\tau) - A_3(\tau)B_3(\tau))\omega}{B_1^2(\tau)\omega^4\left(B_2^2(\tau) - 2B_1(\tau)B_3(\tau)\right)\omega^2 + B_3^2(\tau)}, \tag{3.15a}$$

$$\cos \omega\tau = \frac{[B_2(\tau) - A_1(\tau)B_1(\tau)]\omega^4 + [A_1(\tau)B_3(\tau) + A_3(\tau)B_1(\tau) - A_2(\tau)B_2(\tau)]\omega^2}{B_1^2(\tau)\omega^4\left(B_2^2(\tau) - 2B_1(\tau)B_3(\tau)\right)\omega^2 + B_3^2(\tau)}$$

$$- \frac{A_3(\tau)B_3(\tau)}{B_1^2(\tau)\omega^4\left(B_2^2(\tau) - 2B_1(\tau)B_3(\tau)\right)\omega^2 + B_3^2(\tau)}. \tag{3.15b}$$

By the definitions of $P(\lambda,\tau)$, $Q(\lambda,\tau)$ as in (3.3), and applying the property (a), (3.7) can be written as

$$\sin \omega\tau = \operatorname{Im} \frac{P(i\omega,\tau)}{Q(i\omega,\tau)},$$

$$\cos \omega\tau = -\operatorname{Re} \frac{P(i\omega,\tau)}{Q(i\omega,\tau)}, \tag{3.16}$$

which yields

$$|P(i\omega,\tau)|^2 = |Q(i\omega,\tau)|^2. \tag{3.17}$$

Assume that $\mathcal{Q} \subseteq \mathbb{R}_{+0} = \{\zeta \mid \zeta > 0\}$ is the set where $\omega(\tau)$ is a positive root of

$$F(\omega,\tau) = |P(i\omega,\tau)|^2 - |Q(i\omega,\tau)|^2 \tag{3.18}$$

and for $\tau \notin \mathcal{Q}$, $\omega(\tau)$ is not definite. Then for all τ in $\mathcal{Q}$, $\omega(\tau)$ satisfied

$$F(\omega,\tau) = 0. \tag{3.19}$$

The polynomial function F can be written as $F(\omega,\tau) = f(\omega^2,\tau)$, where f is a third-degree polynomial, defined by

$$f(z,\tau) = z^3 + a_1 z^2 + a_2 z + a_3. \tag{3.20}$$

Depending on the determinant of equation

$$f(z, \tau) = z^3 + a_1 z^2 + a_2 z + a_3 = 0, \tag{3.21}$$

$M = (q/2)^2 + (p/3)^3$, where $p = a_2 - (1/3)a_1^2$, $q = (2/27)a_1^3 - (1/3)a_1 a_2 + a_3$, there are three cases for the solutions of (3.21).

(i) If $M > 0$, (3.21) has a real root and a pair of conjugate complex roots. The real root is positive and is given by

$$\nu_1 = \sqrt[3]{-\frac{q}{2} + \sqrt{M}} + \sqrt[3]{-\frac{q}{2} - \sqrt{M}} - \frac{1}{3}a_1. \tag{3.22}$$

(ii) If $M = 0$, (3.21) has three real roots, of which two are equal. In particular, if $a_1 > 0$, there exists only one positive root, $\nu_1 = 2\sqrt[3]{-q/2} - a_1/3$; If $a_1 < 0$, there exists a positive root $\nu_1 = 2\sqrt[3]{-q/2} - a_1/3$ for $\sqrt[3]{-q/2} > -a_1/3$, and there exist three positive roots for $a_1/6 < \sqrt[3]{-q/2} < -a_1/3$, $\nu_1 = 2\sqrt[3]{-q/2} - a_1/3$, $\nu_2 = \mu_3 = -\sqrt[3]{-q/2} - a_1/3$.

