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Research Article

Permanence and Positive Periodic Solutions of a Discrete Delay Competitive System

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A discrete time non-autonomous two-species competitive system with delays is proposed, which involves the influence of many generations on the density of species population. Sufficient conditions for permanence of the system are given. When the system is periodic, by using the continuous theorem of coincidence degree theory and constructing a suitable Lyapunov discrete function, sufficient conditions which guarantee the existence and global attractivity of positive periodic solutions are obtained. As an application, examples and their numerical simulations are presented to illustrate the feasibility of our main results.

1. Introduction

In recent years, the application of theories of functional differential equations in mathematical ecology has developed rapidly. Various delayed models have been proposed in the study of population dynamics, ecology, and epidemic. In fact, more realistic population dynamics should take into account the effect of delay. Also, delay differential equations may exhibit much more complicated dynamic behaviors than ordinary differential equations since a delay could cause a stable equilibrium to become unstable and cause the population to fluctuate (see [1]). One of the famous models for dynamics of population is the delay Lotka-Volterra competitive system. Owing to its theoretical and practical significance, various delay competitive systems have been studied extensively (see [2–8]). Although much progress has been seen for Lotka-Volterra competitive systems, such systems are not well studied in the sense that most results are continuous time versions related. Many authors [9–11] have argued that the discrete time models governed by difference equations are more appropriate than the continuous ones when the populations have non-overlapping generations. Discrete time models can also provide efficient computational models of continuous models for numerical simulations. Therefore, the dynamic behaviors of population models governed by difference equations have been studied by many authors, see [12–18] and the references cited therein. Noting that some studies of the dynamics of natural populations indicate that the density-dependent population regulation probably takes place over many generations [19, 20], many authors have discussed the influence of many past generations on the density of species population and discussed the dynamic behaviors of competitive, predator-prey, and cooperative systems (see [21–24]).

Motivated by the above work [19–24], in this paper we will investigate the following discrete time non-autonomous two-species competitive system with delays:

$$\begin{aligned} x_1(k+1) &= x_1(k)\exp\left[r_1(k) - \sum_{l=0}^{m} a_{1l}(k)x_1(k-l) - \sum_{l=0}^{m} \frac{c_{2l}(k)x_2(k-l)}{1+x_2(k-l)}\right], \\ x_2(k+1) &= x_2(k)\exp\left[r_2(k) - \sum_{l=0}^{m} a_{2l}(k)x_2(k-l) - \sum_{l=0}^{m} \frac{c_{1l}(k)x_1(k-l)}{1+x_1(k-l)}\right], \end{aligned} \tag{1.1}$$

with the initial conditions

$$x_i(-l) \geq 0, \quad x_i(0) > 0, \quad l = 0,1,\ldots,m, \ i = 1,2, \tag{1.2}$$

where $x_i(k)$ represents the density of population x_i at the kth generation, $r_i(k)$ is the intrinsic growth rate of population x_i at the kth generation, $a_{il}(k)$ measures the intraspecific influence of the $(k-l)$th generation of population x_i on the density of its own population, and $c_{jl}(k)$ stands for the interspecific influence of the $(k-l)$th generation of population x_j on population x_i, $i,j = 1,2$ and $i \neq j$. The coefficients $\{r_i(k)\}$, $\{a_{il}(k)\}$, and $\{c_{il}(k)\}$ $(i = 1,2)$ are bounded nonnegative sequences. The exponential form of the equations in system (1.1) ensures that any forward trajectory $\{(x_1(k),x_2(k))^{\top}\}$ of system (1.1) with initial conditions (1.2) remains positive for all $k \in \{0,1,2,\ldots\}$. For the investigations of some continuous versions of (1.1) we refer to [8, 25, 26] and the references cited therein.

The principle aim of this paper is to study the dynamic behaviors of system (1.1), such as permanence, existence, and global attractivity of positive periodic solutions. To the best of our knowledge, no work has been done for the discrete non-autonomous difference system (1.1). The paper is organized as follows. In Section 2, we obtain sufficient conditions which guarantee the permanence of system (1.1). In Section 3, a good understanding of the existence and global attractivity of positive periodic solutions of system (1.1) is gained by using the method of coincidence degree theory and a Lyapunov discrete function. Some illustrative examples are given to demonstrate the feasibility of the obtained results in Section 4. To do this, we need to give the following notations and Definitions 1.1 and 1.2.

For the simplicity and convenience of exposition, throughout this paper we let $\mathbf{Z}, \mathbf{Z}^{+}, \mathbf{R}^{+}$, and $\mathbf{R}^2$ denote the sets of all integers, nonnegative integers, nonnegative real numbers and two-dimensional Euclidian vector space, respectively. Meanwhile, we denote that $b^{*} = \sup_{k\in\mathbf{Z}^{+}} b(k)$, $b_{*} = \inf_{k\in\mathbf{Z}^{+}} b(k)$ for any bounded sequence $b(k)$.

Definition 1.1. System (1.1) is said to be permanent if there exist positive constants m_i and M_i such that

$$m_i \leq \liminf_{k\to+\infty} x_i(k) \leq \limsup_{k\to+\infty} x_i(k) \leq M_i, \quad i = 1,2 \tag{1.3}$$

for any positive solution $\{(x_1(k), x_2(k))^{\top}\}$ of system (1.1).

Definition 1.2. A positive periodic solution $\{(\tilde{x}_1(k), \tilde{x}_2(k))^{\top}\}$ of system (1.1) is said to be globally attractive if each other solution $\{(x_1(k), x_2(k))^{\top}\}$ of system (1.1) satisfies

$$\lim_{k\to+\infty} |x_1(k) - \tilde{x}_1(k)| = 0, \qquad \lim_{k\to+\infty} |x_2(k) - \tilde{x}_2(k)| = 0. \tag{1.4}$$

2. Permanence

In this section, we will establish sufficient conditions for the permanence of system (1.1). To do this, we first give two lemmas which will be useful for establishing our main result in this section.

Lemma 2.1 (see [27, Lemma 1]). *Assume that $\{x(k)\}$ satisfies $x(k) > 0$ and*

$$x(k+1) \leq x(k)\exp[r(k)(1-\alpha x(k))] \tag{2.1}$$

for $k \in [k_1, +\infty)$, where α is a positive constant and $k_1 \in \mathbf{Z}^+$. Then

$$\limsup_{k\to+\infty} x(k) \leq \frac{1}{\alpha r^*}\exp(r^* - 1). \tag{2.2}$$

Lemma 2.2 (see [27, Lemma 2]). *Assume that $\{x(k)\}$ satisfies $x(k) > 0$ and*

$$x(k+1) \geq x(k)\exp[r(k)(1-\alpha x(k))] \tag{2.3}$$

for $k \in [k_2, +\infty)$, $\limsup_{k\to+\infty} x(k) \leq M$, and $x(k_2) > 0$, where α is a constant such that $\alpha M > 1$ and $k_2 \in \mathbf{Z}^+$. Then

$$\liminf_{n\to+\infty} x(k) \geq \frac{1}{\alpha}\exp[r^*(1-\alpha M)]. \tag{2.4}$$

Before stating Theorem 2.3, for the sake of convenience, we set

$$M_i \stackrel{\text{def}}{=} \frac{\exp(r_i^* - 1)}{\sum_{l=0}^{m} a_{il*} \exp(-r_i^* l)}, \qquad \Delta_i \stackrel{\text{def}}{=} \lim_{\varepsilon \to 0} \Delta_i^{\varepsilon}, \qquad \Delta_i^{\varepsilon} \stackrel{\text{def}}{=} \frac{\sum_{l=0}^{m} a_{il}^* \widehat{G}_i}{r_{i*} - \sum_{l=0}^{m} c_{jl}^*},$$
$$\widehat{G}_i \stackrel{\text{def}}{=} \exp\left\{ \left[(M_i + \varepsilon) \sum_{l=0}^{m} a_{il}^* + \sum_{l=0}^{m} c_{jl}^* - r_{i*} \right] l \right\}, \tag{2.5}$$

where $i, j = 1, 2$, $i \neq j$, $\varepsilon > 0$ is a sufficiently small constant.

We are now in a position to state our main result of this section on the permanence of system (1.1).

