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Research Article

Neimark-Sacker Bifurcation in a Discrete-Time Financial System

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A discrete-time financial system is proposed by using forward Euler scheme. Based on explicit Neimark-Sacker bifurcation (also called Hopf bifurcation for map) criterion, normal form method and center manifold theory, the system's existence, stability and direction of Neimark-Sacker bifurcation are studied. Numerical simulations are employed to validate the main results of this work. Some comparison of bifurcation between the discrete-time financial system and its continuous-time system is given.

1. Introduction

Economic dynamics have recently become more prominent in mainstream economics [1]. The real financial and economic systems show a lot of complex dynamical phenomena, such as, business cycle, financial crisis, irregular growth, and bullwhip effect. Many nonlinear dynamical models of economics and finance [2–9] present various complex dynamical behaviors such as, chaos, fractals, and bifurcation.

Bifurcation refers to a class of phenomena in dynamic systems such that the dynamic properties of the system cause a sudden "qualitative" or topological change when the parameter values (the bifurcation parameters) cross a boundary. Bifurcation boundaries, for example, Hopf bifurcations [10–13], have been discovered in many macroeconomic systems [14]. Hopf bifurcations occur at points where the system has a nonhyperbolic equilibrium with a pair of purely imaginary eigenvalues, but without zero eigenvalues. For a financial or economic system, there can be disequilibrium thresholds where society decides it cannot afford the increasing cost of misallocated resources as disequilibrium increases. Such a threshold then forces a restructuring of the market system. This concept of restructuring to maintain the survival of the system is known as bifurcation theory. A bifurcation in a financial or economic system is a point (or threshold) where the system is restructured to operate at a more acceptable or stable level of disequilibrium. Bifurcations do not usually lead to equilibrium conditions, only to a stable or comfortable disequilibrium condition under which the system can continue to survive [15].

Huang and Li [16] proposed a nonlinear financial model as follows:

$$\begin{aligned} \dot{x} &= z + (y - a)x, \\ \dot{y} &= 1 - by - x^2, \\ \dot{z} &= -x - cz, \end{aligned} \tag{1.1}$$

where x denotes the interest rate, y denotes the investment demand, z denotes the price index, a is the saving amount, b is the cost per investment, c is the demand elasticity of commercial markets, and all three constants $a, b, c \geq 0$.

Chen [1] and Ma et al. [11–13] studied some complex dynamics in system (1.1), such as, a steady state, a periodic oscillation, a quasiperiodic motion and a chaotic motion. In this paper, we will apply the forward Euler scheme to system (1.1) in order to obtain an autonomous discrete-time financial system as follows:

$$\begin{aligned} x_{n+1} &= x_n + \delta(z_n + (y_n - a)x_n), \\ y_{n+1} &= y_n + \delta\left(1 - by_n - x_n^2\right), \\ z_{n+1} &= z_n + \delta(-x_n - cz_n), \end{aligned} \tag{1.2}$$

where $0 < \delta < 1$ is the step size.

An arduous task in the study of nonlinear dynamical systems like system (1.2) is to identify different types of complex nonlinear behaviors and to present how the behavior evolves as a system parameter varies [17, 18]. Thereinto, bifurcation is a very important nonlinear behavior which can indicate a qualitative change of system properties as a system parameter changes. Neimark-Sacker bifurcations give rise to closed invariant curves which present more interesting complex behaviors. The criterion of Hopf bifurcation in continuous-time system can be stated in terms of eigenvalues or the coefficients of characteristic polynomial [19, 20]. The later method, called Schur-Cohn stability criterion, which is more convenient and efficient for detecting the existence of Hopf bifurcation in high-order and multiparameters systems was also demonstrated in discrete dynamical systems [21–23].

The remainder of this paper is organized as follows. In Section 2, we present some preliminaries. In Section 3, we prove stabilities of the fixed points in system (1.2). In Section 4, we analyze the existence of Neimark-Sacker bifurcation in system (1.2) by means of Wen's Neimark-Sacker bifurcation criterion. In Section 5, we study the stability and direction of Neimark-Sacker bifurcation in system (1.2) by utilizing Kuznetsov's normal form method and center manifold theory. In Section 6, we illustrate the Neimark-Sacker bifurcation in system (1.2). In Section 7, we give some comparison of bifurcation between the continuous-time system (1.1) and the discrete-time system (1.2). Finally conclusions in Section 8 close the paper.