(iii) If $M < 0$, there are three distinct real roots, $\nu_1 = 2\sqrt{(|p|/3)} \cos(\varphi/3) - a_1/3$, $\nu_2 = 2\sqrt{(|p|/3)} \cos(\varphi/3 + 2\pi/3) - a_1/3$, $\nu_3 = 2\sqrt{(|p|/3)} \cos(\varphi/3 + 4\pi/3) - a_1/3$, where $\cos\varphi = -q/2\sqrt{(|p|/3)^3}$. Furthermore, if $a_1 > 0$, there exists only one positive root. Otherwise, if $a_1 < 0$, there may exist either one or three positive real roots. If there is only one positive real root, it is equal to $\max(\nu_1, \nu_2, \nu_3)$.

Clearly, the number of positive real roots of (3.21) depends on the sign of a_1. When $a_1 \geq 0$, (3.21) has only one positive real root. Otherwise, there may exist three positive real roots.

It is easy to know that

$$\begin{aligned} a_1 &= A_1^2(\tau) - 2A_2(\tau) - B_1^2(\tau) \\ &= \delta^2 + \mu^2 + \frac{c^2 m^2}{(m+h)^2} + \left(\frac{rS_2}{K}\right)^2 + \frac{2\mu cm}{(m+h)^2} - \frac{2\delta ch}{(m+h)^2} - \beta^2 (e^{-\mu_1 \tau})^2 S_2^2 > 0 \end{aligned} \tag{3.23}$$

if condition

(H_4):

$$\delta^2 + \mu^2 + \frac{c^2 m^2}{(m+h)^2} + \left(\frac{rS_2}{K}\right)^2 + \frac{2\mu cm}{(m+h)^2} > \frac{2\delta ch}{(m+h)^2} + \beta^2 (e^{-\mu_1 \tau})^2 S_2^2 \tag{3.24}$$

holds.

Hence, (3.21) has only one positive real root if condition (H_4) holds. We denote by z_+ this positive real root. Thus, (3.21) has only one positive real root $\omega = \sqrt{z_+}$, and the critical values of τ and $\omega(\tau)$ are impossible to solve explicitly, so we will use the procedure described in Beretta and Kuang [29]. According to this procedure, we define $\theta(\tau) \in [0, 2\pi)$ such that
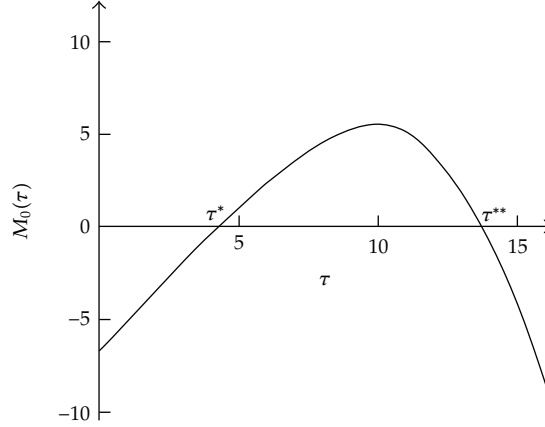
Figure 1: The graph of the function $M_0(\tau)$. Two critical values of τ (i.e., τ^* and τ^{**}), for which stability switches can occur, appear.

$\sin\theta(\tau)$ and $\cos\theta(\tau)$ are given by the right-hand sides of (3.15a) and (3.15b), respectively, with $\theta(\tau)$ given by (3.21). This define $\theta(\tau)$ in a form suitable for numerical evaluation using standard software.

And the relation between the argument $\theta(\tau)$ and $\omega(\tau)\tau$ in (3.19) for $\tau > 0$ must be

$$\omega(\tau)\tau = \theta(\tau) + 2n\pi, \quad n = 0, 1, 2, \ldots. \tag{3.25}$$

Hence we can define the maps: $\tau_n : \mathcal{D} \to \mathbb{R}_{+0}$ given by

$$\tau_n(\tau) := \frac{\theta(\tau) + 2n\pi}{\omega(\tau)}, \quad \tau_n > 0, \; n = 0, 1, 2, \ldots, \tag{3.26}$$