Theorem 2.3. *If the following assumptions:*

$$\min\left\{ r_{1*} - \sum_{l=0}^{m} c_{2l}^*, r_{2*} - \sum_{l=0}^{m} c_{1l}^* \right\} > 0,$$
$$\min\{M_1 \Delta_1, M_2 \Delta_2\} > 1 \tag{2.6}$$

hold, then system (1.1) *is permanent.*

Proof. Clearly, any solution $\{(x_1(k), x_2(k))^{\top}\}$ of system (1.1) satisfies $x_1(k) > 0$, $x_2(k) > 0$. The following two steps are considered.

Step 1. According to Definition 1.1, we will prove that any positive solution of system (1.1) satisfies $\limsup_{k \to +\infty} x_i(k) \leq M_i$ for $i = 1, 2$.

It follows from the first equation of system (1.1) that

$$x_1(k+1) \leq x_1(k) \exp[r_1(k)]. \tag{2.7}$$

We will make a convention that $\prod_{i=k_1}^{k_2} b(i) = 1$ if $k_1 > k_2$ for any bounded sequence $b(i)$. For $k \geq m$, $l = 0, 1, \ldots, m$, we can obtain

$$\prod_{i=k-l}^{k-1} x_1(i+1) \leq \prod_{i=k-l}^{k-1} x_1(i) \exp[r_1(i)], \tag{2.8}$$

that is,

$$x_1(k) \leq x_1(k-l) \exp\left[\sum_{i=k-l}^{k-1} r_1(i)\right], \tag{2.9}$$

in other words,

$$x_1(k-l) \geq x_1(k) \exp\left[-\sum_{i=k-l}^{k-1} r_1(i)\right]. \tag{2.10}$$

Consequently, we have

$$\begin{aligned} x_1(k+1) &\leq x_1(k)\exp\left\{r_1(k)-\sum_{l=0}^{m}\left[a_{1l}(k)x_1(k)\exp\left[-\sum_{i=k-l}^{k-1}r_1(i)\right]\right]\right\} \\ &\leq x_1(k)\exp\left\{r_1(k)-\sum_{l=0}^{m}\left[a_{1l*}\exp\left(-r_1^{*}l\right)\right]x_1(k)\right\} \\ &\leq x_1(k)\exp\left\{r_1(k)\left[1-\frac{\sum_{l=0}^{m}a_{1l*}\exp\left(-r_1^{*}l\right)}{r_1^{*}}x_1(k)\right]\right\}. \end{aligned} \tag{2.11}$$

By Lemma 2.1, we can derive that

$$\limsup_{k\to+\infty}x_1(k)\leq\frac{\exp\left(r_1^{*}-1\right)}{\sum_{l=0}^{m}a_{1l*}\exp\left(-r_1^{*}l\right)}=M_1. \tag{2.12}$$

Similar to the above argument, we can verify that

$$\limsup_{k\to+\infty}x_2(k)\leq\frac{\exp\left(r_2^{*}-1\right)}{\sum_{l=0}^{m}a_{2l*}\exp\left(-r_2^{*}l\right)}=M_2. \tag{2.13}$$

Step 2. By a similar procedure to Step 1, we will prove that any positive solution of system (1.1) satisfies $\liminf_{k\to+\infty}x_i(k)\geq m_i$, where

$$m_i\overset{\text{def}}{=}\frac{\exp\left[\left(r_{i*}-\sum_{l=0}^{m}c_{jl}{}^{*}\right)\left(1-\Delta_iM_i\right)\right]}{\Delta_i},\quad i,j=1,2,\ i\neq j. \tag{2.14}$$

For any sufficiently small $\varepsilon>0$, according to (2.5), there exists a positive integer k_0 such that $x_1(k)\leq M_1+\varepsilon$ for all $k\geq k_0$. Thus, for $k\geq k_0+m$, it follows from the first equation of system (1.1) that

$$x_1(k+1)\geq x_1(k)\exp\left[r_1(k)-\left(M_1+\varepsilon\right)\sum_{l=0}^{m}a_{1l}(k)-\sum_{l=0}^{m}c_{2l}(k)\right]. \tag{2.15}$$

Therefore, for all $l=0,1,\ldots,m$ and $k\geq k_0+m$, it follows that

$$\prod_{i=k-l}^{k-1}x_1(i+1)\geq\prod_{i=k-l}^{k-1}\left\{x_1(i)\exp\left[r_1(i)-\left(M_1+\varepsilon\right)\sum_{l=0}^{m}a_{1l}(i)-\sum_{l=0}^{m}c_{2l}(i)\right]\right\}, \tag{2.16}$$

that is,

$$x_1(k-l)\leq x_1(k)\exp\left\{\sum_{i=k-l}^{k-1}\left[\left(M_1+\varepsilon\right)\sum_{l=0}^{m}a_{1l}(i)+\sum_{l=0}^{m}c_{2l}(i)-r_1(i)\right]\right\}=x_1(k)G_1, \tag{2.17}$$

where

$$G_1 \overset{\text{def}}{=} \exp\left\{\sum_{i=k-l}^{k-1}\left[(M_1+\varepsilon)\sum_{l=0}^{m}a_{1l}(i)+\sum_{l=0}^{m}c_{2l}(i)-r_1(i)\right]\right\}. \tag{2.18}$$

Combining (2.5) and (2.17) with the first equation of system (1.1) leads to

$$\begin{aligned}
x_1(k+1) &\geq x_1(k)\exp\left[r_{1*}-\sum_{l=0}^{m}c_{2l}^{*}-\sum_{l=0}^{m}a_{1l}^{*}x_1(k-l)\right]\\
&\geq x_1(k)\exp\left[r_{1*}-\sum_{l=0}^{m}c_{2l}^{*}-\sum_{l=0}^{m}a_{1l}^{*}G_1x_1(k)\right]\\
&\geq x_1(k)\exp\left\{\left(r_{1*}-\sum_{l=0}^{m}c_{2l}^{*}\right)\left[1-\frac{\sum_{l=0}^{m}a_{1l}^{*}\widehat{G}_1}{r_{1*}-\sum_{l=0}^{m}c_{2l}^{*}}x_1(k)\right]\right\}\\
&= x_1(k)\exp\left\{\left(r_{1*}-\sum_{l=0}^{m}c_{2l}^{*}\right)\left[1-\Delta_1^{\varepsilon}x_1(k)\right]\right\}.
\end{aligned} \tag{2.19}$$

And hence, by applying Lemma 2.2 and letting $\varepsilon \to 0$, it follows from (2.5)-(2.6) and (2.19) that

$$\liminf_{k\to+\infty} x_1(k) \geq \frac{\exp\left[\left(r_{1*}-\sum_{l=0}^{m}c_{2l}^{*}\right)(1-\Delta_1M_1)\right]}{\Delta_1} = m_1. \tag{2.20}$$

Analogously, from the second equation of system (1.1), we can verify that

$$\liminf_{k\to+\infty} x_2(k) \geq \frac{\exp\left[\left(r_{2*}-\sum_{l=0}^{m}c_{1l}^{*}\right)(1-\Delta_2M_2)\right]}{\Delta_2} = m_2. \tag{2.21}$$

The proof of Theorem 2.3 is completed by combining Steps 1 and 2. □

3. Existence and Global Attractivity of Positive Periodic Solutions

In this section, we will give two main results. We first derive sufficient conditions for the existence of positive periodic solutions of system (1.1). We further assume that $r_i, a_{il}, c_{il} : \mathbf{Z} \to \mathbf{R}^{+}$ are positive ω-periodic for system (1.1), that is,

$$\begin{gathered}
r_i(k+\omega)=r_i(k), \qquad a_{il}(k+\omega)=a_{il}(k),\\
c_{il}(k+\omega)=c_{il}(k), \quad l=0,1,2,\ldots,m,\ i=1,2
\end{gathered} \tag{3.1}$$

for any $k \in \mathbf{Z}$, where ω, a fixed positive integer, denotes the prescribed common period of the parameters in system (1.1).

In order to obtain sufficient conditions for the existence of positive periodic solutions of system (1.1), we will use the method of coincidence degree. For convenience, we will summarize in the following a few concepts and results from [28] that will be useful in this section.