2. Preliminaries

Lemma 2.1 (see [24]). *Let F from $\mathbb{R}^n$ to $\mathbb{R}^n$ be C^2. Assume p_0 is a period-k point. Let $\lambda_1, \lambda_2, \dots, \lambda_n$ be the eigenvalues of $D(F^k)_{(p_0)}$.*

(i) *If all the eigenvalues λ_i of $D(F^k)_{(p_0)}$ have $|\lambda_i| < 1$, then the periodic orbit $\mathcal{O}_F^+(p_0)$ is attracting.*

(ii) *If one eigenvalue λ_{i0} of $D(F^k)_{(p_0)}$ has $|\lambda_{i0}| > 1$, then the periodic orbit $\mathcal{O}_F^+(p_0)$ is unstable.*

(iii) *If all the eigenvalues λ_i of $D(F^k)_{(p_0)}$ have $|\lambda_i| > 1$, then the periodic orbit $\mathcal{O}_F^+(p_0)$ is repelling.*

Next, we will study the stability of a nonlinear discrete dynamical system which can be described as follows:

$$X_{t+1} = F(X_t), \quad X(0) = X_0 = (x_{10}, x_{20}, \dots, x_{n0})^T, \text{ where} \tag{2.1}$$

$$F(X_t) = \begin{pmatrix} f_1(x_{1t}, x_{2t}, \dots, x_{nt}) \\ f_2(x_{1t}, x_{2t}, \dots, x_{nt}) \\ \vdots \\ f_n(x_{1t}, x_{2t}, \dots, x_{nt}) \end{pmatrix}, \tag{2.2}$$

$X_t = (x_{1t}, x_{2t}, \dots, x_{nt})^T \in R^n$.

Theorem 2.2. *Let $\widehat{X} = (\widehat{x}_1, \widehat{x}_2, \dots, \widehat{x}_n)^T$ be a fixed point of system* (2.1)*, that is, $\widehat{X} = F(\widehat{X})$ and $A = (\partial F/\partial X)|_{X=\widehat{X}}$ is the Jacobian matrix at the point $\widehat{X}$; then the necessary condition for asymptotically stability of the point $\widehat{X}$ is that*

(i) $|\operatorname{tr}(A^t)| < n$ *for all* $t > 0$,

(ii) $|\det(A^t)| < 1$ *for all* $t > 0$,

where $\operatorname{tr}(A)$ *denotes the trace of A and* $\det(A)$ *the determinant of A.*

Proof. Assume that the point $\widehat{X}$ is asymptotically stable and let $\lambda_1, \dots, \lambda_1, \lambda_n$ be the eigenvalues of the Jacobian matrix A at the point $\widehat{X}$. Then it follows from Lemma 2.1 that all the eigenvalues satisfy

$$|\lambda_i| < 1, \quad i = 1, 2, \dots, n. \tag{2.3}$$

Thus

$$\begin{aligned} \left|\operatorname{tr}(A^t)\right| &= \left|\sum_{i=1}^n \lambda_i^t\right| \le \sum_{i=1}^n |\lambda_i|^t < n, \quad \forall t > 0, \\ \left|\det(A^t)\right| &= \left|\prod_{i=1}^n \lambda_i^t\right| = \prod_{i=1}^n |\lambda_i|^t < 1, \quad \forall t > 0. \end{aligned} \tag{2.4}$$

The theorem is proved. □

Theorem 2.3. *Let $\widehat{X} = (\widehat{x}_1, \widehat{x}_2, \dots, \widehat{x}_n)^T$ be a fixed point of system* (2.1)*, that is, $\widehat{X} = F(\widehat{X})$ and $A = (\partial F/\partial X)|_{X=\widehat{X}}$ is the Jacobian matrix at the point $\widehat{X}$; then the necessary condition for repellent of the point $\widehat{X}$ is that*

(i) $|\operatorname{tr}(A^t)| > n$ *for all* $t > 0$,

(ii) $|\det(A^t)| > 1$ *for all* $t > 0$,

where $\operatorname{tr}(A)$ *denotes the trace of A and* $\det(A)$ *the determinant of A.*