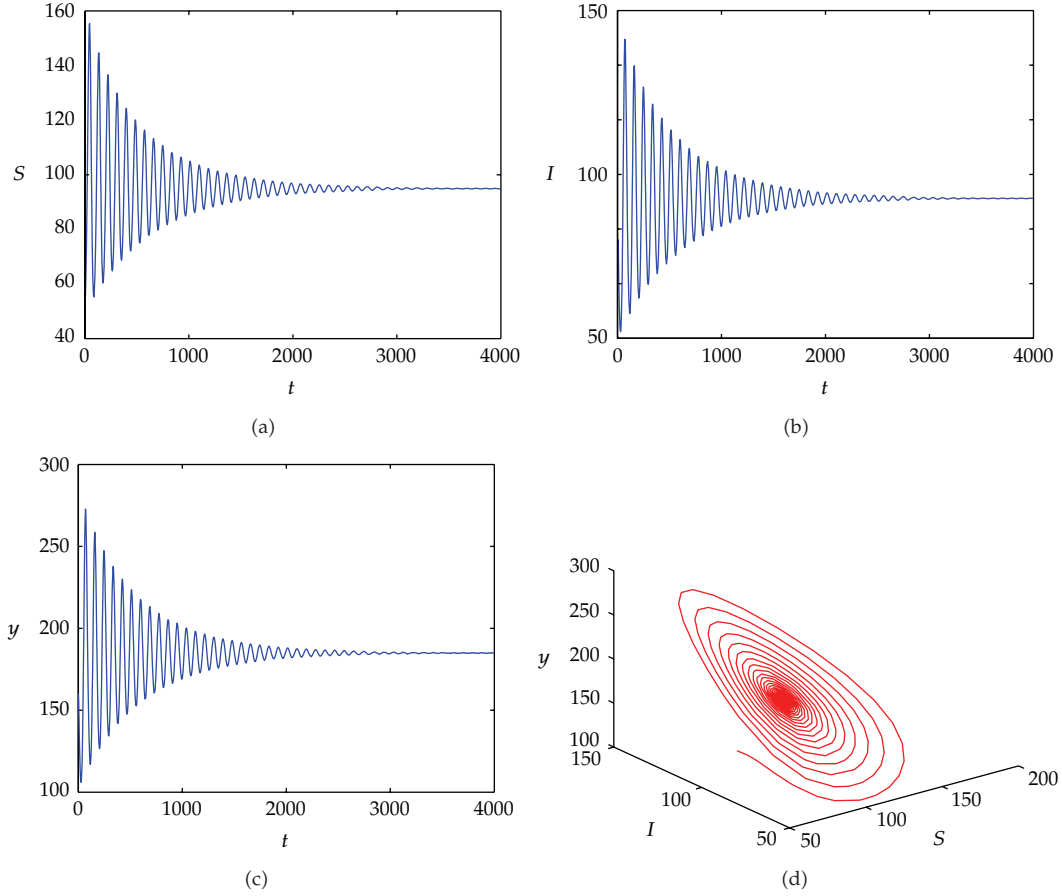where a positive root $\omega(\tau)$ of (3.19) exists in $\mathcal{D}$.

Let us introduce the functions $M_n(\tau) : \mathcal{D} \to \mathbb{R}$,

$$M_n(\tau) = \tau - \frac{\theta(\tau) + 2n\pi}{\omega(\tau)}, \quad n = 0, 1, 2, \ldots. \tag{3.27}$$

that are continuous and differentiable in τ. Thus, we give the following theorem which is due to Beretta and Kuang [29].

Theorem 3.2. *Suppose $\beta > \beta_0$ and condition (H_4) hold. Assume that $\omega(\tau)$ is a positive root of* (3.4) *defined for $\tau \in \mathcal{D}$, $\mathcal{D} \subseteq \mathbb{R}_{+0}$, and at some $\tau^* \in \mathcal{D}$, $M_n(\tau^*) = 0$ for some $n \in \mathbb{N}_0 = \{1, 2, 3, \ldots\}$. Then a pair of simple conjugate pure imaginary roots $\lambda = \pm i\omega$ exists at $\tau = \tau^*$ which crosses the imaginary axis from left to right if $\delta(\tau^*) > 0$ and crosses the imaginary axis from right to left if $\delta(\tau^*) < 0$, where*

$$\delta(\tau^*) = \text{sign}\left\{ \frac{\partial F}{\partial \omega}(\omega\tau^*, \tau^*) \right\} \text{sign}\left\{ \left. \frac{dM_n(\tau)}{d\tau} \right|_{\tau=\tau^*} \right\}. \tag{3.28}$$