Let X and Y be two Banach spaces. Consider an operator equation

$$Lx = \lambda Nx, \quad \lambda \in (0,1), \tag{3.2}$$

where $L : \operatorname{Dom} L \cap X \to Y$ is a linear operator and λ is a parameter. Let P and Q denote two projectors such that

$$P : \mathrm{X} \cap \operatorname{Dom} L \longrightarrow \operatorname{Ker} L, \qquad Q : Y \longrightarrow \frac{Y}{\operatorname{Im} L}. \tag{3.3}$$

Denote that $J : \operatorname{Im} Q \to \operatorname{Ker} L$ is an isomorphism of $\operatorname{Im} Q$ onto $\operatorname{Ker} L$. Recall that a linear mapping $L : \operatorname{Dom} L \cap X \to Y$ with $\operatorname{Ker} L = L^{-1}(0)$ and $\operatorname{Im} L = L(\operatorname{Dom} L)$ will be called a Fredholm mapping if the following two conditions hold:

(i) $\operatorname{Ker} L$ has a finite dimension;

(ii) $\operatorname{Im} L$ is closed and has a finite codimension.

Recall also that the codimension of $\operatorname{Im} L$ is the dimension of $Y/\operatorname{Im} L$, that is, the dimension of the cokernel coker L of L.

When L is a Fredholm mapping, its index is the integer $\operatorname{Ind} L = \dim \operatorname{Ker} L - \operatorname{codim} \operatorname{Im} L$.

We will say that a mapping N is L-compact on Ω if the mapping $QN : \overline{\Omega} \to Y$ is continuous. $QN(\overline{\Omega})$ is bounded and $K_p(I-Q)N : \overline{\Omega} \to X$ is compact, that is, it is continuous and $K_p(I-Q)N(\overline{\Omega})$ is relatively compact, where $K_p : \operatorname{Im} L - \operatorname{Dom} L \cap \operatorname{Ker} P$ is an inverse of the restriction L_p of L to $\operatorname{Dom} L \cap \operatorname{Ker} P$, so that $LK_p = I$ and $K_p = I - P$.

Lemma 3.1 (see [28, Continuation Theorem]). *Let X and Y be two Banach spaces and let L be a Fredholm mapping of index zero. Assume that $N : \overline{\Omega} \to Y$ is L-compact on $\overline{\Omega}$ with Ω open bounded in X. Furthermore assume that*

(a) *for each $\lambda \in (0,1)$, $x \in \partial\Omega \cap \operatorname{Dom} L$, $Lx \neq \lambda Nx$,*

(b) *$QNx \neq 0$ for each $x \in \partial\Omega \cap \operatorname{Ker} L$,*

(c) $\deg\{JQNx, \Omega \cap \operatorname{Ker} L, 0\} \neq 0$.

Then the equation $Lx = Nx$ has at least one solution in $\operatorname{Dom} L \cap \overline{\Omega}$.

In what follows, we will use the following notations:

$$I_\omega = \{0,1,\dots,\omega-1\}, \qquad \overline{f} = \frac{1}{\omega}\sum_{k=0}^{\omega-1} f(k), \qquad f^L = \min_{k\in I_\omega} f(k), \qquad f^U = \max_{k\in I_\omega} f(k), \tag{3.4}$$

where $\{f(k)\}$ is an ω-periodic sequence of real numbers defined for $k \in \mathbf{Z}$.

Lemma 3.2 (see [29, Lemma 3.2]). *Let $f : \mathbf{Z} \to \mathbf{R}$ be ω-periodic, that is, $f(k+\omega) = f(k)$, then for any fixed $k_1, k_2 \in I_\omega$ and any $k \in \mathbf{Z}$, one has*

$$\begin{aligned} f(k) &\leq f(k_1) + \sum_{s=0}^{\omega-1} \left| f(s+1) - f(s) \right|, \\ f(k) &\geq f(k_2) - \sum_{s=0}^{\omega-1} \left| f(s+1) - f(s) \right|. \end{aligned} \tag{3.5}$$

Denote that

$$l_2 = \left\{ x = \{x(k)\} : x(k) \in \mathbf{R}^2,\ k \in \mathbf{Z} \right\}. \tag{3.6}$$

For $a = (a_1, a_2)^\top \in \mathbf{R}^2$, define $|a| = \max\{a_1, a_2\}$. Let $l^\omega \subset l_2$ denote the subspace of all ω-periodic sequences equipped with the usual supremum norm $\|\cdot\|$, that is,

$$\|x\| = \max_{k \in I_\omega} |x(k)| \quad \text{for } x = \{x(k) : k \in \mathbf{Z}\} \in l^\omega. \tag{3.7}$$

Then it follows that l^ω is a finite dimensional Banach space.

Let

$$\begin{aligned} l_0^\omega &= \left\{ x = \{x(k) \in l^\omega\} : \sum_{k=0}^{\omega-1} x(k) = 0 \right\}, \\ l_c^\omega &= \left\{ x = \{x(k) \in l^\omega\} : x(k) = h \in \mathbf{R}^2,\ k \in \mathbf{Z} \right\}. \end{aligned} \tag{3.8}$$

Then it follows that l_0^ω and l_c^ω are both closed linear subspaces of l^ω and

$$l^\omega = l_0^\omega \oplus l_c^\omega, \quad \dim l_c^\omega = 2. \tag{3.9}$$

We are now in a position to state one of the main results of this section on the existence of positive periodic solutions of system (1.1).

Theorem 3.3. *Assume that*

$$\overline{r}_1 > \sum_{l=0}^{m} \overline{c}_{2l}, \qquad \overline{r}_2 > \sum_{l=0}^{m} \overline{c}_{1l}. \tag{3.10}$$

Then system (1.1) *has at least one positive ω-periodic solution.*

Proof. We first make the change of variables

$$x_1(k) = \exp\{y_1(k)\}, \qquad x_2(k) = \exp\{y_2(k)\}. \tag{3.11}$$

By substituting (3.11) into system (1.1), we can get

$$\begin{aligned}
y_1(k+1) - y_1(k) &= r_1(k) - \sum_{l=0}^{m} a_{1l}(k)\exp\{y_1(k-l)\} - \sum_{l=0}^{m} \frac{c_{2l}(k)\exp\{y_2(k-l)\}}{1+\exp\{y_2(k-l)\}}, \\
y_2(k+1) - y_2(k) &= r_2(k) - \sum_{l=0}^{m} a_{2l}(k)\exp\{y_2(k-l)\} - \sum_{l=0}^{m} \frac{c_{1l}(k)\exp\{y_1(k-l)\}}{1+\exp\{y_1(k-l)\}}.
\end{aligned} \tag{3.12}$$

It is easy to see that if system (3.12) has one ω-periodic solution, then system (1.1) has one positive ω-periodic solution. Therefore, to complete the proof, it is only to show that system (3.12) has at least one ω-periodic solution.

Set $X = Y = l^{\omega}$. Denote by $L : X \rightarrow X$ the difference operator given by $Ly = \{(Ly)(k)\}$ with

$$(Ly)(k) = y(k+1) - y(k) \quad \text{for } y \in X,\ k \in \mathbf{Z}, \tag{3.13}$$

and $N : X \rightarrow X$ as follows:

$$N\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} r_1(k) - \sum_{l=0}^{m} a_{1l}(k)\exp\{y_1(k-l)\} - \sum_{l=0}^{m} \dfrac{c_{2l}(k)\exp\{y_2(k-l)\}}{1+\exp\{y_2(k-l)\}} \\ r_2(k) - \sum_{l=0}^{m} a_{2l}(k)\exp\{y_2(k-l)\} - \sum_{l=0}^{m} \dfrac{c_{1l}(k)\exp\{y_1(k-l)\}}{1+\exp\{y_1(k-l)\}} \end{bmatrix} \tag{3.14}$$

for any $(y_1, y_2)^{\top} \in X$ and $k \in \mathbf{Z}$. It is easy to see that L is a bounded linear operator and

$$\operatorname{Ker} L = l_c^{\omega}, \quad \operatorname{Im} L = l_0^{\omega}, \quad \dim \operatorname{Ker} L = 2 = \operatorname{codim} \operatorname{Im} L, \tag{3.15}$$

then we get that L is a Fredholm mapping of index zero.