Proof. Assume that the point $\widehat{X}$ is repelling and let $\lambda_1, \dots, \lambda_1, \lambda_n$ be the eigenvalues of the Jacobian matrix A at the point $\widehat{X}$. Then it follows from Lemma 2.1 that all the eigenvalues satisfy

$$|\lambda_i| > 1, \quad i = 1, 2, \dots, n. \tag{2.5}$$

Thus

$$\begin{aligned} \left|\operatorname{tr}\left(A^t\right)\right| &= \left|\sum_{i=1}^{n} \lambda_i^t\right| \le \sum_{i=1}^{n} |\lambda_i|^t > n, \quad \forall t > 0, \\ \left|\det\left(A^t\right)\right| &= \left|\prod_{i=1}^{n} \lambda_i^t\right| = \prod_{i=1}^{n} |\lambda_i|^t > 1, \quad \forall t > 0. \end{aligned} \tag{2.6}$$

The theorem is proved. □

Lemma 2.4 (An explicit criterion of Neimark-Sacker bifurcation [22]). *For an nth-order discrete-time dynamical system like system* (1.2)*, assume first that at the fixed point x_0, its characteristic polynomial of Jacobian matrix $A = (a_{ij})_{n\times n}$ takes the following form:*

$$p_\mu(\lambda) = \lambda^n + a_1\lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n, \tag{2.7}$$

where $a_j = a_j(\mu, k)$, $j = 1, \dots, n, \mu$, is the bifurcation parameter, and k is the control parameter or the other to be determined. Consider the sequence of determinants $\Delta_0^\pm(\mu, k) = 1$, $\Delta_1^\pm(\mu, k), \dots, \Delta_n^\pm(\mu, k)$, where

$$\Delta_j^\pm(\mu,k) = \left|\begin{pmatrix} 1 & a_1 & a_2 & \cdots & a_{j-1} \\ 0 & 1 & a_1 & \cdots & a_{j-2} \\ 0 & 0 & 1 & \cdots & a_{j-3} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} \pm \begin{pmatrix} a_{n-j+1} & a_{n-j+2} & \cdots & a_{n-1} & a_n \\ a_{n-j+2} & a_{n-j+3} & \cdots & a_n & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n-1} & a_n & \cdots & 0 & 0 \\ a_n & 0 & \cdots & 0 & 0 \end{pmatrix}\right|, \quad j = 1, \dots, n. \tag{2.8}$$

If the following conditions hold,

(H1) *eigenvalue assignment* $\Delta_{n-1}^-(\mu_0, k) = 0$, $p_{\mu_0}(1) > 0$, $(-1)^n p_{\mu_0}(-1) > 0$, $\Delta_{n-1}^+(\mu_0, k) > 0$, $\Delta_j^\pm(\mu_0, k) > 0$, $j = n-3, n-5, \dots, 1$ *(or 2) when n is even (or odd, resp.),*

(H2) *transversality condition* $d\Delta_{n-1}^-(\mu_0, k)/d\mu \neq 0$, (H3) *nonresonance condition* $\cos(2\pi/m) \neq \psi$ *or resonance condition* $\cos(2\pi/m) = \psi$, *where* $m = 3, 4, 5, \ldots$ *and* $\psi = 1 - 0.5p_{\mu_0}(1)\Delta^-_{n-3}(\mu_0, k)/\Delta^+_{n-2}(\mu_0, k)$,

then a Neimark-Sacker bifurcation occurs at μ_0.

3. Stability of the Fixed Points

The fixed points of system (1.2) satisfy the following equations:

$$\begin{aligned} x &= x + \delta(z + (y - a)x), \\ y &= y + \delta\left(1 - by - x^2\right), \\ z &= z + \delta(-x - cz). \end{aligned} \tag{3.1}$$

By the analysis of roots for (3.1), one obtains the following proposition.