Figure 2: Time evolution of all the population for the model (1.4) with $\tau = 3.3$ and initial value $(50, 80, 160)$. (a), (b) and (c) are S, I and y versus t, respectively, and (d) is the projected S-I-y phase plane.
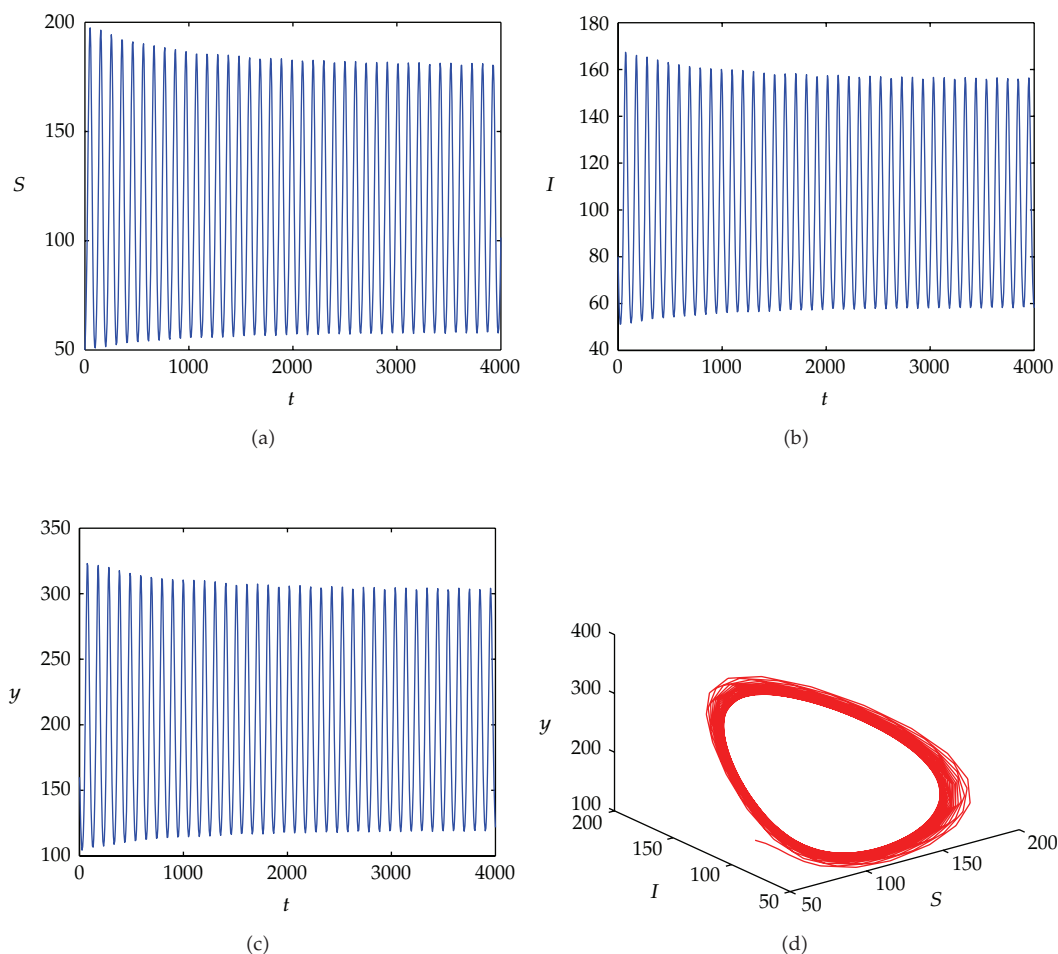
Since

$$\frac{\partial F}{\partial \omega}(\omega, \tau) = 2\omega \frac{\partial f}{\partial z}(\omega^2, \tau),\tag{3.29}$$