Define

$$P\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = Q\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \dfrac{1}{\omega}\sum_{s=0}^{\omega-1} y_1(s) \\ \dfrac{1}{\omega}\sum_{s=0}^{\omega-1} y_2(s) \end{bmatrix}, \quad \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \in X = Y. \tag{3.16}$$

It is not difficult to show that P and Q are continuous projectors such that

$$\operatorname{Im} P = \operatorname{Ker} L, \qquad \operatorname{Ker} Q = \operatorname{Im} L = \operatorname{Im}(I - Q). \tag{3.17}$$

Furthermore, the inverse (to L) $K_p : \operatorname{Im} L \rightarrow \operatorname{Dom} L \cap \operatorname{Ker} P$ exists and is given by

$$K_p(y) = \sum_{s=0}^{k-1} y(s) - \frac{1}{\omega}\sum_{s=0}^{\omega-1}(\omega - s)y(s). \tag{3.18}$$

Then $QN : X \to Y$ and $K_p(I-Q)N : X \to X$ are given by

$$QNy = \frac{1}{\omega}\sum_{s=0}^{\omega-1} Ny(s),$$
$$K_p(I-Q)Ny = \sum_{s=0}^{k-1} Ny(s) - \frac{1}{\omega}\sum_{s=0}^{\omega-1}(\omega-s)Ny(s) - \left(\frac{k}{\omega} - \frac{1+\omega}{2\omega}\right)\sum_{s=0}^{\omega-1} Ny(s). \tag{3.19}$$

In order to apply Lemma 3.1, we need to search for an appropriately open, bounded subset Ω.

Corresponding to the operator equation $Ly = \lambda Ny$, $\lambda \in (0,1)$, we have

$$y_1(k+1) - y_1(k) = \lambda\left[r_1(k) - \sum_{l=0}^{m} a_{1l}(k)\exp\{y_1(k-l)\} - \sum_{l=0}^{m}\frac{c_{2l}(k)\exp\{y_2(k-l)\}}{1+\exp\{y_2(k-l)\}}\right],$$
$$y_2(k+1) - y_2(k) = \lambda\left[r_2(k) - \sum_{l=0}^{m} a_{2l}(k)\exp\{y_2(k-l)\} - \sum_{l=0}^{m}\frac{c_{1l}(k)\exp\{y_1(k-l)\}}{1+\exp\{y_1(k-l)\}}\right]. \tag{3.20}$$

Suppose that $y = \{y(k)\} = \{(y_1(k), y_2(k))^\top\} \in X$ is a solution of (3.20) for a certain $\lambda \in (0,1)$. Summing both sides of (3.20) from 0 to $\omega - 1$ with respect to k, we can derive

$$\bar{r}_1\omega = \sum_{k=0}^{\omega-1}\sum_{l=0}^{m} a_{1l}(k)\exp\{y_1(k-l)\} + \sum_{k=0}^{\omega-1}\sum_{l=0}^{m}\frac{c_{2l}(k)\exp\{y_2(k-l)\}}{1+\exp\{y_2(k-l)\}},$$
$$\bar{r}_2\omega = \sum_{k=0}^{\omega-1}\sum_{l=0}^{m} a_{2l}(k)\exp\{y_2(k-l)\} + \sum_{k=0}^{\omega-1}\sum_{l=0}^{m}\frac{c_{1l}(k)\exp\{y_1(k-l)\}}{1+\exp\{y_1(k-l)\}}. \tag{3.21}$$

Since $y = \{y(k)\} \in X$, there exist $\xi_i \in I_\omega$ such that

$$y_i(\xi_i) = \min_{k\in I_\omega}\{y_i(k)\}, \quad i = 1,2. \tag{3.22}$$

It follows from (3.21) that

$$\begin{aligned}
\bar{r}_i\omega &\geq \sum_{k=0}^{\omega-1}\sum_{l=0}^{m} a_{il}(k)\exp\{y_i(\xi_i)\} + \sum_{k=0}^{\omega-1}\sum_{l=0}^{m}\frac{c_{jl}(k)\exp\{y_j(\xi_i)\}}{1+\exp\{y_j(\xi_i)\}} \\
&> \sum_{k=0}^{\omega-1}\sum_{l=0}^{m} a_{il}(k)\exp\{y_i(\xi_i)\} \\
&= \omega\sum_{l=0}^{m}\bar{a}_{il}\exp\{y_i(\xi_i)\}, \quad i,j = 1,2,\ i \neq j,
\end{aligned} \tag{3.23}$$

which implies

$$y_i(\xi_i) < \ln \frac{\overline{r}_i}{\sum_{l=0}^{m} \overline{a}_{il}} = \ln \frac{\overline{r}_i}{A_i}, \quad i = 1, 2, \tag{3.24}$$

where $A_i \overset{\text{def}}{=} \sum_{l=0}^{m} \overline{a}_{il}$, besides, from (3.20) and (3.21)

$$\begin{aligned}\sum_{k=0}^{\omega-1} \left| y_i(k+1) - y_i(k) \right| &\leq \lambda \left\{ \sum_{k=0}^{\omega-1} \left[r_i(k) + \sum_{l=0}^{m} a_{il}(k) \exp\{y_i(k-l)\} + \sum_{l=0}^{m} \frac{c_{jl}(k) \exp\{y_j(k-l)\}}{1 + \exp\{y_j(k-l)\}} \right] \right\} \\ &\leq 2\overline{r}_i \omega, \quad i, j = 1, 2, \ i \neq j.\end{aligned} \tag{3.25}$$

By (3.24), (3.25), and Lemma 3.2, we have

$$y_i(k) \leq y_i(\xi_i) + \sum_{s=0}^{\omega-1} \left| y_i(s+1) - y_i(s) \right| \leq \ln \frac{\overline{r}_i}{A_i} + 2\overline{r}_i \omega. \tag{3.26}$$

On the other hand, there also exist $\eta_i \in I_\omega$ such that

$$y_i(\eta_i) = \max_{k \in I_\omega} \{y_i(k)\}, \quad i = 1, 2. \tag{3.27}$$

In view of (3.21), we can obtain

$$\begin{aligned}\overline{r}_i \omega &\leq \sum_{k=0}^{\omega-1} \sum_{l=0}^{m} a_{il}(k) \exp\{y_i(\eta_i)\} + \sum_{k=0}^{\omega-1} \sum_{l=0}^{m} c_{jl}(k) \\ &= \left[\sum_{l=0}^{m} \overline{a}_{il} \exp\{y_i(\eta_i)\} + \sum_{l=0}^{m} \overline{c}_{jl} \right] \omega, \quad i, j = 1, 2, \ i \neq j.\end{aligned} \tag{3.28}$$

Therefore,

$$\overline{r}_i \leq \sum_{l=0}^{m} \overline{a}_{il} \exp\{y_i(\eta_i)\} + \sum_{l=0}^{m} \overline{c}_{jl}, \quad i, j = 1, 2, \ i \neq j. \tag{3.29}$$

Then

$$\exp\{y_i(\eta_i)\} > \frac{\overline{r}_i - \sum_{l=0}^{m} \overline{c}_{jl}}{\sum_{l=0}^{m} \overline{a}_{il}} \overset{\text{def}}{=} B_i, \quad i, j = 1, 2, \ i \neq j. \tag{3.30}$$

That is,

$$y_i(\eta_i) > \ln B_i, \quad i = 1, 2. \tag{3.31}$$

By (3.25), (3.31), and Lemma 3.2, we have

$$y_i(k) \geq y_i(\eta_i) - \sum_{s=0}^{\omega-1} |y_i(s+1) - y_i(s)| \geq \ln B_i - 2\overline{r}_i\omega, \quad i = 1,2. \tag{3.32}$$

Inequalities (3.26) and (3.32) imply

$$|y_i(k)| \leq \max\left\{\left|\ln \frac{\overline{r}_i}{A_i} + 2\overline{r}_i\omega\right|, |\ln B_i - 2\overline{r}_i\omega|\right\} \overset{\text{def}}{=} H_i, \quad i = 1,2. \tag{3.33}$$