Proposition 3.1. *(1) If* $c - b - abc \leq 0$, *system* (1.2) *has only one fixed point* $P_0 = (0, 1/b, 0)$.
(2) If $c - b - abc \geq 0$, *system* (1.2) *has three fixed points:*

$$P_1 = \left(0, \frac{1}{b}, 0\right), \qquad P_{2,3} = \left(\pm\sqrt{\frac{c - b - abc}{c}}, \frac{1 + ac}{c}, \mp\frac{1}{c}\sqrt{\frac{c - b - abc}{c}}\right). \tag{3.2}$$

The Jacobian matrix $J(P)$ *of system* (1.2) *evaluated at the fixed point* $P(x^*, y^*, z^*)$ *is given by*

$$J(P) = \begin{pmatrix} 1 + \delta(y^* - a) & \delta x^* & \delta \\ -2\delta x^* & 1 - \delta b & 0 \\ -\delta & 0 & 1 - \delta c \end{pmatrix}. \tag{3.3}$$

Following from Theorem 2.2, it is easy to obtain the following propositions.

Proposition 3.2. *When* $c - b - abc \leq 0$, *the fixed point* P_0 *is not asymptotically stable if*

$$1 - ab - b^2 - bc > 0 \quad \text{or} \quad \left(1 - ab - b^2 - bc\right)h + 6b < 0. \tag{3.4}$$

Proposition 3.3. *When* $c - b - abc \geq 0$,

(1) *the fixed point* P_1 *is not asymptotically stable if*

$$1 - ab - b^2 - bc > 0 \quad \text{or} \quad \left(1 - ab - b^2 - bc\right)h + 6b < 0; \tag{3.5}$$

(2) *the fixed points $P_{2,3}$ are not asymptotically stable if*

$$1 - bc - c^2 > 0 \quad or \quad \left(1 - bc - c^2\right)h + 6c < 0. \tag{3.6}$$

That is, if one of Propositions 3.2 and 3.3 holds, it is possible that bifurcation occurs in system (1.2).

The Jacobian matrix $J(P_0)$ of the system (1.2) *evaluated at the fixed point $P_0 = (0, 1/b, 0)$ is given by*

$$J(P_0) = \begin{pmatrix} 1 + \delta\left(\dfrac{1}{b} - a\right) & 0 & \delta \\ 0 & 1 - \delta b & 0 \\ -\delta & 0 & 1 - \delta c \end{pmatrix}. \tag{3.7}$$

Its eigenvalues can be written as

$$\lambda_{1,2} = \frac{1}{2b}\left(\delta + 2b - ab\delta - bc\delta \pm i\delta\sqrt{4b^2 - (ab - bc - 1)^2}\right), \qquad \lambda_3 = 1 - b\delta. \tag{3.8}$$

Following from Theorem 2.2, it is easy to obtain the following propositions.

Proposition 3.4. *When $c - b - abc \le 0$,*

(1) *the fixed point P_0 is asymptotically stable if $a > (c\delta - b\delta + bc - 1)/b(\delta - 1)$;*

(2) *the fixed point P_0 is unstable if $a < (c\delta - b\delta + bc - 1)/b(\delta - 1)$.*

(3) *a bifurcation occurs at the fixed point P_0 if $a = (c\delta - b\delta + bc - 1)/b(\delta - 1)$.*

4. Existence of Neimark-Sacker Bifurcation

The main task of this paper is to determine the value of bifurcation parameter when the system (1.2) has only one fixed point $P_0 = (0, 1/b, 0)$ with $c - b - abc < 0$.

The characteristic polynomial of the Jacobian matrix (3.7) is

$$p(\lambda) = \lambda^3 + p_2\lambda^2 + p_1\lambda + p_0 = 0, \tag{4.1}$$

where

$$\begin{aligned} p_1 &= \delta\left(a + b + c - \frac{1}{b}\right) - 3, \\ p_2 &= \delta^2\left(ab + ac + bc - \frac{c}{b}\right) - 2\delta\left(a + b + c - \frac{1}{b}\right) + 3, \\ p_3 &= \delta^3(abc + b - c) - \delta^2\left(ab + ac + bc - \frac{c}{b}\right) + \delta\left(a + b + c - \frac{1}{b}\right) - 1. \end{aligned} \tag{4.2}$$