condition (3.28) *is equivalent to*

$$\mathrm{sign}\left\{\left.\frac{d\,\mathrm{Re}\,\lambda}{d\tau}\right|_{\lambda=i\omega(\tau^*)}\right\} = \mathrm{sign}\left\{\frac{\partial f}{\partial z}\left(\omega^2(\tau^*), \tau^*\right)\right\}\mathrm{sign}\left\{\left.\frac{dM_n(\tau)}{d\tau}\right|_{\tau=\tau^*}\right\}.\tag{3.30}$$

We can easily observe that $M_n(0) < 0$. Moreover, for all $\tau \in \mathcal{J}$, $M_n(\tau) > M_{n+1}(\tau)$, when $n \in \mathbb{N}$. Therefore, if M_0 has no root in $\mathcal{J}$, then the M_n functions have no root in $\mathcal{J}$ and, if the function $M_n(\tau)$ has positive roots $\tau \in \mathcal{J}$ for some $n \in \mathbb{N}$, there exists at least one root satisfying

$$\frac{dM_n}{d\tau}(\tau) > 0.\tag{3.31}$$

(a)

(b)

(c)

(d)

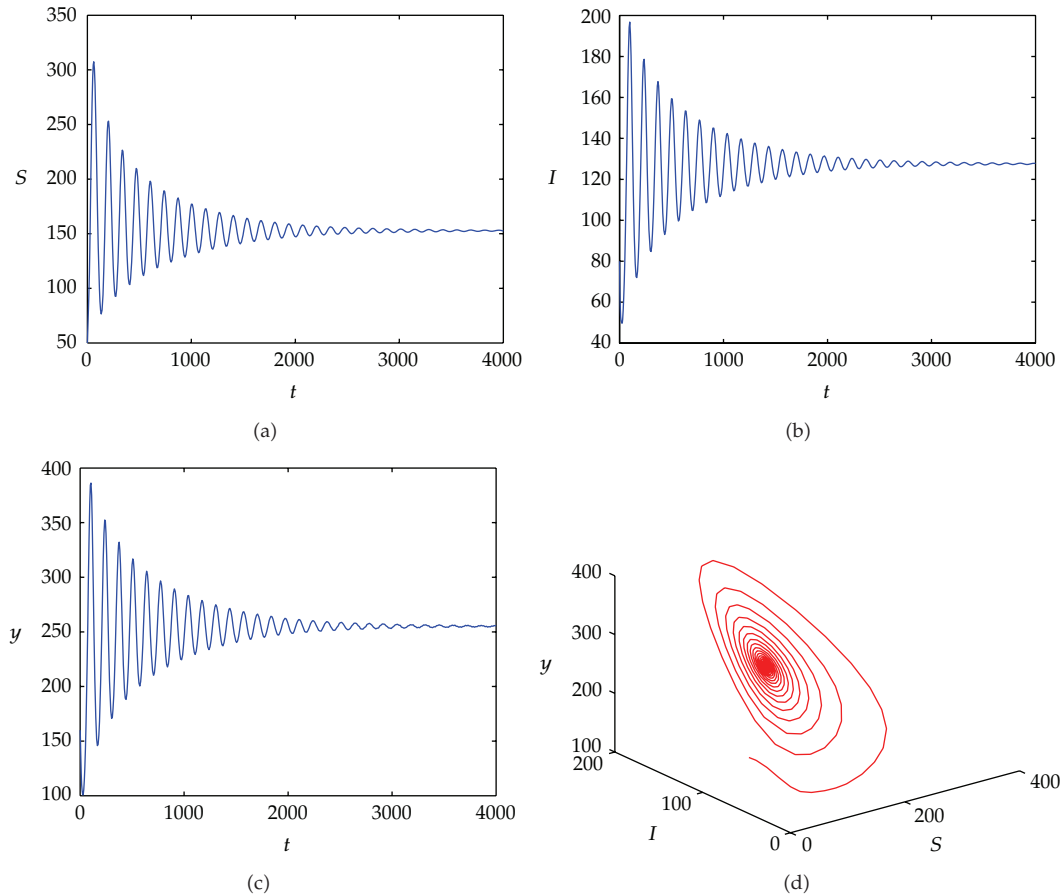**Figure 3:** Time evolution of all the population for the model (1.4) with $\tau = 5.6$ and initial value $(50, 80, 160)$. (a), (b) and (c) are S, I and y versus t, respectively, and (d) is the projected S-I-y phase plane.