Obviously, B_i, A_i, and H_i in (3.33) are independent of λ, respectively. Denote $H = H_1 + H_2 + h_0$, where h_0 is taken sufficiently large such that any solution $\{(\tilde{y}_1, \tilde{y}_2)^\top\}$ of the system of algebraic equations

$$\begin{gathered}
\sum_{l=0}^{m} \overline{a}_{1l} \exp(y_1) + \sum_{l=0}^{m} \frac{\overline{c}_{2l} \exp(y_2)}{1+\exp(y_2)} = \overline{r}_1, \\
\sum_{l=0}^{m} \overline{a}_{2l} \exp(y_2) + \sum_{l=0}^{m} \frac{\overline{c}_{1l} \exp(y_1)}{1+\exp(y_1)} = \overline{r}_2
\end{gathered} \tag{3.34}$$

satisfies $\|(\tilde{y}_1, \tilde{y}_2)^\top\| = \max\{|\tilde{y}_1|, |\tilde{y}_2|\} < h_0$ (If system (3.34) has at least one solution). Let $\Omega \overset{\text{def}}{=} \{y : (y_1, y_2) \in X \mid \|y\| < H\}$, thus condition (a) in Lemma 3.1 holds. When $y \in \partial\Omega \cap \operatorname{Ker} L = \partial\Omega \cap \mathbf{R}^2$, $y = \{(y_1, y_2)^\top\}$, $(y_1, y_2)^\top$ is a constant vector in $\mathbf{R}^2$ with $\|y\| = H$. If system (3.34) has at least one solution, then

$$QN\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \overline{r}_1 - \sum_{l=0}^{m} \overline{a}_{1l} \exp(y_1) - \sum_{l=0}^{m} \frac{\overline{c}_{2l} \exp(y_2)}{1+\exp(y_2)} \\ \overline{r}_2 - \sum_{l=0}^{m} \overline{a}_{2l} \exp(y_2) - \sum_{l=0}^{m} \frac{\overline{c}_{1l} \exp(y_1)}{1+\exp(y_1)} \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{3.35}$$

If system (3.34) does not have one solution, then it is obvious that

$$QN\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{3.36}$$

This implies that condition (b) in Lemma 3.1 is satisfied.

Now we prove that condition (c) in Lemma 3.1 holds. Define $\Phi : \operatorname{Dom} L \times [0,1] \to X$ as follows:

$$\Phi(y_1, y_2, \mu) = \begin{bmatrix} \overline{r}_1 - \sum_{l=0}^{m} \overline{a}_{1l} \exp(y_1) \\ \overline{r}_2 - \sum_{l=0}^{m} \overline{a}_{2l} \exp(y_2) \end{bmatrix} + \mu \begin{bmatrix} -\sum_{l=0}^{m} \frac{\overline{c}_{2l} \exp(y_2)}{1+\exp(y_2)} \\ -\sum_{l=0}^{m} \frac{\overline{c}_{1l} \exp(y_1)}{1+\exp(y_1)} \end{bmatrix}, \tag{3.37}$$

where μ is a parameter with $\mu \in [0,1]$. When $y = \{(y_1, y_2)^\top\} \in \partial\Omega \cap \operatorname{Ker} L = \partial\Omega \cap \mathbf{R}^2$, $(y_1, y_2)^\top$ is a constant vector in $\mathbf{R}^2$ with $\|y\| = H$. We will show that $\{(y_1, y_2)^\top\} \in \partial\Omega \cap \operatorname{Ker} L$, $\Phi(y_1, y_2, \mu) \neq 0$. If the conclusion is not true, then there is a constant vector $(y_1, y_2)^\top \in \mathbf{R}^2$ with $\|y\| = H$ satisfying $\Phi(y_1, y_2, \mu) = 0$, that is,

$$\begin{aligned}
&\overline{r}_1 - \sum_{l=0}^{m} \overline{a}_{1l} \exp(y_1) - \mu \sum_{l=0}^{m} \frac{\overline{c}_{2l} \exp(y_2)}{1 + \exp(y_2)} = 0, \\
&\overline{r}_2 - \sum_{l=0}^{m} \overline{a}_{2l} \exp(y_2) - \mu \sum_{l=0}^{m} \frac{\overline{c}_{1l} \exp(y_1)}{1 + \exp(y_1)} = 0.
\end{aligned} \tag{3.38}$$

A similar argument to the above shows that $\|y\| < H$, which is a contradiction. Using the property of topological degree and taking $J = I : \operatorname{Im} Q \to \operatorname{Ker} L$, $(y_1, y_2)^\top \to (y_1, y_2)^\top$, we have

$$\begin{aligned}
&\deg\left\{ JQN(y_1, y_2)^\top, \Omega \cap \operatorname{Ker} L, (0,0)^\top \right\} \\
&\quad = \deg\left\{ \Phi(y_1, y_2, 1)^\top, \Omega \cap \operatorname{Ker} L, (0,0)^\top \right\} \\
&\quad = \deg\left\{ \Phi(y_1, y_2, 0)^\top, \Omega \cap \operatorname{Ker} L, (0,0)^\top \right\} \\
&\quad = \deg\left\{ \left(\overline{r}_1 - \sum_{l=0}^{m} \overline{a}_{1l} \exp(y_1), \overline{r}_2 - \sum_{l=0}^{m} \overline{a}_{2l} \exp(y_2) \right)^\top, \Omega \cap \operatorname{Ker} L, (0,0)^\top \right\}.
\end{aligned} \tag{3.39}$$

Obviously, the following equations:

$$\overline{r}_1 - \sum_{l=0}^{m} \overline{a}_{1l} \exp(u_1) = 0, \qquad \overline{r}_2 - \sum_{l=0}^{m} \overline{a}_{2l} \exp(u_2) = 0 \tag{3.40}$$

have the unique solution $(\tilde{u}_1, \tilde{u}_2)^\top \in \mathbf{R}^2$. Therefore, we have

$$\deg\left\{ JQN(y_1, y_2)^\top, \Omega \cap \operatorname{Ker} L, (0,0)^\top \right\} = \operatorname{sign}\left\{ \sum_{l=0}^{m} \overline{a}_{1l} \exp(\tilde{u}_1) \sum_{l=0}^{m} \overline{a}_{2l} \exp(\tilde{u}_2) \right\} = 1 \neq 0. \tag{3.41}$$

Finally, we will show that N is L-compact on $\overline{\Omega}$. For any $y \in \overline{\Omega}$, we have

$$\begin{aligned}
\|QNy\| &= \left\| \frac{1}{\omega} \sum_{s=0}^{\omega-1} Ny(x) \right\| \\
&\le \max\left\{ r_1^U + \sum_{l=0}^{m} a_{1l}^U \exp(H_1) + \sum_{l=0}^{m} c_{2l}^U, r_2^U + \sum_{l=0}^{m} a_{2l}^U \exp(H_2) + \sum_{l=0}^{m} c_{1l}^U \right\} \stackrel{\text{def}}{=} E.
\end{aligned} \tag{3.42}$$

Hence, $QN(\overline{\Omega})$ is bounded. Obviously, $QNy : \overline{\Omega} \to Y$ is continuous.

It is easy to see that

$$\|K_p(I-Q)Ny\| \leq \sum_{s=0}^{\omega-1}\|Ny(s)\| + \frac{1}{\omega}\sum_{s=0}^{\omega-1}(\omega-s)\|Ny(s)\| + \frac{1+3\omega}{2\omega}\sum_{s=0}^{\omega-1}\|Ny(s)\| \leq \frac{1+7\omega}{2}E. \tag{3.43}$$

For any $y \in \overline{\Omega}$, $k_1, k_2 \in I_\omega$, without loss of generality, let $k_2 > k_1$, then we have

$$\begin{aligned}
&\left|K_p(I-Q)Ny(k_2) - K_p(I-Q)Ny(k_1)\right| \\
&\quad = \left|\sum_{s=k_1}^{k_2-1}Ny(s) - \frac{k_2-k_1}{\omega}\sum_{s=0}^{\omega-1}Ny(s)\right| \\
&\quad \leq \sum_{s=k_1}^{k_2-1}\left|Ny(s)\right| + \frac{k_2-k_1}{\omega}\sum_{s=0}^{\omega-1}\left|Ny(s)\right| \leq 2E|k_2-k_1|.
\end{aligned} \tag{3.44}$$

Thus, the set $\{K_p(I-Q)Ny \mid y \in \overline{\Omega}\}$ is equicontinuous and uniformly bounded. By using the Arzela-Ascoli theorem, we see that $K_p(I-Q)N : \overline{\Omega} \to X$ is compact. Consequently, N is L-compact.