According to Lemma 2.4, for $n = 3$, we can get the following equalities and inequalities:

$$\begin{gathered}
\Delta_2^-(a) = \left| \begin{pmatrix} 1 & p_1 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} p_2 & p_3 \\ p_3 & 0 \end{pmatrix} \right| = 1 + \frac{1}{b^2}(\delta b - 1)^2 \left(\delta^2 b - \delta bc - \delta ab + \delta + b + \delta^2 abc - \delta^2 c\right)^2 > 0, \\
p_a(1) = \delta^3 (abc + b - c) > 0, \\
(-1)^3 p_a(-1) = \frac{1}{b}(2 - b\delta)\left(4b - 2bc\delta + 2\delta - c\delta^2 - 2ab\delta + abc\delta^2 + b\delta^2\right) > 0, \\
\Delta_2^+(a) = \left| \begin{pmatrix} 1 & p_1 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} p_2 & p_3 \\ p_3 & 0 \end{pmatrix} \right| = 1 - \frac{1}{b^2}(\delta b - 1)^2 \left(\delta^2 b - \delta bc - \delta ab + \delta + b + \delta^2 abc - \delta^2 c\right)^2 > 0.
\end{gathered} \tag{4.3}$$

According to (4.3), the critical value of Neimark-Sacker bifurcation of system (1.2) can be obtained as

$$a^* = \frac{c\delta - b\delta + bc - 1}{b(\delta - 1)}. \tag{4.4}$$

Thus, it follows (3.8) that the eigenvalues modules $|\lambda_{1,2}| = 1$, $|\lambda_3| = 1 - b\delta$ satisfy the condition (H1) in Lemma 2.4, that is, Neimark-Sacker bifurcation occurs at the fixed point $P_0 = (0, 1/b, 0)$.

5. Direction and Stability of the Neimark-Sacker Bifurcations

In this section, we will use Kuznetsov's normal form method and center manifold theory [25] to investigate the direction and stability of the Neimark-Sacker bifurcations in system (1.2).

Since the fixed point $P_0 = (0, 1/b, 0)$ is not the origin $O(0, 0, 0)$, the P_0 needs to be transformed to the origin by the change of variables

$$x = u, \qquad y = \frac{1}{b} + v, \qquad z = w. \tag{5.1}$$

This transforms system (1.2) into the following equivalent system:

$$\begin{gathered}
u_{n+1} = u_n + \delta\left(w_n + \left(v_n - a + \frac{1}{b}\right)u_n\right), \\
v_{n+1} = v_n + \frac{1}{b} - \delta\left(bv_n + u_n^2\right), \\
w_{n+1} = w_n - \delta(u_n + cw_n),
\end{gathered} \tag{5.2}$$

This system can be written as

$$X_{n+1} = JX_n + \frac{1}{2}B(X_n, X_n) + \frac{1}{6}C(X_n, X_n, X_n) + O\left(X_n^4\right), \tag{5.3}$$

where J is the Jacobin matrix of system (5.2) evaluated at the origin $O(0,0,0)$ as follows.

$$J(O) = \begin{pmatrix} \sqrt{2\delta^2 + 2c\delta - c^2\delta^2 + 1} & 0 & \delta \\ 0 & 1 - \delta b & 0 \\ -d & 0 & 1 - \delta c \end{pmatrix}. \tag{5.4}$$

And the multilinear functions $B : \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}^2$ and $C : \mathbb{R}^3 \times \mathbb{R}^3 \rightarrow \mathbb{R}^3$ are defined, respectively, by

$$\begin{aligned} B_i(x,y) &= \sum_{j,k=1}^{n} \frac{\partial^2 X_i(\xi,0)}{\partial \xi_j \partial \xi_k}\bigg|_{\xi=0} x_j y_k, \quad i = 1,2,3, \\ C_i(x,y,z) &= \sum_{j,k,l=1}^{n} \frac{\partial^3 X_i(\xi,0)}{\partial \xi_j \partial \xi_k \partial \xi_l}\bigg|_{\xi=0} x_j y_k z_l, \quad i = 1,2,3. \end{aligned} \tag{5.5}$$

For the system (5.2),

$$B(\xi,\eta) = \begin{pmatrix} \delta\xi_1\eta_2 \\ -\delta\xi_1\eta_1 \\ 0 \end{pmatrix}, \qquad C(\xi,\eta,\zeta) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{5.6}$$

The eigenvalues of the matrix $J(O)$ are

$$\lambda_{1,2} = \frac{1}{2b}\left(\delta + 2b - a^* b\delta - bc\delta \pm ih\sqrt{4b^2 - (a^* b - bc - 1)^2}\right) = e^{\pm i\theta_0}, \quad \text{where } 0 < \theta_0 < \pi. \tag{5.7}$$