Using Theorem 3.1, we can conclude the existence of a Hopf bifurcation as stated in the following theorem.

Theorem 3.3. *Assume the following.*

(1) *If the function $M_0(\tau)$ has no positive root in $\mathcal{I}$, then the positive equilibrium E_2 is locally asymptotically stable for all $\tau \geq 0$.*

(2) *If the function $M_0(\tau)$ has at least one positive root in $\mathcal{I}$, then there exists $\tau^* \in \mathcal{I}$, such that the positive equilibrium E_2 is locally asymptotically stable for $0 \leq \tau < \tau^*$ and becomes unstable for $\tau \geq \tau^*$, with a Hopf bifurcation occurring when $\tau = \tau^*$, if and only if*

$$\text{sign}\left\{\frac{\partial f}{\partial z}\left(\omega^2, \tau\right)\right\} > 0. \tag{3.32}$$

(a)

(b)

(c)

(d)

Figure 4: Time evolution of all the population for the model (1.4) with $\tau = 15.2$ and initial value $(50, 80, 160)$. (a), (b) and (c) are S, I and y versus t, respectively, and (d) is the projected S-I-y phase plane.

4. Numerical Simulations

To determine whether it is possible that there are parameter values such that a stability switch is possible, we choose a set of parameters as follows: $r = 0.1$, $K = 500$, $\beta = 0.001$, $\mu = 0.03$, $c = 8$, $\delta = 0.2$, $m = 150$, $h = 0.5$, $\mu_1 = 0.04$.

Using Maple 10, we compute the coefficients $a_1(\tau)$, $a_2(\tau)$, $a_3(\tau)$ for the above values. We can get $a_1(\tau) = 0.2829795477 + 0.004706297728e^{0.04\tau} > 0$. Hence the polynomial function f, defined in (3.20), has only one positive real root.

The function M_0 is drawn for $\tau \in \mathcal{D}$ in Figure 1. One can see that there are two critical values of the delay τ for which stability switches occur, that is, $\tau^* = 4.69$ and $\tau^{**} = 13.78$. In particulary from Theorem 3.3, a Hopf bifurcation occurs when τ is approximately equal to 4.69. Thus, periodic solutions appear.

Using DDE23, a Matlab Solver for delay differential equation, we can compute the solutions of (1.4) for the above-mentioned values of the parameters, see Figures 2–4. Before the Hopf bifurcation occurs ($0 \leq \tau < \tau^*$), solutions are asymptotically stable and converge to the positive equilibrium. When $\tau^* \leq \tau \leq \tau^{**}$, periodic solutions appear (See Figure 3). When
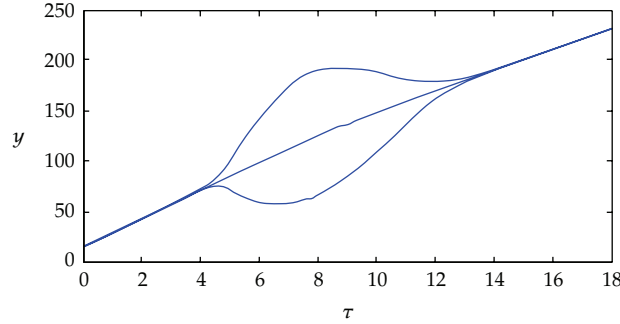
Figure 5: Bifurcation diagram for τ.

τ increases, longer-periods oscillations exist. For $\tau = \tau^{**}(= 13.78)$, a stability switch occurs, that is, the positive equilibrium becomes asymptotically stable again and solutions converge to the positive equilibrium (Figure 4). In Figure 5, using the time delay τ as the bifurcation parameter, we can obtain the bifurcation diagram by Matlab package DDE-BIFTOOL.