By now, we know that Ω verifies all the requirements in Lemma 3.1 and then system (3.12) has at least one ω-periodic solution. By the medium of (3.11), we derive that system (1.1) has at least one ω-periodic solution. This completes the proof of Theorem 3.3. □

Next, by constructing a suitable Lyapunov-like discrete function, we further investigate the global attractivity of positive periodic solutions of system (1.1).

Theorem 3.4. *In addition to* (3.10), *assume further that there exists a constant $\eta > 0$ such that*

$$\min\left\{a_{i0*}, \frac{2}{M_i} - a_{i0}^*\right\} - m a_i^M - (m+1)c_i^M \geq \eta, \quad i = 1,2, \tag{3.45}$$

where M_i (i=1,2) are defined in (2.5) *and*

$$a_i^M = \max\{a_{il}^* : l = 1,2,3,\ldots,m\}, \qquad c_i^M = \max\{c_{il}^* : l = 0,1,2,\ldots,m\}, \quad i = 1,2. \tag{3.46}$$

Then the positive periodic solution of system (1.1) *is globally attractive.*

Proof. Let $\{(\widetilde{x}_1(k), \widetilde{x}_2(k))^{\top}\}$ be a positive periodic solution of system (1.1). To finish the proof of Theorem 3.4, we will consider the following two steps.

Step 1. Let $V_{11}(k) = |\ln x_1(k) - \ln \widetilde{x}_1(k)|$, then it follows from the first equation of system (1.1) that

$$\begin{aligned}
V_{11}(k+1) &= |\ln x_1(k+1) - \ln \widetilde{x}_1(k+1)| \\
&= \Bigg| \left[\ln x_1(k) + r_1(k) - \sum_{l=0}^{m} a_{1l}(k) x_1(k-l) - \sum_{l=0}^{m} \frac{c_{2l}(k) x_2(k-l)}{1 + x_2(k-l)} \right] \\
&\quad - \left[\ln \widetilde{x}_1(k) + r_1(k) - \sum_{l=0}^{m} a_{1l}(k) \widetilde{x}_1(k-l) - \sum_{l=0}^{m} \frac{c_{2l}(k) \widetilde{x}_2(k-l)}{1 + \widetilde{x}_2(k-l)} \right] \Bigg| \\
&\le |\ln x_1(k) - \ln \widetilde{x}_1(k) - a_{10}(k)[x_1(k) - \widetilde{x}_1(k)]| \\
&\quad + \sum_{l=1}^{m} a_{1l}(k) |x_1(k-l) - \widetilde{x}_1(k-l)| + \sum_{l=0}^{m} c_{2l}(k) |x_2(k-l) - \widetilde{x}_2(k-l)|.
\end{aligned} \tag{3.47}$$

By the mean value theorem, we have

$$x_1(k) - \widetilde{x}_1(k) = \exp[\ln x_1(k)] - \exp[\ln \widetilde{x}_1(k)] = \xi_1(k) \ln \frac{x_1(k)}{\widetilde{x}_1(k)}, \tag{3.48}$$

that is,

$$\ln \frac{x_1(k)}{\widetilde{x}_1(k)} = \frac{1}{\xi_1(k)} [x_1(k) - \widetilde{x}_1(k)], \tag{3.49}$$

where $\xi_1(k)$ lies between $x_1(k)$ and $\widetilde{x}_1(k)$. Then we have

$$\begin{aligned}
&\left| \ln \frac{x_1(k)}{\widetilde{x}_1(k)} - a_{10}(k)[x_1(k) - \widetilde{x}_1(k)] \right| \\
&\quad = \left| \ln \frac{x_1(k)}{\widetilde{x}_1(k)} \right| - \left| \ln \frac{x_1(k)}{\widetilde{x}_1(k)} \right| + \left| \ln \frac{x_1(k)}{\widetilde{x}_1(k)} - a_{10}(k)[x_1(k) - \widetilde{x}_1(k)] \right| \\
&\quad = \left| \ln \frac{x_1(k)}{\widetilde{x}_1(k)} \right| - \frac{1}{\xi_1(k)} |x_1(k) - \widetilde{x}_1(k)| + \left| \frac{1}{\xi_1(k)} [x_1(k) - \widetilde{x}_1(k)] - a_{10}(k)[x_1(k) - \widetilde{x}_1(k)] \right| \\
&\quad = \left| \ln \frac{x_1(k)}{\widetilde{x}_1(k)} \right| - \frac{1}{\xi_1(k)} |x_1(k) - \widetilde{x}_1(k)| + \left| \frac{1}{\xi_1(k)} - a_{10}(k) \right| \times |x_1(k) - \widetilde{x}_1(k)| \\
&\quad = \left| \ln \frac{x_1(k)}{\widetilde{x}_1(k)} \right| - \left[\frac{1}{\xi_1(k)} - \left| \frac{1}{\xi_1(k)} - a_{10}(k) \right| \right] \times |x_1(k) - \widetilde{x}_1(k)|.
\end{aligned} \tag{3.50}$$

And hence it follows from (3.47) and (3.50) that

$$\begin{aligned}\Delta V_{11}(k) &= V_{11}(k+1) - V_{11}(k)\\ &\leq -\left[\frac{1}{\xi_1(k)} - \left|\frac{1}{\xi_1(k)} - a_{10}(k)\right|\right] \times |x_1(k) - \tilde{x}_1(k)|\\ &\quad + \sum_{l=1}^{m} a_{1l}(k)|x_1(k-l) - \tilde{x}_1(k-l)| + \sum_{l=0}^{m} c_{2l}(k)|x_2(k-l) - \tilde{x}_2(k-l)|.\end{aligned} \tag{3.51}$$

Step 2. Let

$$V_{12}(k) = \sum_{l=1}^{m}\sum_{s=k-l}^{k-1} a_{1l}(s+l)|x_1(s) - \tilde{x}_1(s)| + \sum_{l=0}^{m}\sum_{s=k-l}^{k-1} c_{2l}(s+l)|x_2(s) - \tilde{x}_2(s)|. \tag{3.52}$$

For the sake of convenience, we will make a convention that $\prod_{i=k_1}^{k_2} b(i) = 1$ if $k_1 > k_2$ for any bounded sequence $b(i)$. By a simple calculation, it derives that

$$\begin{aligned}\Delta V_{12}(k) &= V_{12}(k+1) - V_{12}(k)\\ &= \sum_{l=1}^{m}\sum_{s=k-l+1}^{k} a_{1l}(s+l)|x_1(s) - \tilde{x}_1(s)| + \sum_{l=0}^{m}\sum_{s=k-l+1}^{k} c_{2l}(s+l)|x_2(s) - \tilde{x}_2(s)|\\ &\quad - \sum_{l=1}^{m}\sum_{s=k-l}^{k-1} a_{1l}(s+l)|x_1(s) - \tilde{x}_1(s)| - \sum_{l=0}^{m}\sum_{s=k-l}^{k-1} c_{2l}(s+l)|x_2(s) - \tilde{x}_2(s)|\\ &= \sum_{l=1}^{m} a_{1l}(k+l)|x_1(k) - \tilde{x}_1(k)| - \sum_{l=1}^{m} a_{1l}(k)|x_1(k-l) - \tilde{x}_1(k-l)|\\ &\quad + \sum_{l=0}^{m} c_{2l}(k+l)|x_2(k) - \tilde{x}_2(k)| - \sum_{l=0}^{m} c_{2l}(k)|x_2(k-l) - \tilde{x}_2(k-l)|.\end{aligned} \tag{3.53}$$