Let $q \in C^3$ be a complex eigenvector of the matrix J corresponding to λ_1 given by (5.7), and satisfy

$$Jq = e^{i\theta_0} q. \tag{5.8}$$

Let $p \in C^3$ be a complex eigenvector of the transposed matrix J corresponding to λ_2 given by (5.7), and satisfy

$$J^T p = e^{-i\theta_0} p. \tag{5.9}$$

Then we can obtain

$$q \sim \left(\frac{1}{h}(1 - hc - \lambda_1), 0, 1\right)^T, \qquad p \sim \left(\frac{1}{h}(1 - hc - \lambda_2), 0, 1\right)^T. \tag{5.10}$$

For the eigenvector $q = (1/h(1 - hc - \lambda_1), 0, 1)^T$, to normalize p, let

$$p = \left(\frac{-2K^2}{|K^2|(c - a + (1/b) + (K_A/b\delta)) - bK\delta}, 0, \frac{-4b\delta}{K(c - a + (1/b) + (K_A/b\delta)) - 4b\delta} \right)^T, \tag{5.11}$$

where

$$K_A = ih\sqrt{4b^2 - (ab - bc - 1)^2}, \quad K = ab\delta - \delta - bc\delta + K_A. \tag{5.12}$$

We have $\langle p, q \rangle = 1$, where $\langle \cdot, \cdot \rangle$ means the standard scalar product in $\mathbb{C}^2 : \langle p, q \rangle = \overline{p}_1 q_1 + \overline{p}_2 q_2$

So the coefficients of the normal of the system (5.2) can be computed by the formulas as follows:

$$\begin{aligned}
g_{20} &= \langle p, B(q, q) \rangle, \\
g_{11} &= \langle p, B(q, \overline{q}) \rangle, \\
g_{02} &= \langle p, B(\overline{q}, \overline{q}) \rangle, \\
g_{21} &= \langle p, C(q, q, \overline{q}) \rangle + 2\left\langle p, B\left(q, (I_n - J)^{-1} B(q, \overline{q})\right) \right\rangle + \left\langle p, B\left(\overline{q}, \left(e^{2i\theta_0} I_n - J\right)^{-1} B(q, q)\right) \right\rangle \\
&\quad + \frac{e^{-i\theta_0}(1 - 2e^{i\theta_0})}{1 - e^{i\theta_0}} g_{20} g_{11} + \frac{2}{1 - e^{-i\theta_0}} |g_{11}|^2 + \frac{e^{i\theta_0}}{e^{3i\theta_0} - 1} |g_{02}|^2.
\end{aligned} \tag{5.13}$$

The direction coefficient of bifurcation of a closed invariant curve can be obtained by following formula

$$d = \mathrm{Re}\left(\frac{e^{-i\theta_0} g_{21}}{2} \right) - \mathrm{Re}\left(\frac{e^{-2i\theta_0}(1 - 2e^{i\theta_0})}{2(1 - e^{i\theta_0})} g_{20} g_{11} \right) - \frac{1}{2}|g_{11}|^2 - \frac{1}{4}|g_{02}|^2. \tag{5.14}$$

Thus we can obtain the theorem as follows.

Proposition 5.1. *The direction and stability of Neimark-Sacker bifurcation of system* (1.2) *can be determined by the sign of d. If $d < 0$ (> 0), then the Neimark-Sacker bifurcation of system* (1.2) *at $a^* = (c\delta - b\delta + bc - 1)/b(\delta - 1)$ is supercritical (subcritical), and the unique closed invariant curve bifurcating from $P_0 = (0, 1/b, 0)$ is asymptotically stable (unstable).*

6. Numerical Simulations

In this section, we will give an example to illustrate above analytic results.