5. Discussion

In this paper, we consider the ecoepidemiology model with time delay and Leslie-Gower schemes. The existence of a Hopf bifurcation at a positive steady-state is obtained through the study of an exponential polynomial characteristic equation with delay-dependent coefficients. That is to say, the positive steady-state of this model depends on the maturation period τ through a factor $e^{-\mu_1\tau}$, where μ_1 is the rate of disease-induced death. We can find stability switch occurs from stability to instability. Our computer simulation can easily confirm this. So the time delay plays a stabilizing role for our model.

In addition, if $\mu_1 = 0$, the steady state does not depend on time delay τ. From [28], we can see that the stability switch occurs toward instability if it is stable at $\tau = 0$, and further increase of time delay does not cause the occurrence of stability switch. This implies that both time delay and disease-induced death rate are necessary for the asymptotical stability of system (1.4).

Finally, although we give the reason why we just think of such situation by citing that the rate of the predation on infected prey may be 31 times higher compared to that on susceptible prey [17], it is not usual in real world. In fact, if we deal with a huge population of the prey, then the susceptible population may be too large to ignore their role as prey of the predator. Thus, we can present the more realistic model as follows:

$$\frac{dS(t)}{dt} = rS(t)\left(1 - \frac{S(t)}{K}\right) - \beta e^{-\mu_1\tau}S(t)I(t-\tau) - \mu_1 S(t),$$

$$\frac{dI(t)}{dt} = \beta e^{-\mu_1\tau}S(t)I(t-\tau) - \frac{cy(t)I(t)}{my(t) + I(t)} - \mu_2 I(t), \tag{5.1}$$

$$\frac{dy(t)}{dt} = \delta y(t)\left(1 - \frac{hy(t)}{I(t) + \sigma S(t)}\right),$$

where $\mu_1 < \mu_2$ and $0 \leq \sigma < 1$. Study on the dynamic behavior of system (5.1) will be obtained in the future.

Acknowledgments

References

[1] W. O. Kermack and A. G. McKendrick, "Contributions to the mathematical theory of epidemics. I," *Proceedings of the Royal Society of Edinburgh. Section A*, vol. 115, pp. 700–721, 1927.

[2] A. J. Lotka, *Elements of Physical Biology*, Williams & Wilkins, Baltimore, Md, USA, 1925.

[3] V. Volterra, "Variazioni e fluttuazioni del numero d'individui in specie animali conviventi," *Memorie della Reale Accademia dei Lincei*, vol. 2, pp. 31–113, 1926.

[4] R. Quentin-Grafton and J. Silva-Echenique, "How to manage nature? Strategies, predator-prey models, and chaos," *Marine Resource Economics*, vol. 12, pp. 127–143, 1997.

[5] M. Holyoak and S. P. Lawler, "Persistence of an extinction-prone predator-prey interaction through metapopulation dynamics," *Ecology*, vol. 77, no. 6, pp. 1867–1879, 1996.

[6] Y. Kuang and E. Beretta, "Global qualitative analysis of a ratio-dependent predator-prey system," *Journal of Mathematical Biology*, vol. 36, no. 4, pp. 389–406, 1998.

[7] J. Hofbauer and K. Sigmund, "Adaptive dynamics and evolutionary stability," *Applied Mathematics Letters*, vol. 3, no. 4, pp. 75–79, 1990.

[8] B. C. R. Bertram, "Living in groups: predators and prey," in *Behavioural Ecology*, J. R. Krebs and N. B. Davies, Eds., Blackwell Scientific, Oxford, UK, 1978.

[9] J. Chattopadhyay, P. D. N. Srinivasu, and N. Bairagi, "Pelicans at risk in Salton Sea—an eco-epidemiological model-II," *Ecological Modelling*, vol. 167, no. 1-2, pp. 199–211, 2003.

[10] H. W. Hethcote, W. Wang, L. Han, and Z. Ma, "A predator-prey model with infected prey," *Theoretical Population Biology*, vol. 66, no. 3, pp. 259–268, 2004.