Now, we are in a position to define $V_1(k)$ by

$$V_1(k) = V_{11}(k) + V_{12}(k). \tag{3.54}$$

Therefore, it follows from (3.51) and (3.53) that

$$\begin{aligned}\Delta V_1(k) &= \Delta V_{11}(k) + \Delta V_{12}(k)\\ &\leq -\left[\frac{1}{\xi_1(k)} - \left|\frac{1}{\xi_1(k)} - a_{10}(k)\right| - \sum_{l=1}^{m} a_{1l}(k+l)\right] \times |x_1(k) - \tilde{x}_1(k)|\\ &\quad + \sum_{l=0}^{m} c_{2l}(k+l)|x_2(k) - \tilde{x}_2(k)|.\end{aligned} \tag{3.55}$$

By a similar argument, we can define $V_2(k)$ by

$$V_2(k) = V_{21}(k) + V_{22}(k), \tag{3.56}$$

where

$$\begin{gathered} V_{21}(k) = |\ln x_2(k) - \ln \widetilde{x}_2(k)|, \\ V_{22}(k) = \sum_{l=1}^{m} \sum_{s=k-l}^{k-1} a_{2l}(s+l)|x_2(s) - \widetilde{x}_2(s)| + \sum_{l=0}^{m} \sum_{s=k-l}^{k-1} c_{1l}(s+l)|x_1(s) - \widetilde{x}_1(s)|. \end{gathered} \tag{3.57}$$

Then it is easy to derive that

$$\begin{aligned} \Delta V_2(k) &= \Delta V_{21}(k) + \Delta V_{22}(k) \\ &\leq -\left[\frac{1}{\xi_2(k)} - \left|\frac{1}{\xi_2(k)} - a_{20}(k)\right| - \sum_{l=1}^{m} a_{2l}(k+l)\right] \times |x_2(k) - \widetilde{x}_2(k)| \\ &\quad + \sum_{l=0}^{m} c_{1l}(k+l)|x_1(k) - \widetilde{x}_1(k)|, \end{aligned} \tag{3.58}$$

where $\xi_2(k)$ is between $x_2(k)$ and $\widetilde{x}_2(k)$.

Now we can define a Lyapunov-like discrete function $V(k)$ by

$$V(k) = V_1(k) + V_2(k). \tag{3.59}$$

It is easy to see that $V(k) \geq 0$ for all $k \in \mathbf{Z}$ and $V(k_0 + m) < +\infty$. For the arbitrariness of ε and by (3.45), we can choose a small enough $\varepsilon > 0$ such that

$$\min\left\{a_{i0*}, \frac{2}{M_i + \varepsilon} - a_{i0}^{*}\right\} - m a_i^M - (m+1)c_i^M \geq \eta, \quad i = 1, 2. \tag{3.60}$$

Therefore, it follows from (3.55)–(3.60) that

$$\begin{aligned}
\Delta V(k) &= \Delta V_1(k) + \Delta V_2(k) \\
&\le -\sum_{i=1}^{2}\left\{\frac{1}{\xi_i(k)} - \left|\frac{1}{\xi_i(k)} - a_{i0}(k)\right| - \sum_{l=1}^{m} a_{il}(k+l) - \sum_{l=0}^{m} c_{il}(k+l)\right\} \times |x_i(k) - \tilde{x}_i(k)| \\
&\le -\sum_{i=1}^{2}\left\{\min\left\{a_{i0*}, \frac{2}{M_i+\varepsilon} - a_{i0}^{*}\right\} - \sum_{l=1}^{m} a_{il}^{*} - \sum_{l=0}^{m} c_{il}^{*}\right\} \times |x_i(k) - \tilde{x}_i(k)| \\
&\le -\sum_{i=1}^{2}\left\{\min\left\{a_{i0*}, \frac{2}{M_i+\varepsilon} - a_{i0}^{*}\right\} - m a_i^{M} - (m+1)c_i^{M}\right\} \times |x_i(k) - \tilde{x}_i(k)| \\
&\le -\eta\sum_{i=1}^{2}|x_i(k) - \tilde{x}_i(k)| \quad \text{for } k \ge k_0 + m.
\end{aligned} \tag{3.61}$$

Thus, by (3.61) we obtain

$$\begin{gathered}
V(k+1) + \eta \sum_{p=k_0+m}^{k} \sum_{i=1}^{2} \left|x_i(p) - \tilde{x}_i(p)\right| \le V(k_0+m) \quad \text{for any } k \ge k_0 + m, \\
\sum_{k=k_0+m}^{+\infty} \sum_{i=1}^{2} |x_i(k) - \tilde{x}_i(k)| \le \frac{V(k_0+m)}{\eta} < +\infty,
\end{gathered} \tag{3.62}$$

from which we conclude that $\sum_{i=1}^{2} |x_i(k) - \tilde{x}_i(k)| = 0$ when $k \to +\infty$, that is,

$$\lim_{k\to+\infty} |x_1(k) - \tilde{x}_1(k)| = 0, \qquad \lim_{k\to+\infty} |x_2(k) - \tilde{x}_2(k)| = 0. \tag{3.63}$$

According to Definition 1.2, this result implies that the positive periodic solution $\{(\tilde{x}_1(k), \tilde{x}_2(k))^{\top}\}$ is globally attractive. This completes the proof of Theorem 3.4. □

4. Example and Numerical Simulation

In this paper, a discrete time non-autonomous two-species competitive system with delays is investigated. By using difference inequality technique, continuous theorem of coincidence degree theory, and Lyapunov discrete function, sufficient conditions for the permanence of system (1.1) and the existence and global attractivity of positive periodic solutions of system (1.1) are obtained, respectively.

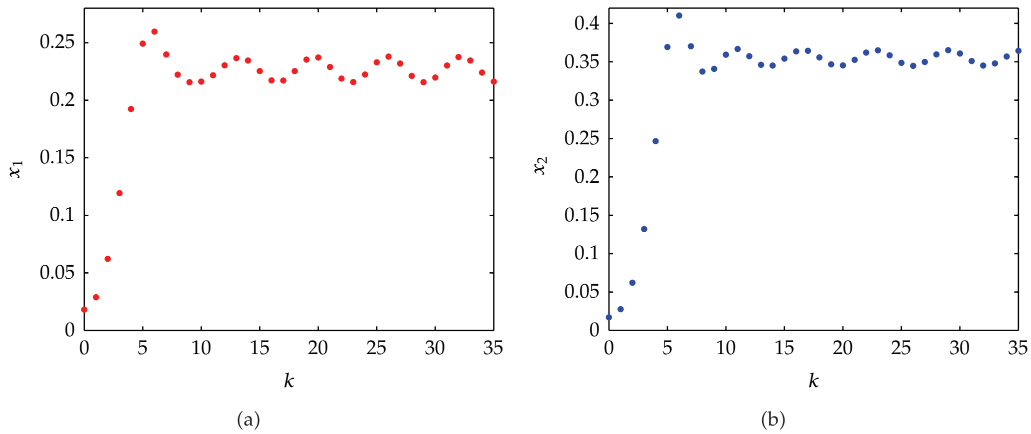


Figure 1: Permanence of system (4.1) with $x_1(-1) = 0.2$, $x_1(0) = 0.018$, $x_2(-1) = 0.3$, and $x_2(0) = 0.017$. (a) Time series of x_1. (b) Time series of x_2.

To substantiate our analytical results, we construct the following example:

$$x_1(k+1) = x_1(k)\exp\left[r_1(k) - a_{10}(k)x_1(k) - a_{11}(k)x_1(k-1) - \frac{c_{20}(k)x_2(k)}{1+x_2(k)} - \frac{c_{21}(k)x_2(k-1)}{1+x_2(k-1)}\right],$$
$$x_2(k+1) = x_2(k)\exp\left[r_2(k) - a_{20}(k)x_2(k) - a_{21}(k)x_2(k-1) - \frac{c_{10}(k)x_1(k)}{1+x_1(k)} - \frac{c_{11}(k)x_1(k-1)}{1+x_1(k-1)}\right]. \tag{4.1}$$

We first verify the sufficient conditions for permanence of system (4.1) and choose the coefficients

$$\begin{aligned} r_1(k) &= 0.85 + 0.04\sin k, & r_2(k) &= 0.87 + 0.01\sin k, \\ a_{10}(k) = a_{11}(k) &= 1.83 + 0.02\sin k, & a_{20}(k) = a_{21}(k) &= 1.20 + 0.03\sin k, \\ c_{10}(k) = c_{11}(k) &= 0.05 + 0.01\sin k, & c_{20}(k) = c_{21}(k) &= 0.04 + 0.01\sin k \end{aligned} \tag{4.2}$$

and the initial conditions

$$x_1(-1) = 0.2, \quad x_1(0) = 0.018, \quad x_2(-1) = 0.3, \quad x_2(0) = 0.017. \tag{4.3}$$

It is easy to see that system (4.1) satisfies assumption (2.6) of Theorem 2.3, and hence system (4.1) is permanent (see Figure 1).