Let $h = 0.3$, $b = 0.6$, and $c = 0.2$ with an initial state $(x_0, y_0, z_0) = (0.4, 0.6, 0.8)$; we can obtain the critical saving amount $a^* = (c\delta - b\delta + bc - 1)/b(\delta - 1) = 1.773$. By substituting this into (5.14), we have $d = -1.83 < 0$. It follows from Proposition 5.1 that system (1.2) undergoes

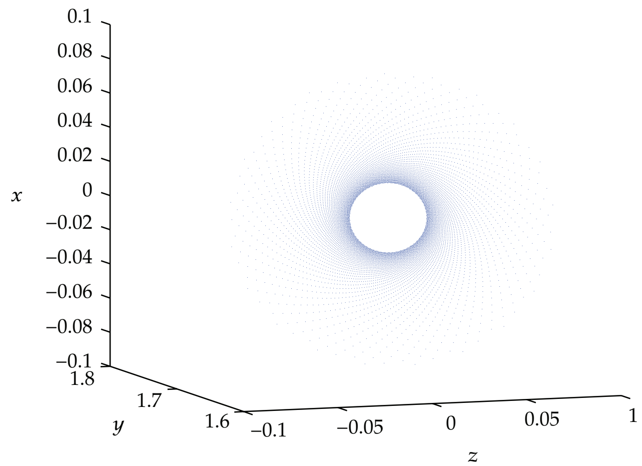

Figure 1: Phase portrait with $a = 1.7731$

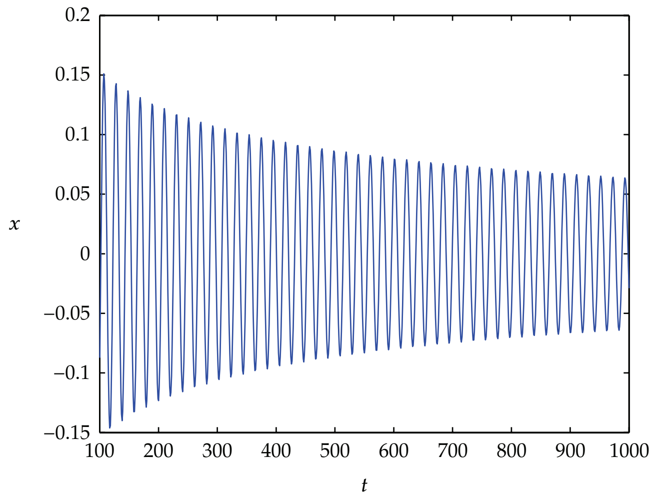

Figure 2: Time histories with $a = 1.7731$

a supercritical Neimark-Sacker bifurcation at $a = 1.773$. When we give a small perturbation $\Delta a = 0.0001$, a sufficiently small positive real number, that is, $a = a^* + \Delta a = 1.773 + 0.0001 = 1.7731$, system (1.2) has a stable closed invariant curve around the equilibrium (quasiperiodic solution), as shown in Figures 1 and 2.

7. Comparison

For system (1.1) at the fixed point $P_0 = (0, 1/b, 0)$ with $c - b - abc < 0$, Ma and Chen [11] gave the critical value of Hopf bifurcation as follows.

$$a^* = \frac{1}{b} - c. \tag{7.1}$$

By simple calculation, we can get the following conclusions.

Proposition 7.1. *Hopf bifurcations of continuous-time system* (1.1) *and discrete-time system* (1.2) *occur simultaneously at* $a = 1/b - 1$ *when* $c = 1$.

Proposition 7.2. *The continuous-time system* (1.1) *undergoes Hopf bifurcation earlier than the discrete-time system* (1.2) *when*

(I) $c < 1$ *and* $(b + \delta)/b(1 - \delta) < -1$

or

(II) $c > 1$ *and* $(b + \delta)/b(1 - \delta) < 1$.

Proposition 7.3. *The discrete-time system* (1.2) *undergoes Hopf bifurcation earlier than the continuous-time system* (1.1) *when*

(I) $c < 1$ *and* $(b + \delta)/b(1 - \delta) < 1$

or

(II) $c > 1$ *and* $(b + \delta)/b(1 - \delta) > 1$.

8. Conclusion

In this paper, we introduce a discrete-time financial system obtained by Euler method. The existence of Neimark-Sacker bifurcation is studied by means of Wen's Neimark-Sacker bifurcation criterion. The stability and direction of Neimark-Sacker bifurcation are proved by utilizing Kuznetsov's normal form method and center manifold theory. Numerical simulations are used to illustrate the above main results. We give Some comparison of bifurcation between the discrete-time financial system and its continuous-time system.

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