[11] Y. Xiao and L. Chen, "Modeling and analysis of a predator-prey model with disease in the prey," *Mathematical Biosciences*, vol. 171, no. 1, pp. 59–82, 2001.

[12] J. Chattopadhyay and N. Bairagi, "Pelicans at risk in Salton sea—an eco-epidemiological model," *Ecological Modelling*, vol. 136, no. 2-3, pp. 103–112, 2001.

[13] D. Greenhalgh and M. Haque, "A predator-prey model with disease in the prey species only," *Mathematical Methods in the Applied Sciences*, vol. 30, no. 8, pp. 911–929, 2007.

[14] M. Haque and E. Venturino, "The role of transmissible diseases in the Holling-Tanner predator-prey model," *Theoretical Population Biology*, vol. 70, no. 3, pp. 273–288, 2006.

[15] J. M. Tchuenche and A. Nwagwo, "Local stability of an SIR epidemic model and effect of time delay," *Mathematical Methods in the Applied Sciences*, vol. 32, no. 16, pp. 2160–2175, 2009.

[16] P. J. Hudson, A. P. Dobson, and D. Newborn, "Do parasites make prey vulnerable to predation? Red grouse and parasites," *Journal of Animal Ecology*, vol. 61, no. 3, pp. 681–692, 1992.

[17] K. D. Lafferty and A. K. Morris, "Altered behavior of parasitized killifish increases susceptibility to predation by bird final hosts," *Ecology*, vol. 77, no. 5, pp. 1390–1397, 1996.

[18] D. L. Murray, J. R. Cary, and L. B. Keith, "Interactive effects of sublethal nematodes and nutritional status on snowshoe hare vulnerability to predation," *Journal of Animal Ecology*, vol. 66, no. 2, pp. 250–264, 1997.

[19] R. M. May, *Stability and Complexity in Model Ecosystems*, Princeton University Press, Princeton, NJ, USA, 1973.

[20] J. D. Murray, *Mathematical Biology*, vol. 19 of *Biomathematics*, Springer, Berlin, Germany, 1989.

[21] A. F. Nindjin, M. A. Aziz-Alaoui, and M. Cadivel, "Analysis of a predator-prey model with modified Leslie-Gower and Holling-type II schemes with time delay," *Nonlinear Analysis: Real World Applications*, vol. 7, no. 5, pp. 1104–1118, 2006.

[22] P. H. Leslie, "Some further notes on the use of matrices in population mathematics," *Biometrika*, vol. 35, pp. 213–245, 1948.

[23] J. T. Tanner, "The stability and the intrinsic growth rates of prey and predator populations," *Ecology*, vol. 56, pp. 855–867, 1975.

[24] C. Jost, O. Arino, and R. Arditi, "About deterministic extinction in ratio-dependent predator-prey models," *Bulletin of Mathematical Biology*, vol. 61, no. 1, pp. 19–32, 1999.

[25] H. I. Freedman and R. M. Mathsen, "Persistence in predator-prey systems with ratio-dependent predator influence," *Bulletin of Mathematical Biology*, vol. 55, no. 4, pp. 817–827, 1993.

[26] R. Arditi and L. R. Ginzburg, "Coupling in predator-prey dynamics: ratio-dependence," *Journal of Theoretical Biology*, vol. 139, no. 3, pp. 311–326, 1989.

[27] X. Yang, L. Chen, and J. Chen, "Permanence and positive periodic solution for the single-species nonautonomous delay diffusive models," *Computers & Mathematics with Applications*, vol. 32, no. 4, pp. 109–116, 1996.

[28] X. Zhou, X. Shi, and X. Song, "Analysis of a delay prey-predator model with disease in the prey species only," *Journal of the Korean Mathematical Society*, vol. 46, no. 4, pp. 713–731, 2009.

[29] E. Beretta and Y. Kuang, "Geometric stability switch criteria in delay differential systems with delay dependent parameters," *SIAM Journal on Mathematical Analysis*, vol. 33, no. 5, pp. 1144–1165, 2002.