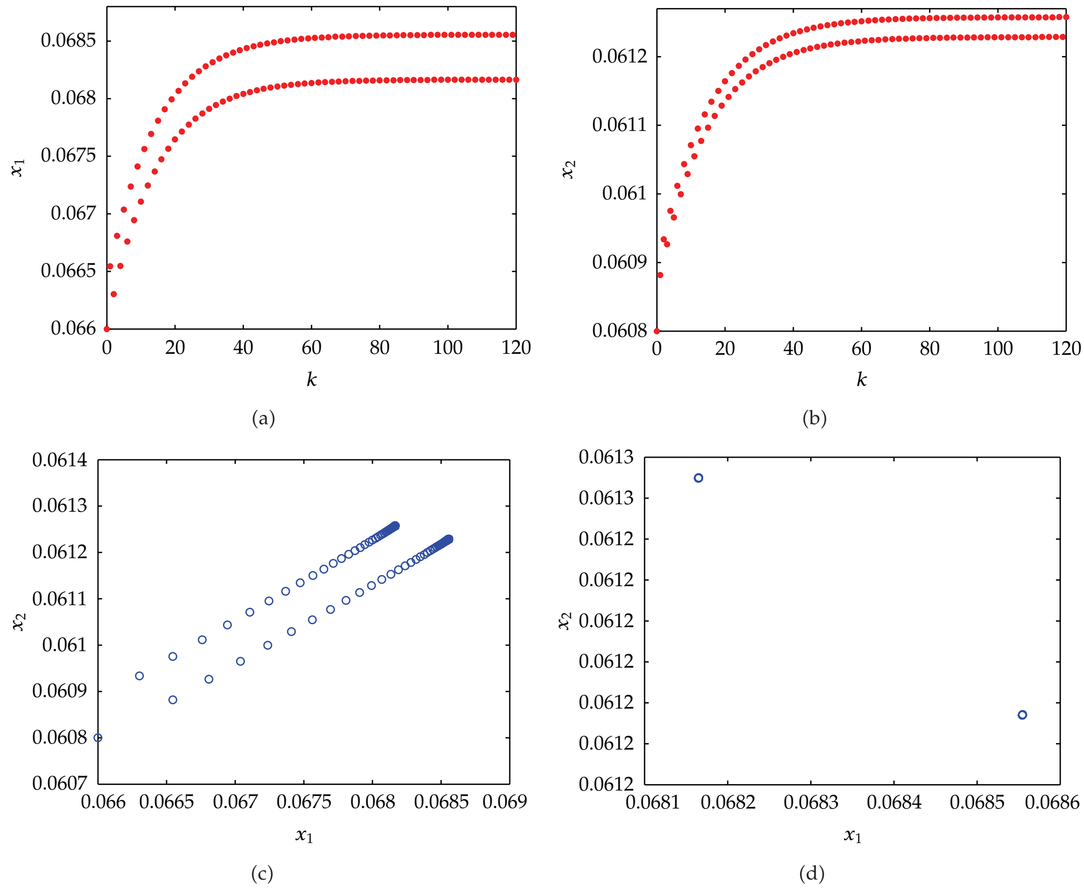
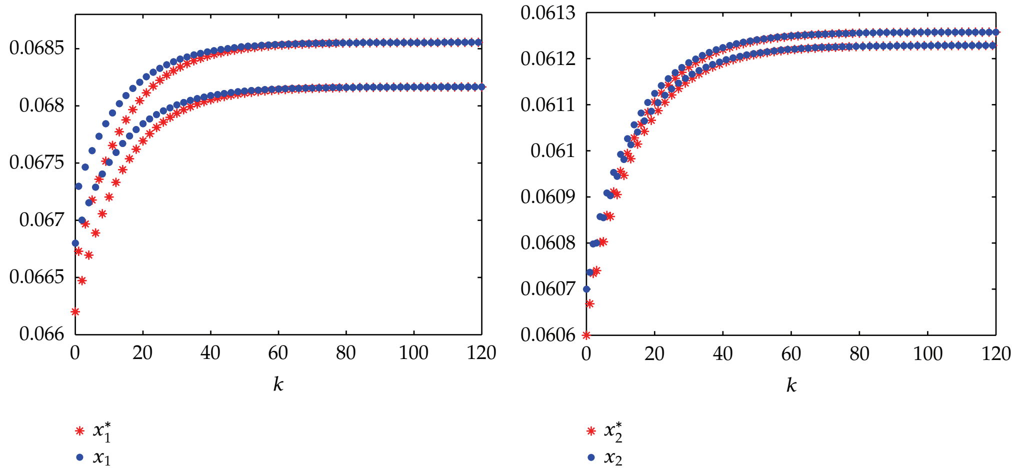


Figure 2: Dynamic behaviors of system (4.1) with $x_1(-1) = 0.02$, $x_1(0) = 0.0660$, $x_2(-1) = 0.03$, and $x_2(0) = 0.0608$. (a) Time series of x_1. (b) Time series of x_2. (c) Phase portrait of x_1 and x_2 with k over $[0, 120]$. (d) Phase portrait of x_1 and x_2 with k over $[110, 120]$.

In the following, we will consider the existence and global attractivity of positive periodic solutions of system (4.1). We assume that

$$
\begin{aligned}
r_1(k) &= 0.0695 + 0.0055\cos \pi k, & r_2(k) &= 0.0665 - 0.0005\cos \pi k,\\
a_{10}(k) &= 1.0080 + 0.0010\cos \pi k, & a_{20}(k) &= 1.0425 - 0.0005\cos \pi k,\\
a_{11}(k) &= 0.0080 - 0.0010\cos \pi k, & a_{21}(k) &= 0.0425 - 0.0005\cos \pi k,\\
c_{10}(k) &= 0.00035 + 0.00015\cos \pi k, & c_{20}(k) &= 0.00035 - 0.00005\cos \pi k,\\
c_{11}(k) &= 0.00045 + 0.00015\cos \pi k, & c_{21}(k) &= 0.00045 - 0.00005\cos \pi k.
\end{aligned}
\tag{4.4}
$$

It is easy to verify that assumptions (3.10) of Theorem 3.3 are satisfied. Figure 2 shows that system (4.1) has a 2-periodic solution $\{(x_1(k), x_2(k))^{\top}\}$, which implies that the two species x_1 and x_2 can coexist. Furthermore, a calculation can show that the assumptions (3.34) of Theorem 3.4 are satisfied, so $\{(x_1(k), x_2(k))^{\top}\}$ is globally attractive, that is, any



(a) (b)

Figure 3: Time series of x_1^*, x_1 and x_2^*, x_2. (a) Time series of x_1^*, x_1 with $x_1^*(-1) = 0.03$, $x_1^*(0) = 0.0662$, $x_1(-1) = 0.02$, and $x_1(0) = 0.0668$. (b) Time series of x_2^*, x_2 with $x_2^*(-1) = 0.04$, $x_2^*(0) = 0.0606$, $x_2(-1) = 0.05$, and $x_2(0) = 0.0607$.

positive solution $\{(x_1^*(k), x_2^*(k))^{\top}\}$ of system (4.1) tends to $\{(x_1(k), x_2(k))^{\top}\}$ (see Figure 3). From Figure 3(a), we see that x_1^* with $x_1^*(-1) = 0.03$ and $x_1^*(0) = 0.0662$ will tend to x_1 with $x_1(-1) = 0.02$ and $x_1(0) = 0.0668$. Similarly, from Figure 3(b), we see that x_2^* with $x_2^*(-1) = 0.04$ and $x_2^*(0) = 0.0606$ will tend to x_2 with $x_2(-1) = 0.05$ and $x_2(0) = 0.0607$.

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